Azerbaijani Astronomical Journal AAJ **21(1)**, 59–61 (2026) Accepted 31 May 2026 https://doi.org/10.59849/2078-4163.2026.1.59

RELATIONSHIP BETWEEN THE CONFORMAL GROUP AND MAXIMAL GROUPS OF MOTIONS IN COSMOLOGY

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Last updated 2026 March 9; in original form 2026 May 28

ABSTRACT

The maximal groups of motions of the Minkowski, de Sitter, and anti-de Sitter spacetime are the Poincaré group, the inhomogeneous Lorentz group, and the de Sitter groups of the first and second kind, respectively. It has been proven that these groups are subgroups of the conformal group locally and globally.

Key words: $ISO_0(3,1)-$ Poincaré group, $SO(3,2), SO(4,1)-$ de Sitter groups, $SO(4,2)-$ conformal group.
AMS Subject Classification: 22-22E, 83-83C, 83-83F

1 INTRODUCTION

There are only three cosmological models with maximal groups of motions - the Minkowski, de Sitter, and anti-de Sitter worlds, Tolman (1969); Weinberg (1972). These worlds' maximal groups of motions are the Poincaré group, i.e., the inhomogeneous Lorentz group, and the de Sitter groups of the 1st and 2nd kind, respectively. Recall that according to Wigner's theorem, the main quantum numbers of elementary particles are determined by the invariants of the maximal groups of motions.

It is well known that these groups are local subgroups of the conformal group. In other words, the Lie algebras of the Poincaré group, de Sitter groups of the 1st and 2nd kind are subalgebras of the Lie algebra of the conformal group.Moreover, a stronger statement is true: the Poincaré and de Sitter groups of both kinds are subgroups of the conformal group not only locally, but also globally. The proof is based on the decomposition of the elements of the conformal group by the stationary subgroup of the 5-dimensional cone, which is a transitive surface.

2 THE VACUUM SOLUTIONS OF EINSTEIN'S EQUATIONS

According to Einstein's general relativity, the metric properties of space-time are determined by the distribution and motion of matter, Einstein (1965-1967); Moeller (1972):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}; \quad R = R^{\mu}_{\mu} \tag{1}$$

where

$R_{\mu\nu}$ – Ricci tensor, $g_{\mu\nu}$ – metric tensor, Λ – cosmological constant, G – the gravitational constant of Newton, $T_{\mu\nu}$ – energy-momentum tensor and $\mu, \nu = 0, 1, 2, 3..$

This equation can be rewritten in an equivalent form:

$$R_{\mu\nu} = \Lambda g_{\mu\nu} - \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T); \quad T = T^{\mu}_{\mu} \tag{2}$$

Einstein's equations for vacuum are as follows:

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{3}$$

It is clear from (3) that the Λ-term, even in the absence of matter ($T_{\mu\nu} = 0$), changes the space-time geometry and $g_{kl}\Lambda$ is the energy-momentum tensor of the vacuum.

For the zero value of the cosmological constant, the solution of this equation is the flat world of Minkowski, for which the Poincaré group is the maximal group of motion.

$$ds^2 = dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) - c^2dt^2. \tag{4}$$

In the general case, the solutions of the Einstein equations do not have a maximal group of motions.

In 1917 Willem de Sitter found two solutions with different global groups of movements:

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) - \left(1 - \frac{r^2}{R^2}\right)c^2dt^2, \tag{5}$$

if $\Lambda > 0$;

$$ds^2 = \frac{dr^2}{1 + r^2/R^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) - \left(1 + \frac{r^2}{R^2}\right)c^2dt^2, \tag{6}$$

if $\Lambda < 0$.

Here R is the radius of curvature and is related to the cosmological constant Λ:

$$\Lambda = \pm\frac{3}{R^2}.$$

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Using stereographic projections:

$$\xi_1 = r\cos\varphi; \quad \xi_2 = r\sin\vartheta\cos\varphi; \quad \xi_3 = r\sin\vartheta\sin\varphi;$$
$$\xi_4 = R\sqrt{1-\frac{r^2}{R^2}}\cosh\left(\frac{ct}{R}\right); \quad \xi_0 = R\sqrt{1-\frac{r^2}{R^2}}\sinh\left(\frac{ct}{R}\right)$$

and

$$\eta_1 = r\cos\varphi; \quad \eta_2 = r\sin\vartheta\cos\varphi; \quad \eta_3 = r\sin\vartheta\sin\varphi;$$
$$\eta_4 = R\sqrt{1+\frac{r^2}{R^2}}\cosh\left(\frac{ct}{R}\right); \quad \eta_5 = R\sqrt{1+\frac{r^2}{R^2}}\sinh\left(\frac{ct}{R}\right)$$

de Sitter solutions can be isometrically embedded as submanifolds in 5−dimensional pseudo-Euclidean spaces:

$$\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2 = -R^2 \tag{7}$$

and

$$\eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_4^2 - \eta_5^2 = -R^2, \tag{8}$$

respectively.

These are the worlds of de Sitter of the 1st and 2nd kind, in other words, the worlds of de Sitter and anti-de Sitter, (Rajabov 2021).

These spaces have global symmetry groups $SO(4,1)$ and $SO(3,2)$, which leave the metric unchanged and are called de Sitter groups. These are the worlds of de Sitter of the 1st and 2nd kind, in other words, the worlds of de Sitter (DS) and anti-de Sitter (AdS), respectively.

Of particular interest is the connection of these groups with the conformal group, which H.Weyl introduced in connection with the conformally invariant theory of gravitation.

3 $SO(4,2)$−CONFORMAL GROUP AND ITS SUBGROUPS

The group $SO_0(4,2)$ is a connected component of the unit of the group of motions of the 6-dimensional pseudo-Euclidean space, which leaves the quadratic form invariant, Barut and Raczka (1977); Gursey (1964):

$$[k,k] = k_1^2 + k_2^2 + k_3^2 + k_4^2 - k_5^2 - k_6^2.$$

Elements of $g \in SO(4,2)$ are represented by $6\times 6-$matrices that perform linear transformations over $k = (k_1,k_2,k_3,k_4,k_5,k_6)$. From the definition it follows that the following relations are satisfied:

$$g^\top \eta g = \eta, \qquad \det g = 1, \tag{9}$$

where the $\top$ means transposition and η is the diagonal $6\times 6-$matrix:

$$\eta = \mathrm{diag}(1,1,1,1,-1,-1)$$

We will number the matrix elements with indices: $a,b,c = 1,2,3,4,5,6$.

The groups $SO(4,1)$ and $SO(3,2)$ are subgroups of the conformal group $SO(4,2)$. Let us show that the Poincaré group is also a subgroup of $SO(4,2)$ globally.

Transitivity surfaces of the group $SO_0(4,2)$:

$$[k,k] = k_1^2 + k_2^2 + k_3^2 + k_4^2 - k_5^2 - k_6^2 = c,$$

pass into themselves under transformations from the group $SO_0(4,2)$. Depending on the sign of c, there are three types of transitive surfaces: a hyperboloid, if $c>0$ or $c<0$, and a cone with a punctured vertex, if $c=0$, because the point $k_1=k_2=k_3=k_4=k_5=k_6=0$ is the vertex of the cone and forms a homogeneous space.

A cone with a punctured vertex plays an exceptional role in our study, so we restrict ourselves to the study of invariant measures and coordinate systems on the cone.

We introduce a spherical coordinate system on the cone $[k,k]=0$:

$$k = \omega\boldsymbol{n}, \qquad \boldsymbol{n} = (\xi,\zeta), \qquad 0<\omega<\infty, \qquad [\boldsymbol{n},\boldsymbol{n}] = 0.$$

Here ξ and ζ are 4-dimensional and 2-dimensional unit vectors, respectively[1]:

$$\begin{aligned}
&\zeta = (\sin\psi,\cos\psi), \qquad \zeta^2 = 1, \qquad 0\leqslant\psi<2\pi;\\
&\xi = (\cos\chi,\ \sin\chi\cos\vartheta, \qquad , \qquad \xi^2 = 1;\\
&\qquad\qquad \sin\chi\sin\vartheta\cos\varphi,\\
&\qquad\qquad \sin\chi\sin\vartheta\sin\varphi)\\
&\quad 0\leqslant\varphi<2\pi, \qquad 0\leqslant\vartheta,\chi\leqslant\pi.
\end{aligned} \tag{10}$$

As a fixed point $\overset{\circ}{k}$ it is convenient to choose point with spherical coordinates $\omega=1$; $\psi=\vartheta=\varphi=\chi=0$:

$$\overset{\circ}{k} = (1,0,0,0,0,1).$$

The transformation $h(k)$ (the so-called "boost" or the Wigner operator) that maps a fixed point $\overset{\circ}{k}$ to a given point k is defined as follows:

$$h(k) = g_{34}(\varphi)g_{23}(\vartheta)g_{12}(\chi)g_{56}(\psi)g_{16}(\beta), \quad \beta = \ln\omega \tag{11}$$

The stationary group W of the point $\overset{\circ}{k}$ is isomorphic to the group of motions of the 4-dimensional pseudo-Euclidean space M_4, that is, Poincarè group $ISO(3,1)$:

$$W \simeq ISO(3,1).$$

Each element of $w\in W$ uniquely represented as:

$$w = R(r)B(\vec{z}), \tag{12}$$

where $\vec{z}$ is a 4-dimensional vector, and r is the pseudo-orthogonal 4×4-matrix:

$$r^\top\tilde{\eta}r = \tilde{\eta}; \qquad \tilde{\eta} = \mathrm{diag}\,(1,1,1,-1)\,.$$

The matrices $R(r)$ and $B(\vec{z})$ are elements of the rotation and translation subgroups, respectively:

$$R(r) \in SO(3,1) \subset SO(4,2),$$

$$B(\vec{z}) \in W \simeq ISO(3,1), \quad z = \vec{z}^{\,2} = \vec{z}^{\,\top}\tilde{\eta}\vec{z} = z_1^2+z_2^2+z_3^2-z_4^2.$$

The matrices $B(\vec{z})$ form an Abelian subgroup, which is a normal divisor of the stationary subgroup $W\simeq ISO(3,1)$:

[1] The spherical coordinate system does not constitute an atlas. Therefore, all of the following relations hold almost everywhere or for almost all elements of the group.

$$B(\vec{z}_1\,)B(\vec{z}_2\,) = B(\vec{z}_1+\vec{z}_2\,), \qquad R(r^{-1})B(\vec{z}\,)R(r) = B(r^{-1}\vec{z}\,).$$

Thus, for $g \in SO(4,2)$ we have:

$$g = h(k)w. \tag{13}$$

Using the last formulas, we obtain the expansion for the elements of the group $SO(4,2)$:

$$g = g_{34}(\varphi)g_{23}(\vartheta)g_{12}(\chi)g_{56}(\psi)g_{16}(\beta)R(r)B(\vec{z}\,). \tag{14}$$

Thus, we have proven the following proposition:

The groups $SO(4,1)$, $SO(3,2)$, and $ISO(3,1)$ are, up to isomorphism, subgroups of the conformal group $SO(4,2)$, globally.

4 CONCLUSION

The principle of general covariance in general relativity makes the interpretation of solutions and the definition of the energy-momentum tensor ambiguous. The complexity of this problem is related, among other things, to the non-trivial symmetry properties of vacuum solutions to Einstein's equations, which create serious difficulties in defining the quantum vacuum, Rajabov (2022).

Without the maximal group of motions, using the methods of relativistic quantum field theory is impossible. Thus, the gravitational field is not amenable to quantization, and the unification of general relativity with quantum mechanics is a pressing problem in modern physics. A reasonable question arises: is it possible to apply quantum mechanics on the scale of the Universe? All this forces us to look for other possibilities for solving the problems of high-energy physics and cosmology, Rajabov (2021).

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