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Triangular Representation of the Solution to the Schrödinger Equation with an Additional Linear Potential

A.Kh. Khanmamedov*, D.H. Orudjov

Abstract. This work considers the Schrödinger equation with an additional linear potential on the whole axis. For a potential with a finite first moment, we prove the validity of the triangular representation of the Jost solution with the condition on $+\infty$. Estimates for the kernel of the triangular representation are obtained.

Key Words and Phrases: Schrödinger equation, additional linear potential, triangular representation, the Jost solution, inverse problem.

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1. Introduction and main result

Let us consider the Schrödinger equation of the form

$$-y'' + xy + q(x)y = \lambda y, \qquad -\infty < x < +\infty. \tag{1}$$

where $q(x)$ is the real potential which satisfies the condition

$$\int_{-\infty}^{+\infty} (1 + |x|) |q(x)| \, dx < \infty. \tag{2}$$

The inverse spectral problem for equation (1) was studied in [1]. Furthermore, on the basis of the formal triangular representation of the solution of equation (1), the main integral equation of Gelfand-Levitan was obtained. Later in [2], an accurate justification of the validity of the above mentioned triangular representation (see also [3]) was given for the class of potentials

$$q(x) \in C^{(1)}(-\infty, +\infty), \int_{-\infty}^{+\infty} (1 + |x|) |q(x)| \, dx < \infty.$$

*Corresponding author.

However, the proof of Lemma 1.3 in [2], which plays an essential role in the proof of the main result cannot be considered satisfactory. Indeed, it follows from the integral equation (1.8) in [2] that

$$\frac{\partial \tilde{K}(\xi_0,\eta_0)}{\partial \xi_0} = \frac{1}{4} q(\xi_0) - \frac{1}{4}\int_0^{\eta_0} q\left(\frac{\xi_0-\eta}{2}\right)\tilde{K}(\xi_0,\eta)\,d\eta - \\ -\frac{1}{4}\int_0^{\eta_0}\int_{\xi_0}^{\infty}\frac{\partial V(\xi_0,\eta_0,\xi,\eta)}{\partial \xi_0} q\left(\frac{\xi-\eta}{2}\right)\tilde{K}(\xi,\eta)\,d\xi d\eta.$$

The last equation shows that to establish the asymptotes $\frac{\partial \tilde{K}(\xi_0,\eta_0)}{\partial \xi_0} = O\left(\eta_0^2\right)$ and $\frac{\partial^2 \tilde{K}(\xi_0,\eta_0)}{\partial \xi_0^2} = O\left(\eta_0^4\right)$ (see formula (1.15) in [2]), additional restrictions on functions $q(\xi_0)$ and $q'(\xi_0)$ when $\xi_0 \to \infty$ are required. It should be noted that such questions also apply to the work [3], where the existence of a triangular representation of the solution to equation (1) is established for another class of potentials. Moreover, in addition to the above shortcomings, the arguments presented in [2, 3] are not sufficient to assert the validity of the triangular representation of the Jost solution of equation (1) in the case of a potential $q(x)$ from the class (2).

In this paper, we prove the validity of the triangular representation of the Jost solution of equation (1) for potentials from the class (2). The results of this work also justify the spectral problems for equations (1) studied in [1, 2, 3]. Moreover, the main result (see the theorem below) is also valid in the case of a complex potential from the class (2). Note that various spectral problems for equation (1) have recently been actively studied by many authors (see [4, 5, 6, 7] and references therein).

Let us formulate the main result of our work. Let $\sigma(x) = \int_x^{\infty} |q(t)|\,dt, \sigma_1(x) = \int_x^{\infty}\sigma(t)\,dt$, $f_0(x,\lambda) = Ai(x-\lambda)$, where $Ai(x)$ is the Airy function (see [1, 2, 3]) of the first kind.

Theorem 1. *If the potential satisfies condition* (2)*, then the equation* (1) *has a solution* $f(x,\lambda)$ *that can be represented as follows:*

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^{\infty} K(x,t)\, f_0(t,\lambda)\,dt. \tag{3}$$

For the kernel $K(x,t)$*, the following relations hold:*

$$|K(x,t)| \le \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)e^{\sigma_1(x)}, \tag{4}$$

$$K(x,x) = \frac{1}{2}\int_x^{\infty} q(t)\,dt. \tag{5}$$

2. Proof of the Theorem 1

Without loss of generality, we will assume that $x \geq 0$. Consider the function

$$R(\xi, \eta, \xi_0, \eta_0) = J_0\left(2\sqrt{\left(\eta_0^2 - \eta^2\right)(\xi - \xi_0)}\right)$$

in domain $0 \leq \eta \leq \eta_0 \leq \xi_0 \leq \xi < \infty$, where $J_0(z)$ is a Bessel function of the first kind. The following properties of the function $R(\xi, \eta, \xi_0, \eta_0)$ have been established in [2]:

$$\begin{array}{l} |R| \leq 1, \ \left|\frac{\partial R}{\partial \xi_0}\right| \leq \frac{1}{4}\left(\eta_0^2 - \eta^2\right), \ \left|\frac{\partial R}{\partial \eta_0}\right| \leq \frac{1}{2}\eta_0(\xi - \xi_0), \\ \left|\frac{\partial^2 R}{\partial \xi_0^2}\right| \leq \frac{1}{48}\left(\eta_0^2 - \eta^2\right)^2, \ \left|\frac{\partial^2 R}{\partial \eta_0^2}\right| \leq \frac{1}{3}\eta^2(\xi - \xi_0)^2, \left|\frac{\partial^2 R}{\partial \xi_0 \partial \eta_0}\right| \leq \frac{1}{4}\eta_0 . \end{array} \tag{6}$$

We precede the proof of the theorem with the following lemma.

Lemma 1. *If the function $q(x)$ satisfies condition (2), then the integral equation*

$$U(\xi_0, \eta_0) = \frac{1}{2}\int_{\xi_0}^{\infty} R(\xi, 0, \xi_0, \eta_0)\, q(\xi)\, d\xi +$$

$$+\int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} U(\xi, \eta)\; R(\xi, \eta, \xi_0, \eta_0)\, q(\xi - \eta)\, d\eta \tag{7}$$

has a unique solution in the domain $0 \leq \eta_0 \leq \xi_0$. Additionally, $U(\xi_0, \eta_0)$ satisfies the estimate

$$|U(\xi_0, \eta_0)| \leq \frac{1}{2}\sigma(\xi_0)\, e^{\sigma_1(\xi_0 - \eta_0)}. \tag{8}$$

Proof. We will use the method of successive approximations. Let

$$U_0(\xi_0, \eta_0) = \frac{1}{2}\int_{\xi_0}^{\infty} R(\xi, 0; \xi_0, \eta_0) q(\xi) d\xi,$$

$$U_n(\xi_0, \eta_0) = \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} U_{n-1}(\xi, \eta) q(\xi - \eta) R(\xi, \eta; \xi_0, \eta_0) d\eta.$$

Taking into account the above estimate $|R| \leq 1$, we have

$$|U_0(\xi_0, \eta_0)| \leq \frac{1}{2}\int_{\xi_0}^{\infty} |\, R(\xi, 0; \xi_0, \eta_0)\,|\ |q(\xi)|\, d\xi \leq \frac{1}{2}\int_{\xi_0}^{\infty} |q(\xi)|\, d\xi = \frac{1}{2}\sigma(\xi_0)$$

Further, we find that

$$\begin{array}{l} |U_1(\xi_0, \eta_0)| \leq \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} |U_0(\xi, \eta)| \cdot |q(\xi - \eta)| \cdot r(\xi, \eta;\, \xi_0, \eta_0)\, d\eta \leq \\ \leq \frac{1}{2}\int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} \sigma(\xi)\ \ |q(\xi - \eta)|\, d\eta \leq \frac{1}{2}\int_{\xi_0}^{\infty} \sigma(\xi)\, d\xi \int_0^{\eta_0} |q(\xi - \eta)|\ d\eta \leq \\ \leq \frac{\sigma(\xi_0)}{2}\int_{\xi_0}^{\infty} d\xi\ \int_0^{\eta_0} |q(\xi - \eta)|\ d\eta = \frac{\sigma(\xi_0)}{2}\int_{\xi_0}^{\infty} d\xi\ \int_{\xi - \eta_0}^{\xi} |q(\alpha)|\ d\alpha \leq \\ \leq \frac{\sigma(\xi_0)}{2}\int_{\xi_0}^{\infty} d\xi\ \int_{\xi - \eta_0}^{\infty} |q(\alpha)|\, d\alpha\ \leq \frac{\sigma(\xi_0)}{2}\int_{\xi_0}^{\infty} \sigma(\xi - \eta_0)\, d\xi\ = \frac{\sigma(\xi_0)}{2}\sigma_1(\xi_0 - \eta_0). \end{array}$$

Let now

$$|U_{n-1}(\xi_0,\eta_0)| \le \frac{1}{2}\,\sigma\left(\xi_0\right)\frac{\left(\sigma\left(\xi_0-\eta_0\right)\right)^{n-1}}{(n-1)!}.$$

Then we have

$$\begin{array}{l}|U_n(\xi_0,\eta_0)| \le \int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|q(\xi-\eta)r(\xi,\eta,\,\xi_0,\eta_0)U_{n-1}(\xi,\eta)|\,d\eta\ \le \\ \frac{1}{2}\,\sigma\left(\xi_0\right)\int_{\xi_0}^{\infty}\frac{\left(\sigma_1(\xi-\eta_0)\right)^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\xi}|q\left(\alpha\right)|\,d\alpha d\xi \le \\ \frac{1}{2}\,\sigma\left(\xi_0\right)\int_{\xi_0}^{\infty}\frac{\left(\sigma_1(\xi-\eta_0)\right)^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\infty}|q\left(\alpha\right)|\,d\alpha d\xi \le \\ = -\frac{1}{2}\,\sigma\left(\xi_0\right)\int_{\xi_0}^{\infty}\frac{\left(\sigma_1(\xi-\eta_0)\right)^{n-1}}{(n-1)!}d\sigma_1\left(\xi-\eta_0\right) = \frac{1}{2}\,\sigma\left(\xi_0\right)\frac{\left(\sigma_1(\xi_0-\eta_0)\right)^{n}}{n!}.\end{array}$$

Hence it obviously follows that the series $U\left(\xi_0,\eta_0\right)=\sum_{n=0}^{\infty}U_n\left(\xi_0,\eta_0\right)$ converges absolutely and uniformly, and its sum is a solution to equation (7) and $U\left(\xi_0,\eta_0\right)$ satisfies the inequality (8).

The lemma is proved. ◀

Suppose now that the function $q\left(x\right)$ is continuously differentiable on the whole axis and satisfies the condition

$$\int_{-\infty}^{+\infty}\left[x^2\left|q\left(x\right)\right|+\left|q'\left(x\right)\right|\right]dx<\infty. \tag{9}$$

Differentiating equation (7) and taking into account (6), (10), we find that the function $U\left(\xi_0,\eta_0\right)$ is twice differentiable for $\xi_0\ge\eta_0$ and the relations

$$\begin{array}{l}\frac{\partial U(\xi_0,\eta_0)}{\partial\xi_0} = -\frac{1}{2}q\left(\xi_0\right)+\frac{1}{2}\int_{\xi_0}^{\infty}\frac{\partial R(\xi,0,\xi_0,\eta_0)}{\partial\xi_0}q\left(\xi\right)d\xi- \\ -\int_0^{\eta_0}q\left(\xi_0-\eta\right)U\left(\xi_0,\eta\right)d\eta+\int_0^{\eta_0}\int_{\xi_0}^{\infty}\frac{\partial R(\xi_0,\eta_0,\xi,\eta)}{\partial\xi_0}q\left(\xi-\eta\right)U\left(\xi,\eta\right)d\xi d\eta.\end{array} \tag{10}$$

$$\begin{array}{l}\frac{\partial U(\xi_0,\eta_0)}{\partial\eta_0} = \frac{1}{2}\int_{\xi_0}^{\infty}\frac{\partial R(\xi,0,\xi_0,\eta_0)}{\partial\eta_0}q\left(\xi\right)d\xi+\int_{\xi_0}^{\infty}q\left(\xi-\eta_0\right)U\left(\xi,\eta_0\right)d\xi+ \\ +\int_0^{\eta_0}\int_{\xi_0}^{\infty}\frac{\partial R(\xi_0,\eta_0,\xi,\eta)}{\partial\eta_0}q\left(\xi-\eta\right)U\left(\xi,\eta\right)d\xi d\eta\end{array} \tag{11}$$

are true. Using (6), (8), from the last equations we find that

$$\begin{array}{l}\left|\frac{\partial U(\xi_0,\eta_0)}{\partial\xi_0}+\frac{1}{2}q\left(\xi_0\right)\right| \le \frac{\eta_0^2}{8}\int_{\xi_0}^{\infty}|q\left(s\right)|\,ds+\frac{1}{2}\sigma\left(\xi_0\right)e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0-\eta_0}^{\infty}|q\left(s\right)|\,ds+ \\ +\frac{1}{4}\left(\eta_0^2-\eta^2\right)\frac{1}{2}\sigma\left(\xi_0\right)e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0}^{\infty}\sigma\left(\xi-\eta_0\right)d\xi \le \\ \le\frac{\eta_0^2}{8}\sigma\left(\xi_0\right)+\frac{1}{2}\sigma\left(\xi_0\right)e^{\sigma_1(\xi_0-\eta_0)}\left[\sigma\left(\xi_0-\eta_0\right)+\frac{\eta_0^2}{4}\sigma_1\left(\xi_0-\eta_0\right)\right],\end{array} \tag{12}$$

$$\begin{array}{l}\left|\frac{\partial U(\xi_0,\eta_0)}{\partial \eta_0}\right| \le \frac{\eta_0}{4}\int_{\xi_0}^{\infty}(\xi-\xi_0)\left|q(\xi)\right|d\xi+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0-\eta_0}^{\infty}\left|q(s)\right|ds+\\ +\frac{\eta_0}{4}e^{\sigma_1(\xi_0-\eta_0)}\int_0^{\eta_0}\int_{\xi_0}^{\infty}(\xi-\xi_0)\,\sigma(\xi)\left|q(\xi-\eta)\right|d\xi d\eta\\ =\frac{\eta_0}{4}\sigma_1(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\sigma(\xi_0-\eta_0)+\\ +\frac{\eta_0}{4}e^{\sigma_1(\xi_0-\eta_0)}\int_0^{\eta_0}\int_{\xi_0}^{\infty}(\xi-\xi_0)\int_{\xi}^{\infty}\left|q(s)\right|ds\left|q(\xi-\eta)\right|d\xi d\eta\\ \le\frac{\eta_0}{4}\sigma_1(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\sigma(\xi_0-\eta_0)+\\ +\frac{\eta_0}{4}e^{\sigma_1(\xi_0-\eta_0)}\int_0^{\eta_0}\int_{\xi_0}^{\infty}\int_{\xi}^{\infty}(s-\xi_0)\left|q(s)\right|ds\left|q(\xi-\eta)\right|d\xi d\eta\\ \le\frac{\eta_0}{4}\sigma_1(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\sigma(\xi_0-\eta_0)+\\ +\frac{\eta_0}{4}e^{\sigma_1(\xi_0-\eta_0)}\sigma_1(\xi_0)\,\sigma_1(\xi_0-\eta_0)\,.\end{array} \qquad (13)$$

Further, differentiating equations (10) and (11), we obtain

$$\begin{array}{l}\frac{\partial^2 U(\xi_0,\eta_0)}{\partial \xi_0^2}=-\frac{1}{2}q'(\xi_0)-\frac{1}{2}\frac{\partial R(\xi_0,0,\xi_0,\eta_0)}{\partial \xi_0}q(\xi_0)+\\ +\frac{1}{2}\int_{\xi_0}^{\infty}\frac{\partial^2 R(\xi,0,\xi_0,\eta_0)}{\partial \xi_0^2}q(\xi)\,d\xi-\int_0^{\eta_0}q'(\xi_0-\eta)\,U(\xi_0,\eta)\,d\eta+\\ -\int_0^{\eta_0}q(\xi_0-\eta)\frac{\partial U(\xi_0,\eta)}{\partial \xi_0}d\eta-\int_0^{\eta_0}\frac{\partial R(\xi_0,\eta_0,\xi_0,\eta)}{\partial \xi_0}q(\xi_0-\eta)\,U(\xi_0,\eta)\,d\eta\\ +\int_0^{\eta_0}\int_{\xi_0}^{\infty}\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial \xi_0^2}q(\xi-\eta)\,U(\xi,\eta)\,d\xi d\eta.\end{array} \qquad (14)$$

$$\begin{array}{l}\frac{\partial^2 U(\xi_0,\eta_0)}{\partial \eta_0^2}=\frac{1}{2}\int_{\xi_0}^{\infty}\frac{\partial^2 R(\xi,0,\xi_0,\eta_0)}{\partial \eta_0^2}q(\xi)\,d\xi-\int_{\xi_0}^{\infty}q'(\xi-\eta_0)\,U(\xi,\eta_0)\,d\xi+\\ +\int_{\xi_0}^{\infty}q(\xi-\eta_0)\frac{\partial U(\xi,\eta_0)}{\partial \eta_0}d\xi+\int_{\xi_0}^{\infty}\frac{\partial R(\xi_0,\eta_0,\xi,\eta_0)}{\partial \eta_0}q(\xi-\eta_0)\,U(\xi,\eta_0)\,d\xi+\\ +\int_0^{\eta_0}\int_{\xi_0}^{\infty}\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial \eta_0^2}q(\xi-\eta)\,U(\xi,\eta)\,d\xi d\eta.\end{array} \qquad (15)$$

Moreover, we have

$$\begin{array}{l}\frac{\partial^2 U(\xi_0,\eta_0)}{\partial \xi_0\partial \eta_0}-2\eta_0 U(\xi_0,\eta_0)=\\ =\frac{1}{2}\int_{\xi_0}^{+\infty}\left[\frac{\partial^2 R(\xi_0,\eta_0,\xi,0)}{\partial \xi_0\partial \eta_0}-2\eta_0 R(\xi_0,\eta_0,\xi,0)\right]q(\xi)\,d\xi-\\ +\int_{\xi_0}^{+\infty}d\xi\int_0^{\eta_0}\left[\left[\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial \xi_0\partial \eta_0}-2\eta_0 R(\xi_0,\eta_0,\xi,\eta)\right]\right]U(\xi,\eta)\,q(\xi-\eta)\,d\eta-\\ -q(\xi_0-\eta_0)\,U(\xi_0,\eta_0)=-q(\xi_0-\eta_0)\,U(\xi_0,\eta_0)\,.\end{array} \qquad (16)$$

From the last equations, we get

$$\begin{array}{l}\left|\frac{\partial^2 U(\xi_0,\eta_0)}{\partial \xi_0^2}+\frac{1}{2}q'(\xi_0)\right|\le\frac{\eta_0^2}{8}q(\xi_0)+\frac{\eta_0^4}{96}\sigma(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0-\eta_0}^{\infty}\left|q'(s)\right|ds+\\ +\frac{1}{2}\left[\frac{\eta_0^2}{8}\sigma(\xi_0)+\left|q(\xi_0)\right|+\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\left[\sigma(\xi_0-\eta_0)+\frac{\eta_0^2}{4}\sigma_1(\xi_0-\eta_0)\right]\right]\\ \int_{\xi_0-\eta_0}^{\infty}\left|q(s)\right|ds+\frac{\eta_0^2}{4}\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0-\eta_0}^{\infty}\left|q(s)\right|ds+\\ +\frac{1}{48}\eta_0^4\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0-\eta_0}^{\infty}\left|q(s)\right|ds=\frac{\eta_0^2}{8}q(\xi_0)+\frac{\eta_0^4}{96}\sigma(\xi_0)+\frac{1}{2}\left[\frac{\eta_0^2}{8}\sigma(\xi_0)+\right.\\ \left.+\left|q(\xi_0)\right|+\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\left[\sigma(\xi_0-\eta_0)+\frac{\eta_0^2}{4}\sigma_1(\xi_0-\eta_0)\right]\right]\sigma(\xi_0-\eta_0)+\\ \left[\frac{\eta_0^2}{4}\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}+\frac{1}{48}\eta_0^4\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\right]\sigma_1(\xi_0-\eta_0)+\\ +\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0-\eta_0}^{\infty}\left|q'(s)\right|ds\end{array}$$

$$\left|\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\eta_0^2}\right| \le \frac{\eta_0^2}{6}\int_{\xi_0}^{\infty}(\xi-\xi_0)^2\left|q(\xi)\right|d\xi + \frac{1}{4}\int_{\xi_0}^{\infty}(\xi-\xi_0)\left|q(\xi)\right|d\xi +$$
$$+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0-\eta_0}^{\infty}\left|q'(s)\right|ds +$$
$$+\left[\frac{\eta_0}{4}\sigma_1(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\sigma(\xi_0-\eta_0)+\right.$$
$$\left.+\frac{\eta_0}{4}e^{\sigma_1(\xi_0-\eta_0)}\sigma_1(\xi_0)\,\sigma_1(\xi_0-\eta_0)\right]\sigma(\xi_0-\eta_0)+$$
$$+\frac{\eta_0}{4}e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0}^{\infty}(\xi-\xi_0)\,\sigma(\xi)\left|q(\xi-\eta_0)\right|d\xi +$$
$$+\frac{\eta_0^2}{6}e^{\sigma_1(\xi_0-\eta_0)}\int_0^{\eta_0}\int_{\xi_0}^{\infty}(\xi-\xi_0)^2\sigma(\xi)\left|q(\xi-\eta)\right|d\xi d\eta +$$
$$+\frac{1}{4}e^{\sigma_1(\xi_0-\eta_0)}\int_0^{\eta_0}\int_{\xi_0}^{\infty}(\xi-\xi_0)\,\sigma(\xi)\left|q(\xi-\eta)\right|d\xi d\eta \le$$
$$\le \frac{\eta_0^2}{6}\int_{\xi_0}^{\infty}(\xi-\xi_0)^2\left|q(\xi)\right|d\xi + \frac{1}{4}\int_{\xi_0}^{\infty}(\xi-\xi_0)\left|q(\xi)\right|d\xi +$$
$$+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\int_{\xi_0-\eta_0}^{\infty}\left|q'(s)\right|ds +$$
$$+\left[\frac{\eta_0}{4}\sigma_1(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}\sigma(\xi_0-\eta_0)+\right.$$
$$\left.\frac{\eta_0}{4}e^{\sigma_1(\xi_0-\eta_0)}\sigma_1(\xi_0)\,\sigma_1(\xi_0-\eta_0)\right]\sigma(\xi_0-\eta_0)+$$
$$+\frac{\eta_0}{4}e^{\sigma_1(\xi_0-\eta_0)}\sigma_1(\xi_0)\,\sigma(\xi_0-\eta_0)+$$
$$+\frac{\eta_0^2}{3}e^{\sigma_1(\xi_0-\eta_0)}\sigma_2(\xi_0)\,\sigma_1(\xi_0-\eta_0)+$$
$$\frac{1}{4}e^{\sigma_1(\xi_0-\eta_0)}\sigma_1(\xi_0)\,\sigma_1(\xi_0-\eta_0),$$

$$\left|\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0\partial\eta_0}\right| \le \eta_0\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}+\frac{1}{2}\left|q(\xi_0-\eta_0)\right|\sigma(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}.$$

Since the function $q(x)$ is continuously differentiable and satisfies the condition (9), the relation $\lim_{x\to\infty} q(x)=0$ holds. Moreover, since $\xi_0\ge\eta_0$, the condition $\eta_0\to+\infty$ leads to relation $\xi_0\to+\infty$. From this and the last three inequalities it follows that under the condition $\eta_0\to+\infty$, the following relations hold:

$$\frac{\partial U(\xi_0,\eta_0)}{\partial\xi_0}+\frac{1}{2}q(\xi_0)=o(\eta_0),\ \frac{\partial U(\xi_0,\eta_0)}{\partial\eta_0}=o(\eta_0), \tag{17}$$

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0^2}+\frac{1}{2}q'(\xi_0)=o\left(\eta_0^3\right),\ \frac{\partial^2 U(\xi_0,\eta_0)}{\partial\eta_0^2}=o\left(\eta_0^2\right),\ \frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0\partial\eta_0}=o(1). \tag{18}$$

Further, setting $\eta_0=0$ in (7), we obtain

$$U(\xi_0,0)=\frac{1}{2}\int_{\xi_0}^{+\infty}q(\xi)d\xi. \tag{19}$$

It follows from the last relations and (8), (16), (19) that the function $K(x,t)=U\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ is twice continuously differentiable for $t\ge x$ and satisfies the relations

$$\frac{\partial K(x,t)}{\partial x^2}-\frac{\partial K(x,t)}{\partial t^2}-(x-t+q(x))\quad K(x,t)=0, \tag{20}$$

$$K(x,x) = \frac{1}{2}\int_x^{+\infty} q(t)\,dt, \tag{21}$$

$$|K(x,t)| \le \frac{1}{2}\sigma\left(\frac{t+x}{2}\right)e^{\sigma_1(x)}. \tag{22}$$

In addition, by virtue of (17), (18), we find that if x is fixed, then for $t \to +\infty$, the following relations are true:

$$\begin{array}{l} \frac{\partial K(x,t)}{\partial x} = o(t),\ \frac{\partial K(x,t)}{\partial t} = o(t), \\ \frac{\partial^2 K(x,t)}{\partial x^2} + \frac{1}{2}q'\left(\frac{x+t}{2}\right) = o\left(t^3\right),\ \frac{\partial^2 K(x,t)}{\partial t^2} + \frac{1}{2}q'\left(\frac{x+t}{2}\right) = o\left(t^3\right). \end{array} \tag{23}$$

Since the function $f_0(x,\lambda) = Ai(x-\lambda)$ and its derivative for all λ satisfy the asymptotic equalities

$$f_0(x,\lambda) \sim \frac{1}{2\sqrt{\pi}}x^{-\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}},\ f_0'(x,\lambda) \sim -\frac{1}{2\sqrt{\pi}}x^{\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}, x \to +\infty, \tag{24}$$

it follows from (20), (23), (24) that the function

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^{+\infty} K(x,t)\,f_0(t,\lambda)\,dt$$

is a solution to equation (1). Moreover, the kernel $K(x,t)$ satisfies conditions (21), (22).

Now let only the condition (2) be satisfied, so that the functions $U(\xi,\eta)$ and $K(x,t)$ may lack second derivatives. In this case, the kernel $K(x,t)$ satisfies the estimate (14). Moreover, it follows from (10)-(13) that the function $U(\xi,\eta)$ and thus $K(x,t)$ have the first partial derivatives with respect to both variables almost everywhere; moreover, if x is fixed, then the following relations hold for $t \to +\infty$:

$$\frac{\partial K(x,t)}{\partial x} + \frac{1}{2}q\left(\frac{x+t}{2}\right) = O\left((t-x)\,\sigma_1\left(\frac{x+t}{2}\right)\right),$$

$$\frac{\partial K(x,t)}{\partial t} + \frac{1}{2}q\left(\frac{x+t}{2}\right) = O\left((t-x)\,\sigma_1\left(\frac{x+t}{2}\right)\right).$$

Let us construct a sequence of continuously differentiable functions $q_n(x)$ such that the relations

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}(1+|x|)\,|q_n(x)-q(x)|\,dx = 0,$$

$$\int_{-\infty}^{+\infty} x^2\,|q_n(x)|\,dx < \infty,\ \int_{-\infty}^{+\infty}\left|q_n'(x)\right|dx < \infty$$

hold. Without loss of generality, we can assume that a sequence of functions $q_n(x)$ also converges to a function $q(x)$ almost everywhere (otherwise we would have to choose some subsequence of this sequence). Then, as shown above, the equation

$$U^{(n)}(\xi_0,\eta_0) = \frac{1}{2}\int_{\xi_0}^{+\infty} R(\xi,0,\xi_0,\eta_0)\, q_n(\xi)\, d\xi +$$

$$+\int_{\xi_0}^{+\infty} d\xi \int_0^{\eta_0} U^{(n)}(\xi,\eta)\, R(\xi,\eta,\xi_0,\eta_0)\, q_n(\xi-\eta)\, d\eta \qquad (25)$$

has a unique solution satisfying relations similar to (8) and (11). Using equations (7) and (25), we find that

$$\begin{aligned} &V^{(n)}(\xi_0,\eta_0) = \tfrac{1}{2}\textstyle\int_{\xi_0}^{+\infty} R(\xi,0,\xi_0,\eta_0)\, p_n(\xi)\, d\xi + \\ &+\textstyle\int_{\xi_0}^{+\infty} d\xi \int_0^{\eta_0} R(\xi,\eta,\xi_0,\eta_0)\, p_n(\xi-\eta)\, U(\xi,\eta)\, d\eta \\ &\textstyle\int_{\xi_0}^{+\infty} d\xi \int_0^{\eta_0} R(\xi,\eta,\xi_0,\eta_0)\, p_n(\xi-\eta)\, V^{(n)}(\xi,\eta)\, d\eta, \end{aligned} \qquad (26)$$

where $p_n(\xi) = q_n(\xi) - q(\xi)$, $V^{(n)}(\xi,\eta) = U^{(n)}(\xi,\eta) - U(\xi,\eta)$. To handle the equation (26), we use the method of successive approximations. Let

$$\rho_n(\xi) = \int_{\xi}^{+\infty} |p_n(s)|\, ds,\ \rho_{1,n}(\xi) = \int_{\xi}^{+\infty} \rho_n(s)\, ds.$$

$$V_0^{(n)}(\xi_0,\eta_0) = \frac{1}{2}\int_{\xi_0}^{+\infty} R(\xi,0,\xi_0,\eta_0)\, p_n(\xi)\, d\xi +$$

$$+\int_{\xi_0}^{+\infty} d\xi \int_0^{\eta_0} R(\xi,\eta,\xi_0,\eta_0)\, p_n(\xi-\eta)\, U(\xi,\eta)\, d\eta$$

$$V_k^{(n)}(\xi_0,\eta_0) = \int_{\xi_0}^{+\infty} d\xi \int_0^{\eta_0} R(\xi,\eta,\xi_0,\eta_0)\, p_n(\xi-\eta)\, V_{k-1}^{(n)}(\xi,\eta)\, d\eta,\, k = 1,2,...$$

Similar to above, it is established that

$$\left|V_0^{(n)}(\xi_0,\eta_0)\right| \le \frac{1}{2}\rho_n(\xi_0) + \frac{1}{2}\sigma(\xi_0)\, \rho_{1,n}(\xi_0-\eta_0),$$

$$\left|V_1^{(n)}(\xi_0,\eta_0)\right| \le \frac{1}{2}\rho_n(\xi_0)\, \rho_{1,n}(\xi_0-\eta_0) + \frac{1}{2}\sigma(\xi_0)\, \frac{[\rho_{1,n}(\xi_0-\eta_0)]^2}{2!},$$

$$\left|V_k^{(n)}(\xi_0,\eta_0)\right| \le \frac{1}{2}\rho_n(\xi_0)\, \frac{[\rho_{1,n}(\xi_0-\eta_0)]^k}{k!} + \frac{1}{2}\sigma(\xi_0)\, \frac{[\rho_{1,n}(\xi_0-\eta_0)]^{k+1}}{(k+1)!}.$$

Therefore, for the sum $V^{(n)}(\xi_0,\eta_0)=\sum_{k=0}^{\infty}V_k^{(n)}(\xi_0,\eta_0)$ we obtain

$$\left|V^{(n)}(\xi_0,\eta_0)\right|\leq\frac{1}{2}\rho_n(\xi_0)\,e^{\rho_{1,n}(\xi_0-\eta_0)}+\frac{1}{2}\sigma(\xi_0)\left[e^{\rho_{1,n}(\xi_0-\eta_0)}-1\right].$$

The last estimate shows that the sequence of functions $U^{(n)}(\xi_0,\eta_0)$ converges uniformly in the domain $\xi_0\geq\eta_0\geq 0$ to the function $U(\xi_0,\eta_0)$. Obviously, the last statement remains true for a sequence of functions $K^{(n)}(x,t)=U^{(n)}\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ in the domain $t\geq x$, the limit function of which is $K(x,t)$. Hence it follows that the sequence

$$f_n(x,\lambda)=f_0(x,\lambda)+\int_x^{\infty}K^{(n)}(x,t)\,f_0(t,\lambda)\,dt,$$

uniformly converges to the function

$$f(x,\lambda)=f_0(x,\lambda)+\int_x^{\infty}K(x,t)\,f_0(t,\lambda)\,dt,$$

for $x\geq 0$ and for λ, taken from any finite region.

In a similar way, differentiating equation (26), we find that the sequences of functions $\frac{\partial U^{(n)}(\xi_0,\eta_0)}{\partial\xi_0}$ and $\frac{\partial U^{(n)}(\xi_0,\eta_0)}{\partial\eta_0}$ converge almost everywhere to the functions $\frac{\partial U(\xi_0,\eta_0)}{\partial\xi_0}$ and $\frac{\partial U(\xi_0,\eta_0)}{\partial\eta_0}$, respectively. It follows that the sequences of functions $\frac{\partial K^{(n)}(x,t)}{\partial x}$, $\frac{\partial K^{(n)}(x,t)}{\partial t}$ converge almost everywhere to functions $\frac{\partial K(x,t)}{\partial x}$, $\frac{\partial K(x,t)}{\partial t}$, respectively. By virtue of (24) and Lebesgue's theorem on the passage to the limit under the integral sign, the sequence

$$f_n'(x,\lambda)=f_0'(x,\lambda)-K^{(n)}(x,x)\,f_0(x,\lambda)+\int_x^{\infty}\frac{\partial K^{(n)}(x,t)}{\partial x}f_0(t,\lambda)\,dt,$$

converges to the function

$$f'(x,\lambda)=f_0'(x,\lambda)-K(x,x)\,f_0(x,\lambda)+\int_x^{\infty}\frac{\partial K(x,t)}{\partial x}f_0(t,\lambda)\,dt,$$

for any x and λ. On the other hand, the functions $f_n(x,\lambda)$ satisfy the equations

$$-y''+q_n(x)\,y=\lambda y$$

as shown above. Passing to the limit in these formulas as $n\to\infty$, we come to the conclusion that the function $f(x,\lambda)$ must satisfy the equation (1). This completes the proof of the theorem. ◄

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Agil Kh.Khanmamedov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ 1141, Baku, Azerbaijan
Baku Engineering University, AZ 0101, Baku, Azerbaijan
Baku State University, AZ 1148, Baku, Azerbaijan
Azerbaijan University, Jeyhun Hajibeyli, 71, Baku, Azerbaijan
E-mail: agil_khanmamedov@yahoo.com

Davud H.Orudjov
Baku Engineering University, AZ 0101, Baku, Azerbaijan
E-mail: dorudjov@gmail.com

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