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Generalized Keller Graph

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Abstract. Generalized Keller graph Γ_d^k is defined and its properties are investigated. Moreover, connections between Keller's conjecture and the size of a maximum clique of generalized Keller graph are discussed.

Key Words and Phrases: Keller graph, Keller's conjecture, maximum clique.

2010 Mathematics Subject Classifications: 05C15, 05C45, 05C99

1. Introduction

As the new approach to Keller's conjecture, that was stated in 1930 [6], which says that every cube tiling of the d-dimensional Euclidean space contains a pair of cubes that have a common $(d-1)$-dimensional face, in 1990 Corrádi and Sabó [2] defined Keller graph (a graph in which vertices are all vectors of the length d with entries from the set $\{0, 1, 2, 3\}$ and two vertices are adjacent if and only if they differ by 2 in one coordinate and they are distinct in another coordinate) and showed that if the size of a maximum clique of it is equal to 2^d, then there exists a counterexample to Keller's conjecture in $\mathbb{R}^d$. Next, in 1992 Lagarias and Shor [9] found a maximum clique of the size 2^{10} in Keller graph for $d = 10$, and a few years later Mackey [11] found such a clique for $d = 8$. This implies that the size of a maximum clique of Keller graph is 2^d for $d \geq 8$. In 1940, Perron [12] showed that Keller's conjecture is true for $d \leq 6$. The result of Perron implies that the size of a maximum clique of Keller graphs for $d \leq 6$ is less than 2^d. In 2011, Debroni, Eblen, Langston, Myrvold, Shor and Weerapurage [3] showed that the size of a maximum clique of Keller graph for $d = 7$ is 124. Moreover, it is known that for $d = 2, 3, 4, 5, 6$ the size of a maximum clique of Keller graph is $2, 5, 12, 28, 60$, respectively. In 2016, Jarnicki, Myrvold, Saltzman and Wagon [5], using computer calculations, investigated properties of Keller graph such as Hamiltonian, the independence number, the chromatic number, etc. In 2018,

M. Łysakowska [10] defined extended Keller graph, i.e. graph in which vertices are all vectors of the length d with entries from the set $\{0, 1, 2, 3, 4, 5\}$ and two vertices are adjacent if and only if they differ by 3 in one coordinate and they are distinct in another coordinate, and overtly proved basic properties of this graph.

In this paper, generalized Keller graph Γ_d^k is defined, i.e. the graph in which vertices are all vectors of the length d with entries from the set $\{0, 1, \ldots, 2k-1\}$ and two vertices are adjacent if they meet the appropriate conditions, and properties of these graphs are shown explicitly. In the last section, the conjecture about sizes of maximum cliques in Keller graphs is stated.

The result of Debroni, Elden, Langston, Myrvold, Shor and Weerapurage [3] showing that the size of maximum clique of Γ_7^2 is equal to 124 implies that Keller's conjecture is true in dimension 7 for cubes with centers at points of the set $\{x = (x_1, \ldots, x_7) \colon x_i \in \mathbb{Z} \cup \frac{1}{2}\mathbb{Z}\}$. In [7] Kisielewicz showed that Keller's conjecture in dimension 7 is true for cubes with centers at points of $\{x = (x_1, \ldots, x_7) \colon x_i \in \bigcup_{k=1}^{n} \frac{1}{k}\mathbb{Z},\ n \geq 6\}$ and in [8] Kisielewicz and Łysakowska proved that Keller's conjecture in dimension 7 is also true for cubes with centers in points of $\{x = (x_1, \ldots, x_7) \colon x_i \in \bigcup_{k=1}^{5} \frac{1}{k}\mathbb{Z}\}$. These results imply that the size of a maximum clique of generalized Keller graphs for $d = 7$ and $k \geq 5$ is less than 2^7. Finally, in 2019, Brakensiek, Heule, Mackey and Narváez [1], using computer calculations, showed that the size of a maximum clique of the graphs Γ_7^3, Γ_7^4 and Γ_7^5 is less than $2^7 = 128$ and this implies that Keller's conjecture is true for $d = 7$.

2. Preliminaries

Generalized Keller graph $\Gamma_d^k = (V, E)$, $k \geq 2$, $d \geq 2$, is defined by

$$V = \{v = (v_1, \ldots, v_d) \colon v_i \in \mathbb{Z}_{2k}\},$$

$$E = \{\{u, v\} \colon \exists i \ \ u_i - v_i \equiv k \pmod{2k} \ \ \exists j \neq i \ \ u_j \neq v_j\}.$$

It is easily seen that the graph Γ_d^k has $(2k)^d$ vertices. Moreover, an automorphisms group of Γ_d^k is formed by bijections $f \colon V(\Gamma_d^k) \to V(\Gamma_d^k)$ and permutations $\sigma \colon \{1, \ldots, d\} \to \{1, \ldots, d\}$ such that for all vertices $u = (u_1, \ldots, u_d)$, $v = (v_1, \ldots, v_d) \in V(\Gamma_d^k)$ the following conditions are satisfied: for every $i \in \{1, \ldots, d\}$

- $u_i - v_i \equiv k \pmod{2k}$ if and only if $f(u)_{\sigma(i)} - f(v)_{\sigma(i)} \equiv k \pmod{2k}$;
- $u_i = v_i$ if and only if $f(u)_{\sigma(i)} = f(v)_{\sigma(i)}$.

It can be also noticed that the graph Γ_d^k is vertex transitive and, as a consequence, it is regular.

Let us see that degree Δ of Γ_d^k is equal to $(2k)^d - (2k-1)^d - d$. Indeed, let $v = (v_1, \ldots, v_d) \in V$. Then there is exactly d vertices $u = (u_1, \ldots, u_d) \in V$ such that $u_i - v_i \equiv k \pmod{2k}$ for some $i \in \{1, \ldots, d\}$ and $u_j = v_j$ for all $j \neq i$, and there is exactly $(2k-1)^d$ vertices $w = (w_1, \ldots, w_d) \in V$ such that $v_i \not\equiv w_i \pmod{2k}$ for all $i \in \{1, \ldots, d\}$. This implies that the graph Γ_d^k has $\frac{1}{2}(2k)^d((2k)^d - (2k-1)^d - d)$ edges.

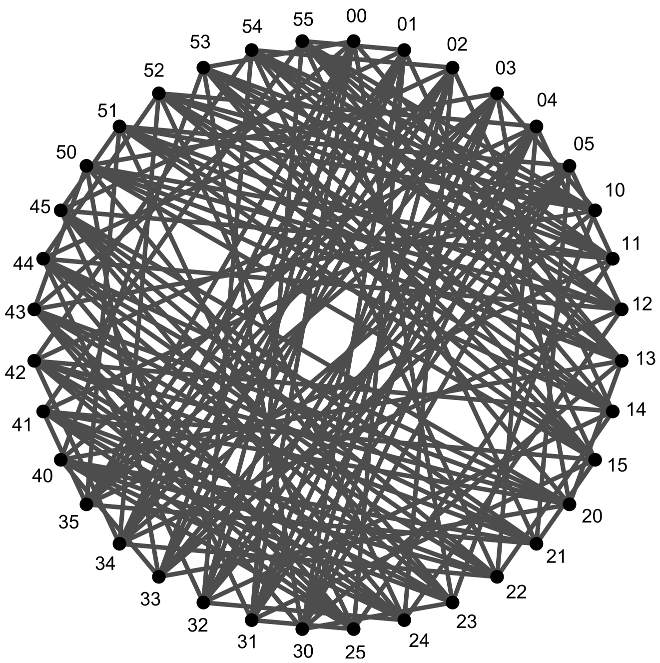

Figure 1
Generalized Keller graph Γ_2^3

A family of vertices $W \subseteq V(\Gamma_d^k)$ is called *a simple class* if for every two vertices $u = (u_1, \ldots, u_d), v = (v_1, \ldots, v_d) \in W$ we have $u_i = v_i$ or $u_i = -v_i$ for all $i \in \{1, \ldots, d\}$. If two vertices from a simple class are neighbours, they are said to be *simple neighbours.* Two vertices $u = (u_1, \ldots, u_d)$, $v = (v_1, \ldots, v_d) \in V$ are called *dichotomous* if there is an $i \in \{1, \ldots, d\}$ such that $u_i - v_i \equiv k \pmod{2k}$ and they are said to be *a twin pair* if they are dichotomous and there are $d-1$ indexes $j \in \{1, \ldots, d\}$ such that $u_j = v_j$.

For example, in the graph Γ_5^4 vertices 05721 and 05321 form a twin pair, as in the third coordinate they differ by 4 and they are equal in the rest coordinates. In the graph Γ_2^3, for instance, the family of vertices $\{11, 14, 41, 44\}$ is a simple class; the vertex 00 has nine neighbours: $13, 23, 31, 32, 33, 34, 35, 43, 45$, whereby 33 is its simple neighbour.

In the graph Γ_d^k with each vertex $w = (w_1, \dots, w_d) \in V$ a vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{+, -\}^d$ defined by

$$\varepsilon_i = \begin{cases} +, & \text{if } w_i \in \{0, 1, \dots, k-1\}, \\ -, & \text{if } w_i \in \{k, k+1, \dots, 2k-1\} \end{cases}$$

is associated. Vectors ε are called *codes*.

To simplify notation, we will often omit brackets and write, for example, a code $+--+$ and a vertex $w_1 \dots w_d$ instead of $\varepsilon = (+, -, -, +)$ and a vertex $w = (w_1, \dots, w_d)$, respectively.

All arithmetic in the paper is done modulo $2k$. The number of vertices and the number of edges of Γ_d^k are denoted by n_v and n_e, correspondingly. Moreover, the independence number and the chromatic number are denoted, in the traditional way, by α and χ, respectively.

Let $A \subseteq (\mathbb{Z}_{2k})^m$ and $v \in (\mathbb{Z}_{2k})^n$. Then the set Av is defined by

$$Av = \{wv = (w_1, \dots, w_m, v_1, \dots, v_n) \colon w \in A\}.$$

3. Properties of Γ_d^k

In this section some basic properties of generalized Keller graph are presented.

In 1952, Dirac [4] proved that a simple graph is Hamiltonian if every vertex of it has degree greater or equal to $n_v/2$. This implies that generalized Keller graph Γ_d^k is Hamiltonian for some k and d, for example, for $k = 2$ and $d \geq 3$, $k = 3$ and $d \geq 4$. In the proof of Theorem 1 a Hamiltonian cycle in all graphs Γ_d^k is given explicitly.

Theorem 1. *All generalized Keller graphs are Hamiltonian.*

Proof. It is easily seen that the cycle

((0,0),(k,2k-1),(0,1),(k,2k-2),(0,2),(k,2k-3),. . . ,(0,k-2),(k,k+1),(0,k-1),(k,0),

(0,k),(k,1),(0,k+1),(k,2),. . . ,(0,2k-2),(k,k-1),(0,2k-1),(k,k),(1,0),(k+1,2k-1),

(1,1),(k+1,2k-2),. . . ,(1,k-2),(k+1,k+1),(1,k-1),(k+1,0),(1,k),(k+1,1),(1,k+1),

(k+1,2),. . . ,(k+1,k-1),(1,2k-1),(k+1,k),. . . . . ,(k-1,0),(2k-1,2k-1),

(k-1,1),(2k-1,2k-2),. . . ,(k-1,k-2),(2k-1,k+1),(k-1,k-1),(2k-1,0),(k-1,k),

(2k-1,1),(k-1,k+1),(2k-1,2),. . . ,(2k-1,k-1),(k-1,2k-1),(2k-1,k)) is a Hamiltonian cycle in Γ_2^k. Let us denote this cycle by H. Then all vertices of the graph Γ_d^k, $d \geq 3$, can be arranged into a cycle in the following way

$$Hv_1, Hv_2, \ldots, Hv_{(2k)^{d-2}},$$

where $v_i \in \Gamma_{d-2}^k$, $i = 1, 2, \ldots, (2k)^{d-2}$, while Γ_1^k denotes $\mathbb{Z}_{2k}$. As a consequence we obtain a Hamiltonian cycle in Γ_d^k. ◀

Next theorem shows that all generalized Keller graphs Γ_d^k can be edge-colored in $\Delta = (2k)^d - (2k-1)^d - d$ colors and in the proof of Theorem 2 the manner of such coloring is given.

Theorem 2. *All generalized Keller graphs are class 1.*

Proof. Let S be the family of all neighbours of the vertex $v_0 = 00\ldots0$, let S_p be the family of all simple neighbours of v_0, and let $S_s = S \setminus S_p$. Then sets S and S_p have $(2k)^d - (2k-1)^d - d = \Delta$ and $2^d - d - 1$ elements, respectively. Moreover, if $s \in S_p$, then $s = -s$, and if $s \in S_s$, then $-s \in S_s$. Let us notice also that as the graph Γ_d^k is vertex transitive, the set of all neighbours of every vertex $v \in V(\Gamma_d^k)$ has the form

$$v + S = \{v + s\colon s \in S\}.$$

For every $s \in S_s$ and $v \in V$ let T_v^s be defined by

$$T_v^s = \{v + ms\colon m \in \mathbb{Z}_{2k}\}.$$

Then

$$\begin{aligned} T_v^{-s} &= \{v, v-s, v-2s, \ldots, v-(2k-2)s, v-(2k-1)s\} = \\ &= \{v, v+(2k-1)s, v+(2k-2)s, \ldots, v+2s, v+s\} = T_v^s. \end{aligned}$$

Additionally, if $w \in T_v^s$, then $w = v + ms$ for some $m \in \mathbb{Z}_{2k}$. Thus $v = w - ms \in T_w^{-s} = T_w^s$. As a result, for every $s \in S_s$ we obtain a partition of vertices of Γ_d^k into $(2k)^{d-1}$ pairwise disjoint classes $\{v+ms\colon m \in \mathbb{Z}_{2k}\}$ with $2k$ elements in each of them.

Edges of the graph Γ_d^k are colored in the following way:

- each color $s \in S_s$ is put on the edge between v and $v + s$;
- each color $s \in S_p$ is put on edges $\{v, v+s\}, \{v+2s, v+3s\}, \ldots, \{v+(2k-2)s, v+(2k-1)s\}$.

Then

$$\frac{1}{2}(2^d - d - 1)\cdot(2k)^d + \frac{1}{2}((2k)^d - (2k-1)^d - d - (2^d - d - 1))\cdot(2k)^d =$$

$$= \frac{(2k)^d}{2}((2k)^d - (2k-1)^d - d) = \frac{1}{2} n_v \cdot \Delta = n_e$$

edges are colored. Therefore in this way all edges of Γ_d^k are colored.

Now we will show that every edge of Γ_d^k is colored exactly once. Let us notice that if the edge $\{u, v\}$ has color s, then $u - v = \pm s$. If $s \in S_p$, then $s = -s$ and, as a consequence, the edge $\{u, v\}$ has exactly one color. If $s \in S_s$, then if the edge $\{u, v\}$ has another color apart from s it must be $-s$. Then the edge $\{u, v\}$ has to be at the same time one edge from sets

$$\{\{v, v+s\}, \{v+2s, v+3s\}, \ldots, \{v+(2k-2)s, v+(2k-1)s\}\}$$

and

$$\{\{v, v-s\}, \{v-2s, v-3s\}, \ldots, \{v-(2k-2)s, v-(2k-1)s\}\} =$$

$$= \{\{v, v+(2k-1)s\}, \{v+(2k-2)s, v+(2k-3)s\}, \ldots, \{v+2s, v+s\}\},$$

what is impossible. As a consequence, every edge of the graph Γ_d^k has exactly one color.

Finally we show that such coloring is proper. Indeed, if the edge $\{u, v\}$ has color s, then $u - v = \pm s$. If $s \in S_p$, then $s = -s$ and the color is chosen uniquely. If $s \in S_s$, then $u = w + ms$, where $w \in T_v^s$, $m \in \mathbb{Z}_{2k}$. If m is even, then $v = w + (m+1)s$, and if m is odd, then $v = w + (m-1)s$. In both cases the choice of v is unique. As a result, the coloring is proper. ◀

In [5], Jarnicki, Myrvold, Saltzman and Wagon showed that for $k = 2$ the independence number of Keller graph is 2^d for $d \geq 3$ and it is 5 for $d = 2$. The next theorem shows that for $k \geq 3$ the independence number of all Keller graphs Γ_d^k is k^d.

Theorem 3. *For $k \geq 3$ the independence number of all generalized Keller graphs Γ_d^k is k^d.*

Proof. For $k = 3$ and $d = 2$ it is not too hard to check that $\alpha(\Gamma_2^3) = 3^2 = 9$ and there are two kinds of maximum independent sets in Γ_2^3:

1. four twin pairs and one additional vertex which is not dichotomous with all of them; each such a set is isomorphic with

   00, 03, 01, 04, 02, 12, 42, 22, 52;

2. all nine vertices are not pairwise dichotomous; each such a set contains vertices with the same code and is isomorphic with

$$00,\ 01,\ 02,\ 10,\ 11,\ 12,\ 20,\ 21,\ 22.$$

Moreover, it is also quite easy to check that $\alpha(\Gamma_3^3) = 3^3$, $\alpha(\Gamma_2^4) = 4^2$, $\alpha(\Gamma_2^5) = 5^2$ and every maximum independent set in the graphs Γ_3^3, Γ_2^4, Γ_2^5 is isomorphic with the set of all vertices with the same code, i.e.

- in Γ_3^3 all maximum independent sets are isomorphic with

$$000,\ 001,\ 010,\ 100,\ 002,\ 020,\ 200,\ 012,\ 102,\ 120,\ 021,\ 201,\ 210,\ 111,$$

$$222,\ 112,\ 121,\ 211,\ 221,\ 212,\ 122,\ 110,\ 101,\ 011,\ 220,\ 202,\ 022;$$

- in Γ_2^4 all maximum independent sets are isomorphic with

$$00,\ 11,\ 22,\ 33,\ 01,\ 10,\ 02,\ 20,\ 03,\ 30,\ 12,\ 21,\ 13,\ 31,\ 23,\ 32;$$

- in Γ_2^5 all maximum independent sets are isomorphic with

$$00,\ 01,\ 02,\ 03,\ 04,\ 10,\ 11,\ 12,\ 13,\ 14,\ 20,\ 21,$$

$$22,\ 23,\ 24,\ 30,\ 31,\ 32,\ 33,\ 34,\ 40,\ 41,\ 42,\ 43,\ 44.$$

Now, let us notice that the set of all vertices of Γ_d^k with the same code is an independent set with k^d elements.

Let us see that for $k = 3$, $d \geq 3$ and $k \geq 4$, $d \geq 2$ a maximum independent set does not contain any twin pair. In fact, suppose that M is a maximum independent set in Γ_d^k with l twin pairs. Then all these twin pairs are not dichotomous with each other. Thus each such a pair lies in another simple class. Without loss of generality, we can assume that one vertex in every twin pair has the code $+\cdots+$. It is easy to see that a maximum independent set containing these twin pairs can be obtained by adding to them all vertices with the code $+\cdots+$ which are not dichotomous with them. Let m_i be a non-negative integer denoting the number of twin pairs that are dichotomous in position i, $i \in \{1,\ldots,d\}$, $\sum_{i=1}^{d} m_i = l$. Then the set M has

$$\prod_{i=1}^{d}(k - m_i) + 2l < k^d$$

elements, what is a contradiction.

As a maximium independent set does not contain any twin pair, every element of it has to lie in other simple class. As we have k^d different simple classes and vertices with the same code form independent set which has a property that each of its elements lies in other simple class, a maximum independent set in Γ_d^k has k^d elements. ◀

Theorem 4. *The chromatic number of all generalized Keller graphs is* 2^d.

Proof. All vertices of the graph Γ_d^k can be put into the array with 2^d rows and k^d columns such that each row of the array contains vertices with the same code and each column contains all vertices from the same simple class. Then all k^d vertices in every row of the array are independent. Thus $\chi(\Gamma_d^k) \leq 2^d$.

On the other hand,

$$\frac{n_v}{\alpha(\Gamma_d^k)} = \frac{(2k)^d}{k^d} = 2^d$$

implies $\chi(\Gamma_d^k) \geq 2^d$. As a result, $\chi(\Gamma_d^k) = 2^d$. ◀

PROPERTIES	Γ_d^k
number of vertices n_v	$(2k)^d$
number of edges v_e	$\frac{1}{2}(2k)^d((2k)^d - (2k-1)^d - d)$
degree Δ	$(2k)^d - (2k-1)^d - d$
the independence number α	5 for $k = 2$ and $d = 2$ k^d for the rest k and d
the chromatic number χ	2^d
Hamiltonian	Yes
class 1	Yes

Table 1

Properties of generalized Keller graphs Γ_d^k

4. Open question

It is easy to check that the size of maximum clique is:

- 2 for Γ_2^3, Γ_2^4, Γ_2^5; there are two kinds of such cliques and both of them are ismorophic with such cliques in Γ_2^2:

$$00,\ 33 \qquad \text{or} \qquad 00,\ 13;$$

- 5 for Γ_3^3; all these cliques are isomorphic with such a clique in Γ_3^2:

000, 032, 320, 203, 222;

- 12 for Γ_4^3; all these cliques are isomorphic with such a clique in Γ_4^2:

0000, 3023, 1203, 2331, 0021, 2003, 0231, 2011, 3233, 1211, 3210, 1323.

These results enforce us to state the following conjecture:

Conjecture 1. *The size of a maximum clique of generalized Keller graphs* Γ_d^k *is:*

- 2 *for* $d = 2$, $k \geq 6$,
- 5 *for* $d = 3$, $k \geq 4$,
- 12 *for* $d = 4$, $k \geq 4$,
- 28 *for* $d = 5$, $k \geq 3$,
- 60 *for* $d = 6$, $k \geq 3$,
- 124 *for* $d = 7$, $k \geq 3$.

Additionally, all these maximum cliques are isomorphic with maximum cliques of Γ_d^2, $d = 2, 3, 4, 5, 6, 7$, *respectively.*

d	**the size of a maximum clique of** Γ_d^k
$d = 2$	2 for $k = 2, 3, 4$ $< 2^2 = 4$ for $k \geq 5$
$d = 3$	5 for $k = 2, 3, 4$ $< 2^3 = 8$ for $k \geq 5$
$d = 4$	12 for $k = 2, 3$ $< 2^4 = 16$ for $k \geq 4$
$d = 5$	28 for $k = 2$ $< 2^5 = 32$ for $k \geq 3$
$d = 6$	60 for $k = 2$ $< 2^6 = 64$ for $k \geq 3$
$d = 7$	124 for $k = 2$ $< 2^7 = 128$ for $k \geq 3$
$d \geq 8$	2^d for $k \geq 2$

Table 2

The size of a maximum clique of generalized Keller graphs Γ_d^k

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