Azerbaijan Journal of Mathematics
V. 13, No 2, 2023, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2023.2.120

Rothe Time-Discretization Method for Nonlinear Parabolic Problems in Weighted Sobolev Space with Variable Exponents

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Abstract. In the present paper, we prove existence and uniqueness of weak solutions for nonlinear parabolic problem whose model is

$$\begin{cases} \frac{\partial v}{\partial t} - \operatorname{div}\left[\omega|\nabla v - \Theta(v)|^{p(x)-2}(\nabla v - \Theta(v))\right] + \beta(v) = f & \text{in} \quad Q_T := (0,T) \times \Omega, \\ v = 0 & \text{on} \quad \Sigma_T := (0,T) \times \partial\Omega, \\ v(\cdot, 0) = v_0 & \text{in} \quad \Omega. \end{cases}$$

The main tool used here is the Rothe time-discretization method combined with the theory of weighted Sobolev spaces with variable exponents.

Key Words and Phrases: nonlinear parabolic problem, existence, weak solution, variable exponent, semi-discretization, uniqueness, Rothe method, weighted Sobolev space.

2010 Mathematics Subject Classifications: 35K55, 35K61, 35J60

1. Introduction

Our goal in this work is to prove the existence and uniqueness results for weak solutions of the following nonlinear parabolic problem:

$$(P^\omega) \begin{cases} \frac{\partial v}{\partial t} - \operatorname{div}(\Phi(\nabla v - \Theta(v))) + \beta(v) = f & \text{in} \quad Q_T := (0,T) \times \Omega, \\ v = 0 & \text{on} \quad \Sigma_T := (0,T) \times \partial\Omega, \\ v(\cdot, 0) = v_0 & \text{in} \quad \Omega. \end{cases}$$

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$\Omega \subset \mathbb{R}^d (d \geq 3)$ is an open bounded domain with Lipschitz boundary $\partial\Omega; T$ is a fixed positive number; ∇v is the gradient of v and $\Phi(\xi) := \omega|\xi|^{p(x)-2}\xi$, for all $\xi \in \mathbb{R}^d$ with $1 < p(x) < d$.

The study of partial differential equations and variational problems with variable exponent has received considerable attention for the first time in 1931, in a work by Orlicz [23], but the field of variable exponent function spaces has witnessed an explosive growth in last years. The evolutions in science lead to a period of intense study of variable exponent spaces. Also observed were problems related to modelling of so-called electrorheological fluids, the study of thermorheological fluids and image processing. For more general application of this kind of problem we refer the reader to [18, 20, 21, 26].

Our problem (P^ω) arises in various physical fields as chemical heterogeneous catalysts, non-Newtonian fluids and as well as the theory of heat conduction in electrically conducting materials (see, for example, [8, 24, 26]. Here we shall refer to one of them which are related to turbulent flows.

Model: Flow through a porous medium in a turbulent regime
This model is governed by the continuity equation

$$\frac{\partial\theta}{\partial t} - div\left(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\right) = 0,$$

where

- θ is the volumetric content of moisture.
- $\varphi(\theta)$ is the hydrostatic potential.
- $K(\theta)$ is the hydraulic conductivity.
- e is the unit vector in the vertical direction.

The problem (P^ω) or special cases of it has been extensively studied by many authors in elliptic or parabolic case, we refer the reader to [1, 3, 4, 5, 9, 11, 14, 15].

We mention that the Euler forward scheme has been used by several authors while studying time discretization of nonlinear parabolic problems, For the more complete references, we refer the reader to [6, 10, 12, 13, 16, 20, 22, 25].

The advantage of our method is that we cannot only obtain the existence and uniqueness of weak solutions to the problem (P^ω), but also compute the numerical approximations. In the particular case, where $\Theta = 0$, the author in [16] proved the existence and uniqueness of entropy solutions in Orlicz spaces by using our Rothe time-discretization method.

This paper is divided into five parts. In Part 1, we introduce the problem (P^ω) and we state the assumptions. In Part 2, we mention some preliminary results and notations, also we state our main result. In Part 3, we discretize the problem (P^ω) by the Euler forward scheme, we prove the existence and uniqueness of weak solution for the discretized problems and we show some stability results. At the last section, we finish this work by treating the convergence and existence for the problem (P^ω), moreover, we confirm the uniqueness of solution.

2. Preliminary results and notations

As the problem (P^ω) depends on the weight ω and the variable $p(x)$, we should use the weighted Lebesgue and Sobolev spaces with variable exponents. We recall some notations and definitions which will be used in this paper.

We consider the following set

$$C^+(\bar{\Omega}) = \left\{p : \bar{\Omega} \to \mathbb{R}^+ : p \text{ is continuous and such that } 1 < p_- < p_+ < \infty\right\},$$

where

$$p_- = \min_{x\in\bar{\Omega}} p(x) \text{ and } p_+ = \max_{x\in\bar{\Omega}} p(x).$$

Let ω be a measurable positive and a.e. finite function defined in $\mathbb{R}^d$ and satisfying the following integrability conditions:

$$\omega \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad \omega^{\frac{-1}{p(x)-1}} \in L^1_{\text{loc}}(\Omega), \tag{1}$$

$$\omega^{-s(x)} \in L^1_{\text{loc}}(\Omega), \text{ where } \quad s(x) \in \left(\frac{d}{p(x)}, \infty\right) \cap \left[\frac{1}{p(x)-1}, \infty\right). \tag{2}$$

For $p(\cdot) \in C^+(\bar{\Omega})$, we define the weighted Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega, \omega)$ by

$$L^{p(\cdot)}(\Omega,\omega) = \left\{v : \Omega \to \mathbb{R} : v \text{ is measurable and } \int_\Omega |v|^{p(x)}\omega(x)\mathrm{d}x < \infty\right\}.$$

We denote by $L^{p'(\cdot)}(\Omega, \omega^*)$ the conjugate space of $L^{p(\cdot)}(\Omega, \omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

and where

$$\omega^*(x) = \omega(x)^{1-p'(x)} \qquad \text{for all } x \in \Omega.$$

On the space $L^{p(\cdot)}(\Omega, \omega)$, we consider the function $\rho_{p(\cdot),\omega} : L^{p(\cdot)}(\Omega,\omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot),\omega}(v) = \rho_{L^{p(\cdot)}(\Omega,\omega)}(v) = \int_\Omega |v(x)|^{p(x)}\omega(x)\mathrm{d}x.$$

The connection between $\rho_{p(\cdot),\omega}$ and $\|\cdot\|_{p(\cdot),\omega}$ is established by the following result. Let v be an element of $L^{p(\cdot)}(\Omega, \omega)$. Then the following assertions hold:

1. $\|v\|_{p(\cdot),\omega} < 1$ (respectively $>, = 1\big) \Leftrightarrow \rho_{p(\cdot),\omega}(v) < 1$ (respectively $>, = 1\big)$.
2. If $\|v\|_{p(\cdot),\omega} < 1$, then $\|v\|^{p_+}_{p(\cdot),\omega} \leq \rho_{p(\cdot),\omega}(v) \leq \|v\|^{p_-}_{p(\cdot),\omega}$.
3. If $\|v\|_{p(\cdot),\omega} > 1$, then $\|v\|^{p_-}_{p(\cdot),\omega} \leq \rho_{p(\cdot),\omega}(v) \leq \|v\|^{p_+}_{p(\cdot),\omega}$.
4. $\|v\|_{p(\cdot),\omega} \to 0 \Leftrightarrow \rho_{p(\cdot),\omega}(v) \to 0$ and $\|v\|_{p(\cdot),\omega} \to \infty \Leftrightarrow \rho_{p(\cdot),\omega}(v) \to \infty$.

Proof. See Proposition 2.1 in [2].

The weighted Sobolev space with variable exponent is defined by

$$W^{1,p(\cdot)}(\Omega,\omega) = \Big\{ v \in L^{p(\cdot)} \text{ and } |\nabla v| \in L^{p(\cdot)}(\Omega,\omega) \Big\},$$

endowed with the norm

$$\|v\|_{1,p(\cdot),\omega} = \|v\|_{p(\cdot)} + \|\nabla v\|_{p(\cdot),\omega}, \quad \forall v \in W^{1,p(\cdot)}(\Omega,\omega).$$

Hereinafter, by $W_0^{1,p(\cdot)}(\Omega,\omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega,\omega)$.

Let $p(\cdot), s(\cdot)$ be two elements of space $C^+(\bar{\Omega})$, where the function $s(\cdot)$ satisfies the condition (2). We define the following functions:

$$p^*(x) = \tfrac{dp(x)}{d-p(x)} \quad \text{for } p(x) < d,$$
$$p_s(x) = \tfrac{p(x)s(x)}{1+s(x)} < p(x),$$
$$p_s^*(x) = \begin{cases} \frac{p(x)s(x)}{(1+s(x))d-p(x)s(x)} & \text{if } d > p_s(x), \\ +\infty & \text{if } d \leq p_s(x). \end{cases}$$

for almost all $x \in \Omega$.

Proposition 1. *Let $\Omega \in \mathbb{R}^d$ be an open set of $\mathbb{R}$, $p(\cdot) \in C^+(\Omega)$ and let* (1) *be satisfied. Then we have*

$$L^{p(\cdot)}(\Omega,\omega) \hookrightarrow L^1_{\mathrm{loc}}(\Omega).$$

Proof. See Proposition 2.8 in [19]. ◀

Proposition 2. *Let condition* (1) *be satisfied. Then the space* $\big(W^{1,p(\cdot)}(\Omega,\omega), \|v\|_{1,p(\cdot),\omega}\big)$ *is a separable and reflexive Banach space.*

Proof. See Theorem 2.10 in [19]. ◀

Proposition 3. *Suppose that conditions* (1) *and* (2) *hold and* $p(\cdot),\ s(\cdot) \in C^+(\bar{\Omega})$. *Then we have the continuous embedding*

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow W^{1,p_s(x)}(\Omega,\omega).$$

Moreover, we have the compact embedding

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega),$$

provided that $r \in C^+(\bar{\Omega})$ *and* $1 \leq r(x) < p_s^*(x)$ *for all* $x \in \Omega$.

Proof. In [19]. ◀

Proposition 4. *(**Holder inequality** in [17]) Let* $p(\cdot),\ p'(\cdot) \in C^+(\bar{\Omega})$ *with* $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. *Then for any* $v_1 \in L^{p(\cdot)}(\Omega)$ *and* $v_2 \in L^{p'(\cdot)}(\Omega)$ *we have*

$$\left|\int_\Omega v_1 \cdot v_2 dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|v_1\|_{p(\cdot)} \|v_2\|_{p'(\cdot)}.$$

We assume that the exponent $p(x)$ *is log-Holder continuous, i.e., there is a constant* C *such that*

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \tag{3}$$

for every x, y *with* $|x-y| \leq \frac{1}{2}$.

Proposition 5. *(**Poincarè type inequality** in [20]) Let* $p(\cdot) \in C_+(\bar{\Omega})$ *satisfy the log-Holder continuity condition* (3). *If* (1) *and* (2) *hold, then the estimate*

$$\|v\|_{L^{p(x)}(\Omega)} \leq C\|\nabla v\|_{L^{p(x)}(\Omega,\omega)},$$

holds, for every $u \in C_0^\infty(\Omega)$ *with a positive constant* C *independent of* v.

Lemma 1. *For* $\xi, \eta \in \mathbb{R}^d$ *and* $1 < p < \infty$, *we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi(\xi - \eta). \tag{4}$$

Proof. Consider the function $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $x \mapsto x^p - px + (p-1)$. We have

$$g(x) \geq \min_{y \in \mathbb{R}^+} g(y) = g(1) = 0 \text{ for all } x \in \mathbb{R}^+.$$

Therefore, we take $x = \frac{|\eta|}{|\xi|}$(if $|\xi| = 0$, the result is obvious) in the inequality above to get the result of the lemma by using Cauchy-Schwarz inequality. ◀

Lemma 2. *([8]) Let p, p' be two real numbers such that $p > 1, p' > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have*

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right|^{p'} \leq$$

$$\leq C\left\{(\xi - \eta)\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\right\}^{\frac{\alpha}{2}}\left\{|\xi|^{p} + |\eta|^{p}\right\}^{1-\frac{\alpha}{2}}, \forall \xi, \eta \in \mathbb{R}^{d},$$

where $\alpha = 2$ if $1 < p \leq 2$ and $\alpha = p'$ if $p \geq 2$.

Remark 1. *Hereinafter, $c_i, (i \in \mathbb{N})$ are positive constants independent of N.*

Definition 1. *A measurable function $v : Q_T \to \mathbb{R}$ is a weak solution to nonlinear parabolic problem (P^{ω}) in Q_T if $v(.,0) = v_0$ in $\Omega, v \in C\left(0, T; L^2(\Omega)\right) \cap L^{p(x)}\left(0, T; W^{1,p(x)}(\Omega, \omega)\right)$, $\frac{\partial v}{\partial t} \in L^2(Q_T)$ and we have*

$$\int_0^T \int_\Omega \frac{\partial v}{\partial t}\varphi dxdt + \int_0^T \int_\Omega \Phi(\nabla v - \Theta(v)).\nabla\varphi dxdt + \int_0^T \int_\Omega \beta(v)\varphi dxdt$$

$$= \int_0^T \int_\Omega f\varphi dxdt, \qquad \forall \varphi \in C^1(Q_T). \tag{5}$$

Given a constant $k > 0$, we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) := \begin{cases} s & \text{if} \quad |s| \leq k, \\ k \quad \text{sign}(s) & \text{if} \quad |s| > k, \end{cases}$$

where

$$\text{sign}(s) := \begin{cases} 1 & \text{if} \quad s > 0, \\ 0 & \text{if} \quad s = 0, \\ -1 & \text{if} \quad s < 0. \end{cases}$$

Here, we state the main result of our paper.

Theorem 1. *Under the hypotheses $(H1)$, $(H2)$ and $(H3)$, there exists a unique weak solution for the nonlinear parabolic problem (P^{ω}).*

3. The semi-discrete problem and stability results

3.1. The semi-discrete problem

In this part, we discretize the problem (P^{ω}) by Euler forward scheme and we study the questions of existence and uniqueness for the discretized problems. We make the following hypotheses.

(H1) β is a non decreasing continuous real function on $\mathbb{R}$, surjective such that $\beta(0)=0$ and $|\beta(x)| \leq M|x|$, where M is a positive constant.

(H2) $f \in L^{\infty}(Q_T)$ and $v_0 \in L^{\infty}(\Omega) \cap W^{1,p(x)}(\Omega,\omega)$.

(H3) Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^d$ and $\Theta(0)=0$ such that $|\Theta(x)-\Theta(y)| \leq \lambda|x-y|$, for all $x,y \in \mathbb{R}$, and λ is a positive constant.

The Euler forward scheme applied to the problem (P^{ω}) yields the following problem:

$$(P^{\omega})\begin{cases} V_n - \tau \operatorname{div}\left(\Phi\left(\nabla V_n - \Theta\left(V_n\right)\right)\right) + \tau\beta\left(V_n\right) = \tau f_n + V_{n-1} & \text{in} \quad \Omega, \\ V_n = 0 & \text{on} \quad \partial\Omega, \\ V_0 = v_0 & \text{in} \quad \Omega, \end{cases}$$

where $N\tau = T, 0<\tau<1$, $1 \leq n \leq N$, $t_n = n\tau$ and

$$f_n(\cdot) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(s,\cdot)ds, \text{in } \Omega.$$

A weak solution to the discretized problem (P_n^{ω}) is a sequence $(V_n)_{0\leq n\leq N}$ such that $V_0 = u_0$ and V_n is defined by induction as a weak solution to the problem

$$\begin{cases} v - \tau \operatorname{div}\left(\Phi\left(\nabla v - \Theta\left(v\right)\right)\right) + \tau\beta\left(v\right) = \tau f_n + V_{n-1} & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6}$$

i.e. for $V_n \in L^{\infty}(\Omega) \cap W^{1,p(x)}(\Omega,\omega)$ and $\forall \varphi \in W^{1,p(x)}(\Omega,\omega), \forall \tau > 0$, we have

$$\int_{\Omega} V_n\varphi dx + \tau\int_{\Omega}\Phi(\nabla V_n - \Theta(V_n)).\nabla\varphi dx + \tau\int_{\Omega}\beta\left(V_n\right)\varphi dx = \int_{\Omega}\left(\tau f_n + V_{n-1}\right)\varphi dx. \tag{7}$$

Theorem 2. *Under the hypotheses* $(H1)$, $(H2)$, $(H3)$, *the problem* (P_n^{ω}) *has a unique weak solution* $(V_n)_{0\leq n\leq N}$ *and for all* $n = 1,\ldots,N, V_n \in L^{\infty}(\Omega) \cap W^{1,p(x)}(\Omega,\omega)$.

For $n=1$, we denote $V = V_1$ and we rewrite the problem (6) as

$$\begin{cases} -\tau \operatorname{div}(\Phi(\nabla V - \Theta(V))) + \bar{\beta}(V) = F & \text{in} \quad \Omega, \\ V = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

Thanks to the hypothesis $(H2)$, the function $F = \tau f_1 + v_0$ is an element of $L^{\infty}(\Omega)$ and the function $\bar{\beta}(s) = \tau\beta(s) + s$ is a non decreasing continuous real function on $\mathbb{R}$ surjective such that $\bar{\beta}(0) = 0$. Therefore, by [7], the problem (8) has a unique weak solution V_1 in $L^{\infty}(\Omega) \cap W^{1,p(x)}(\Omega,\omega)$.

By induction, we deduce by the same way that the problem (P^{ω}) has a unique weak solution $(V_n)_{0\leq n\leq N}$ such that $n = 1,\ldots,N, V_n \in L^{\infty}(\Omega) \cap W^{1,p(x)}(\Omega,\omega)$.

3.2. Stability results

In this part, we prove some a priori estimates for the discrete weak solution $(V_n)_{1\leq n\leq N}$ which we use later to derive convergence results for the Euler forward scheme.

Theorem 3. *Under the hypotheses* $(H1)$, $(H2)$, $(H3)$ *there exists a positive constant* $C\left(V_0, f, F\right)$ *depending on the data but not on* N *such that for all* $n = 1$, *…*, N, *we have*

$$\left\|V_n\right\|_{\infty} \leq C\left(v_0, f, F\right), \tag{9}$$

$$\sum_{i=1}^{n}\left\|V_i - V_{i-1}\right\|_2^2 \leq C\left(v_0, f, F\right), \tag{10}$$

$$\tau\sum_{i=1}^{n}\int_{\Omega}\Phi(\nabla V_i - \Theta(V_i)).\nabla V_i dx \leq C\left(v_0, f, F\right). \tag{11}$$

Proof. **For** (9). For $k > 0$ and $1 \leq n \leq N$, we have $V_n \in L^{\infty}(\Omega)$. So, multiplying (P_n^{ω}) by $\left|V_n\right|^k V_n$ and integrating over Ω, we have

$$\int_{\Omega}\left|V_n\right|^{k+2}dx - \tau\int_{\Omega}\operatorname{div}\left(\Phi\left(\nabla V_n - \Theta\left(V_n\right)\right)\right)\left|V_n\right|^k V_n dx + \tau\int_{\Omega}\beta\left(V_n\right)\left|V_n\right|^k V_n dx$$
$$= \int_{\Omega}\left(\tau f_n + V_{n-1}\right)\left|V_n\right|^k V_n dx. \tag{12}$$

According to Holder's inequality, $(H1)$, $(H2)$, and $(H3)$, using also the fact that $\Phi(\nabla V_i - \Theta(V_i)).\nabla V_i$ is monotone, we get

$$\left\|V_n\right\|_{k+2}^{k+2} \leq \tau c_1\left\|V_n\right\|_{k+1}^{k+1} + \left\|V_{n-1}\right\|_{k+2}\left\|V_n\right\|_{k+2}^{k+1}. \tag{13}$$

We obtain

$$\left\|V_n\right\|_{k+2} \leq \tau c_1\left\|V_n\right\|_{k+1}^{k+1} + \left\|V_{n-1}\right\|_{k+2}. \tag{14}$$

By using simple induction, we get

$$\left\|V_n\right\|_{k+2} \leq N c_2 T + \left\|V_0\right\|_{k+2}. \tag{15}$$

Finally, as $k \to \infty$, we obtain the result (9).

For (10). Let $1 \leq i \leq N$ and let $\varphi = V_i$ as a test function in (7). Then we have

$$\int_{\Omega}\left(V_i - V_{i-1}\right)V_i dx + \tau\int_{\Omega}\Phi(\nabla V_i - \Theta(V_i)).\nabla V_i dx + \tau\int_{\Omega}\beta\left(V_i\right)V_i dx = \int_{\Omega}\tau f_i V_i dx. \tag{16}$$

By the elementary identity

$$a(a-b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2,$$

we get from (16)

$$\frac{1}{2}\|V_i\|_2^2 - \frac{1}{2}\|V_{i-1}\|_2^2 + \frac{1}{2}\|V_i - V_{i-1}\|_2^2 + \tau \int_\Omega \Phi(\nabla V_i - \Theta(V_i)).\nabla V_i dx \le \tau c_3 \|V_i\|_2 . \tag{17}$$

Taking the sum of (17) from $i = 1$ to n, we get

$$\frac{1}{2}\|V_n\|_2^2 - \frac{1}{2}\|V_0\|_2^2 + \frac{1}{2}\sum_{i=1}^n \|V_i - V_{i-1}\|_2^2 + \tau \sum_{i=1}^n \int_\Omega \Phi(\nabla V_i - \Theta(U_i)).\nabla V_i dx \le c_4. \tag{18}$$

So

$$\frac{1}{2}\sum_{i=1}^n \|V_i - V_{i-1}\|_2^2 + \tau \sum_{i=1}^n \int_\Omega \Phi(\nabla V_i - \Theta(V_i)).\nabla V_i dx \le c_4 + \frac{1}{2}\|V_0\|_2^2 . \tag{19}$$

Thus

$$\frac{1}{2}\sum_{i=1}^n \|V_i - V_{i-1}\|_2^2 + \tau \sum_{i=1}^n \int_\Omega \Phi(\nabla V_i - \Theta(V_i)).\nabla V_i dx \le c_5. \tag{20}$$

Hence

$$\frac{1}{2}\sum_{i=1}^n \|V_i - V_{i-1}\|_2^2 \le c_5. \tag{21}$$

This yields the stability result (10).

For (11). By (20) and (10), we have the stability result (11). ◄

Theorem 4. *Let the hypotheses* $(H1)$, $(H2)$, $(H3)$ *hold. Then, there exists a positive constant* $C(u_0, f, F)$ *depending on the data but not on* N *such that for all* $n = 1, \dots, N$, *we have*

$$\tau \sum_{i=1}^n \|\beta(V_i)\|_1 \le C(v_0, f, F), \tag{22}$$

$$\lim_{k\to 0} \sum_{i=1}^n \frac{\tau}{k} \int_{\{|V_i| \le k\}} \Phi(\nabla V_i - \Theta(V_i)).\nabla V_i \le C(v_0, f, F), \tag{23}$$

$$\sum_{i=1}^n \|V_i - V_{i-1}\|_1 \le C(v_0, f, F). \tag{24}$$

Proof. **For** (22) **and** (23).
Let $\varphi = T_k(V_i)$ as a test function in (7). Then, dividing this equality by k and taking limits when k goes to 0, we have

$$\|V_i\|_1+\tau\|\beta(V_i)\|_1+\lim_{k\to 0}\frac{\tau}{k}\int_{\{|V_i|\le k\}}\Phi(\nabla V_i-\Theta(V_i)).\nabla V_i \le \tau\|f_i\|_1+\|V_{i-1}\|_1. \quad (25)$$

Summing (25) from $i=1$ to n, we deduce the stability results (22) and (23).
For (24).
Let $\varphi = T_\tau(V_i - V_{i-1})$ in (7). Then, dividing this equality by τ we get

$$\int_\Omega (V_i - V_{i-1})\frac{T_\tau(V_i - V_{i-1})}{\tau}dx + \int_{B^i_\tau}\Phi(\nabla V_i - \Theta(V_i)).(\nabla V_i - \nabla V_{i-1})dx$$
$$\le \tau\|\beta(V_i)\|_1 + \tau\|f_i\|_1, \quad (26)$$

where $B^i_\tau = \{|V_i - V_{i-1}| \le \tau\}$.

By applying Lemma 2.5, we get

$$\frac{1}{p(x)}|\nabla V_i - \theta(V_i)|^{p(x)} - \frac{1}{p(x)}|\nabla V_{i-1} - \theta(V_i)|^{p(x)} \le$$
$$\le \omega(x)|\nabla V_i - \theta(V_i)|^{p(x)-2}(\nabla V_i - \theta(V_i)).(\nabla V_i - \nabla V_{i-1}).$$

So

$$\int_\Omega (V_i - V_{i-1})\frac{T_\tau(V_i - V_{i-1})}{\tau}dx+$$
$$+\int_{B^i_\tau}(\frac{1}{p(x)}|\nabla V_i - \theta(V_i)|^{p(x)} - \frac{1}{p(x)}|\nabla V_{i-1} - \theta(V_i)|^{p(x)})dx$$
$$\le \tau\|\beta(V_i)\|_1 + \tau\|f_i\|_1.$$

Summing the inequality above from $i=1$ to n, using the stability result (22), we have

$$\sum_{i=1}^n\int_\Omega (V_i - V_{i-1})\frac{T_\tau(V_i - V_{i-1})}{\tau}dx \le$$
$$\le \frac{1}{p(x)}\int_\Omega|\nabla V_0|^{p(x)}dx + c_6 \le \frac{1}{p_-}\int_\Omega|\nabla V_0|^{p(x)}dx + c_6. \quad (27)$$

So, we let τ tend to 0 in the inequality above, and we get the stability result (24).
◄

4. Convergence and existence results

In this part, using the above results, we build a weak solution of problem (P^{ω}) and we show that this solution is unique.

4.1. Proof of existence

Let us introduce a piecewise linear extension, called Rothe function by

$$\begin{cases} v_N(0) := v_0 \\ v_N(t) := V_{n-1} + (V_n - V_{n-1})\,\frac{(t-t_{n-1})}{\tau},\ \forall t \in \left]\, t_{n-1}, t_n \right],\ n = 1, \ldots, N \quad \text{in}\, \Omega, \end{cases} \tag{28}$$

and a piecewise constant function

$$\begin{cases} \bar{v}_N(0) := v_0 \\ \bar{v}_N(t) := V_n \quad \forall t \in \left]\, t_{n-1}, t_n \right], \quad n = 1, \ldots, N \quad \text{in} \quad \Omega, \end{cases} \tag{29}$$

where $t_n := n\tau$. As already shown, for any $N \in \mathbb{N}$, the solution $(V_n)_{1 \le n \le N}$ of problems (P_n^{ω}) is unique. Thus, v_N and $\bar{v}_N$ are uniquely defined and by construction, for any $t \in \left]t_{n-1}, t_n\right]$, $n = 1, \ldots, N$, we have

i) $\frac{\partial v_N(t)}{\partial t} = \frac{(V_n - V_{n-1})}{\tau}$.

ii) $\bar{v}_N(t) - v_N(t) = (V_n - V_{n-1})\,\frac{t_n - t}{\tau}$.

By Theorem 3, for any $N \in \mathbb{N}$, the solution $(V_n)_{1 \le n \le N}$ of problem (6) is unique. Thus, v_N and $\bar{v}_N$ are uniquely defined.

By using the stability results of Theorem 4, we deduce the following a priori estimates concerning the Rothe function v_N and the function $\bar{u}_N$.

Lemma 3. *Under the hypotheses $(H1)$, $(H2)$ and $(H3)$, there exists a positive constant $C\,(T, v_0, f, F)$ not depending on N such that for all $N \in \mathbb{N}$, we have*

$$\|\bar{v}_N - v_N\|^2_{L^2(Q_T)} \le \frac{1}{N} C\,(T, v_0, f, F)\,, \tag{30}$$

$$\|\bar{v}_N\|_{L^{\infty}(0,T,L^2(\Omega))} \le C\,(T, v_0, f, F)\,, \tag{31}$$

$$\|v_N\|_{L^{\infty}(0,T,L^2(\Omega))} \le C\,(T, v_0, f, F)\,, \tag{32}$$

$$\|\bar{v}_N\|_{L^{p(x)}\left(0,T,W^{1,p(x)}(\Omega,\omega)\right)} \le C\,(T, v_0, f, F)\,, \tag{33}$$

$$\|\beta\,(\bar{v}_N)\|_{L^1(Q_T)} \le C\,(T, v_0, f, F)\,, \tag{34}$$

$$\left\|\frac{\partial v_N}{\partial t}\right\|^2_{L^2(Q_T)} \le C\,(T, v_0, f, F)\,. \tag{35}$$

Proof.
For (30)**.** We have

$$\|\bar{v}_N - v_N\|^2_{L^2(Q_T)} = \int_0^T \int_\Omega |\bar{v}_N - v_N|^2 \, dxdt$$

$$\leq \sum_{i=1}^{i=N} \int_{t_{n-1}}^{t_n} \int_\Omega |V_n - V_{n-1}|^2 \left(\frac{t_n - t}{\tau}\right)^2 dxdt$$

$$\leq \frac{1}{N} C\left(T, v_0, f, F\right).$$

Using the same method as above, we prove the estimates (31), (32), (33) and (34).

For (35)**.**
We have for $n = 1, \ldots, N$ and $t \in (t_{n-1}, t_n]$

$$\frac{\partial v_N(t)}{\partial t} = \frac{(V_n - V_{n-1})}{\tau}.$$

This yields

$$\left\|\frac{\partial v_N}{\partial t}\right\|_{L^1(Q_T)} = \int_0^T \int_\Omega \left|\frac{\partial v_N}{\partial t}\right| \mathrm{d}x \ \mathrm{d}t$$

$$= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \frac{1}{\tau} \|V_n - V_{n-1}\|_1$$

$$= \sum_{n=1}^{N} \|V_n - V_{n-1}\|_1 .$$

By the result (24), we deduce the estimate (35). Finally, the proof of Lemma 3 is complete.

Now, using the two results (31) and (32) of Lemma 3, we find that the sequences $(v_N)_{N\in\mathbb{N}}$ and $(\bar{v}_N)_{N\in\mathbb{N}}$ are uniformly bounded in $L^\infty\left(0, T, L^2(\Omega)\right)$. Therefore, there exist two elements v and u in $L^\infty\left(0, T, L^2(\Omega)\right)$ such that

$$\bar{v}_N \to^* v \quad \text{in} \quad L^\infty\left(0, T, L^2(\Omega)\right),$$

$$v_N \to^* u \quad \text{in} \quad L^\infty\left(0, T, L^2(\Omega)\right).$$

And by the result (30) of Lemma 3 it follows that

$$v \equiv u.$$

Furthermore, by Lemma 3 and the hypothesis $(H2)$, we obtain

$$\frac{\partial v_N}{\partial t} \rightarrow \frac{\partial v}{\partial t} \quad \text{in} \quad L^2\left(Q_T\right), \tag{36}$$

$$\bar{v}_N \rightarrow v \quad \text{in} \quad L^{p(x)}\left(0, T, W^{1,p(x)}(\Omega, \omega)\right). \tag{37}$$

By the hypothesis $(H1)$, we know that

$$\beta\left(\bar{v}_N\right) \rightarrow \beta(v) \quad a.e.\ \ in\ \ Q_T,$$

and

$$|\beta(\bar{v}_N)| \leq M|\bar{v}_N| \in L^1\left(Q_T\right).$$

Then, thanks to the Lebesgue dominated convergence theorem, we deduce that

$$\beta\left(\bar{v}_N\right) \rightarrow \beta(v) \quad in \quad L^1\left(Q_T\right). \tag{38}$$

Since $\{\nabla \bar{v}_N - \Theta\left(\bar{v}_N\right)\}$ is equiintegrable by the assumption $(H3)$ and due to the boundedness of $(\bar{v}_N)$, we have

$$\Phi(\nabla \bar{v}_N - \Theta(\bar{v}_N)) \rightarrow \Phi(\nabla v - \Theta(v)) \quad \text{weakly} \quad \text{in} \quad L^1\left(Q_T\right).$$

From the reflexivity of $L^{p'(x)}(\Omega, \omega)$ and the boundedness of $\{\Phi\left(\nabla \bar{v}_N - \Theta\left(\bar{v}_N\right)\right)\}$, we have

$$\Phi(\nabla \bar{v}_N - \Theta(\bar{v}_N)) \rightarrow \Phi(\nabla v - \Theta(v)) \quad \text{weakly} \quad \text{in} \quad \left(L^{p'(x)}\left(Q_T, \omega\right)\right)^d. \tag{39}$$

Thanks to Lemma 3 and Aubin-Simon's compactness result, we obtain

$$v_N \rightarrow v \quad in \quad C\left(0, T, L^2(\Omega)\right). \tag{40}$$

Now, let us show that the limit function u is a weak solution of problem (P^{ω}). Firstly, we have $v_N(0) = V_0 = v_0$ for all $N \in \mathbb{N}$. Then $v(0,.) = v_0$. Now let $\varphi \in C^1(Q_T)$ and rewrite (5) in the form

$$\int_0^T \int_\Omega \frac{\partial v_N}{\partial t}\varphi dxdt + \int_0^T \int_\Omega \Phi(\nabla \bar{v}_N - \Theta(\bar{v}_N)).\nabla\varphi dxdt + \int_0^T \int_\Omega \beta\left(\bar{}_N\right)\varphi dxdt$$

$$= \int_0^T \int_\Omega f_N \varphi dxdt, \tag{41}$$

where

$$f_N(t,x) = f_n(x), \forall t \in]\, t_{n-1}, t_n] \,, n = 1, \ldots, N.$$

Taking limits as $N \to \infty$ in (41) and by the above results, we find that v is a weak solution of nonlinear parabolic problem (P^{ω}). ◀

4.2. Proof of uniqueness

Let v^1 and v^2 be two weak solutions of nonlinear parabolic problem (P^{ω}). Take $\varphi = v^1 - v^2$ as a test function for solution v^1 in (5) and take $\varphi = v^2 - v^1$ as a test function for solution v^2 in (5). Then we obtain

$$\int_0^T \int_\Omega \frac{\partial v^1}{\partial t}(v^1 - v^2)dxdt + \int_0^T \int_\Omega \Phi(\nabla v^1 - \Theta(v^1)).\nabla(v^1 - v^2)dxdt$$

$$+ \int_0^T \int_\Omega \beta(v^1)(v^1 - v^2)dxdt = \int_0^T \int_\Omega f(v^1 - v^2)dxdt,$$

and

$$\int_0^T \int_\Omega \frac{\partial v^2}{\partial t}(v^2 - v^1)dxdt + \int_0^T \int_\Omega \Phi(\nabla v^2 - \Theta(v^2)).\nabla(v^2 - v^1)dxdt$$

$$+ \int_0^T \int_\Omega \beta(v^2)(v^2 - v^1)dxdt = \int_0^T \int_\Omega f(v^2 - v^1)dxdt.$$

By summing up the two above equalities, we get

$$\int_0^T \int_\Omega \frac{\partial(v^1 - v^2)}{\partial t}(v^1 - v^2)dxdt +$$

$$+ \int_0^T \int_\Omega (\Phi(\nabla v^1 - \Theta(v^1)) - \Phi(\nabla v^2 - \Theta(v^2))).\nabla(v^1 - v^2)dxdt$$

$$+\int_0^T \int_\Omega (\beta(v^1) - \beta(v^2))(v^1 - v^2)dxdt = 0.$$

According to the hypotheses $(H1)$, $(H3)$, we get

$$v^1 \equiv v^2.$$

5. Conclusion

In this work, we prove the existence and uniqueness of weak solutions for nonlinear parabolic problem (P^ω) using time discretization technique by Euler forward scheme and Rothe method combined with the theory of variable exponent weighted Sobolev spaces.

Acknowledgements

The authors are grateful to the editor in chief and anonymous referee for their valuable comments which helped to improve the quality of this article.

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Received 10 September 2022
Accepted 16 November 2022