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On Equations of the Form $\Delta u - \frac{\partial u}{\partial t} = f(x, t, u, Du)$

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Abstract. The paper describes an interpolation method for obtaining a priori estimates for strong solutions of semilinear parabolic equations with unbounded singularities on the right-hand side, provided that there is a first a priori estimate in the space of summable functions.

Key Words and Phrases: a priori estimate, semilinear, maximum principle.

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1. Introduction

The initial-boundary value problem for a semilinear second-order parabolic equation is studied. We consider the problem of the existence of a priori estimates for a solution in the Sobolev space in terms of its norm in the Lebesgue space. The growth conditions for the nonlinearities are found for which this problem has a solution.

Let $R_+ \equiv \{t_0 \in R : t_0 \geq 0\}$. Let us introduce the following notations: $x = (x_1, ..., x_n)$ is a point in the space $R^n; \Omega \subset R^n$ is a bounded domain with the boundary $\partial\Omega$ from the class C^2; $Q_\tau = \Omega \times (t_0, t_0 + \tau)$ is a cylindrical domain in $R^{n+1}, t_0, \tau \in R_+; \partial Q_\tau = \partial\Omega \times (t_0, t_0 + \tau)$ is a lateral surface area of the cylinder Q_τ; $Q_T = \Omega \times (0, T)$ is a cylinder of height T and $(Q_T) = \left\{ (x, t) \mid x \in \bar{\Omega}, t = 0 \right\} \bigcup (\partial\Omega \times [0, T])$ is a parabolic boundary of Q_T.

All functions introduced below are assumed to be real-valued functions. The following functional spaces have been used [8]: the space of summable functions $L_p(Q_T)$, $p \geq 1$, with the norm

$$\|u\|_{p;Q_T} = \left(\int_0^T \int_\Omega |u(x,t)|^p \, dxdt \right)^{1/p}, \|u\|_{\infty;Q_T} = \lim_{p\to\infty} \|u\|_{p;Q_T};$$

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anisotropic space of summable functions $L_{q,r}(Q_T)$, $q,r \geq 1$, with the norm

$$\|u\|_{q,r;Q_T} \equiv \|u\|_{L_{q,r}(Q_T)} = \left(\int_0^T \left(\int_\Omega |u(x,t)|^q \, dx \right)^{r/q} dt \right)^{1/r};$$

anisotropic Sobolev space $W_p^{2,1}(Q_T)$ with the norm

$$\|u\|_{W_p^{2,1}(Q_T)} \equiv \|u\|_{p;Q_T} + \sum_{i=1}^n \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{p;Q_T} + \left\| \frac{\partial u}{\partial t} \right\|_{p;Q_T}.$$

2. Basic a priori estimate

Consider the following boundary value problem:

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = f(x,t,u,Du), \quad (x,t) \in Q_T, \\ u|_{\partial Q_T} = \varphi(x,t), \ x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) = \psi(x), \quad x \in \Omega. \end{cases} \tag{1}$$

Here Δ is the Laplace operator, $u = u(x,t)$ and $Du \equiv D_x u(x,t)$ is the gradient of the function $u(x,t)$.

We consider this problem in the class of real functions from the Sobolev space $W_p^{2,1}(Q_T)$ for $p > n+2$, so that the boundary function $\varphi(x,t)$ belongs to the space $W_p^{2-\frac{1}{p},1}(\partial Q_T)$ and $\psi(x) \in W_p^{2-\frac{2}{p}}(\Omega)$ (see [2, p. 389]).

Regarding the function $f(x,t,\xi_0,\xi_1)$, it is assumed that the following conditions are satisfied:

A.1) Let the function $f(x,t,\xi_0,\xi_1)$ be defined on $\bar{Q}_T \times R \times R^n$ with values in R and satisfy the Carathéodory condition, i.e. let it be measurable with respect to (x,t) for all $(\xi_0,\xi_1) \in R \times R^n$ and continuous with respect to (ξ_0,ξ_1) for almost all $(x,t) \in Q_T$.

A.2) Let

$$|f(x,t,\xi_0,\xi_1)| \leq b(x,t,\xi_0) + b_1(x,t,\xi_0) \cdot |\xi_1|^{\mu_1}$$

for almost all $(x,t) \in Q_T$ and for all $\xi_0 \in R$, $\xi_1 \in R^n$, with non-negative carathéodorian functions $b(x,t,\xi_0)$ and $b_1(x,t,\xi_0)$ such that for any $\delta \geq 0$ the function

$$\hat{b}_\delta(x,t) \equiv \sup\{ b(x,t,\xi_0) \mid |\xi_0| \leq \delta \}$$

belongs to $L_p(Q_T)$, $p > 1$ and $p > n+2$; the function

$$\hat{b}_{1,\delta}(x,t) \equiv \sup\{ b_1(x,t,\xi_0) \mid |\xi_0| \leq \delta \}$$

belongs to $L_{q,r}(Q_T)$ with $q \geq p,\ r \geq p$.

A.3) Let

$$\mu \equiv \mu_1 = 2 - \frac{n}{q} - \frac{2}{r}. \tag{2}$$

For elliptic operators, this problem was considered in [1, 2, 3, 4, 5]. In [1, 2, 3, 4], it is assumed that a nonlinear function $f(x, u(x), Du(x))$ is continuous in all its arguments. In this case, the well-known S.N. Bernstein growth condition on a non-linear function is sufficient for the a priori estimate in $\|u\|_{\infty;\Omega}$ to follow from the a priori estimate in $\|Du\|_{\infty;\Omega}$ and, hence, the estimate in $\|u\|_{W_p^2(\Omega)}$. In [5, 6], a boundary value problem for elliptic and parabolic equations, respectively, was considered, where the non-linear function f belongs to the space L_p.

In this paper, we do not assume that the nonlinear function is continuous with respect to all of its arguments. Instead, we require that this non-linear function belongs to the space $L_p(Q_T)$ for an arbitrarily fixed function $u(x,t)$ from $W_p^{2,1}(Q_T)$ with $p > n+2$.

A new exact growth condition for the considered nonlinear function $f(x, t, \xi_0, \xi_1)$ with respect to $\xi_0 \in R$, $\xi_1 \in R^n$ is obtained, under which a priori estimate in $\|Du\|_{\infty;Q_T}$ follows from the a priori estimate in $\|u\|_{\infty;Q_T}$ of the solution of problem (1).

One particular example shows the unimprovability of the corresponding growth index. The theory of solvability of problems of the form (1) is considered under the existence condition for upper and lower solutions of these problems. Here, along with the theorem on a priori estimate in $\|u\|_{W_p^{2,1}(Q_T)}$, the maximum principle is used for Aleksandrov type parabolic equations (see [7, pp. 58-71]).

The maximum principle for the considered problem (1) in the Sobolev space $W_p^{2,1}(Q_T)$ with $p > n+2$. is discussed in examples. Based on the general solvability theorem, particular theorems on the solvability of problems of the form (1) are obtained.

To study the problem (1), we will make essential use of the embedding theorem for the spaces $W_p^{2,1}(Q_T)$ and the multiplicative inequality that follows from it. Let us formulate the embedding theorem [8] in the form we need.

Lemma 1. *Let $u(x,t) \in W_p^{2,1}(Q_T)$, $u|_{(Q_T)} = 0$, the conditions $p > n+2, q \geq p$, $r \geq p$ and A.3) be fulfilled. Then*

$$\|Du\|_{z_x, z_t; Q_T} \leq C_1 \cdot \|u\|^{\theta}_{W_p^{2,1}(Q_T)} \cdot \|u\|^{1-\theta}_{l,m;Q_T} + C_2 \cdot \|u\|_{l,m;Q_T} \tag{3}$$

with positive constants C_1 and C_2, independent of the function $u(x,t)$ from $W_p^{2,1}(Q_T)$, where

$$\theta = \frac{1}{\mu},\ \ \mu = \frac{\chi_0 + \chi_1}{\chi_0},$$

$$\chi_0 = \frac{1}{2}\left[n\left(\frac{1}{l}-\frac{1}{z_x}\right)+1+2\left(\frac{1}{m}-\frac{1}{z_t}\right)\right] > 0,$$
$$\chi_1 = 1-\frac{1}{2}\left[n\left(\frac{1}{p}-\frac{1}{z_x}\right)+1+2\left(\frac{1}{p}-\frac{1}{z_t}\right)\right] > 0,$$

$$z_x = \frac{\mu \cdot p \cdot q}{q-p} \geq l,\ z_t = \frac{\mu \cdot p \cdot r}{r-p} \geq m,\ 1 \leq l,\ \ m \leq \infty \text{ are constants and } \mu > 1. \tag{4}$$

Lemma 2. *Let $u(x,t) \in W_p^{2,1}(Q_T)$ and the conditions A.1)- A.3) be fulfilled. Then the operator*

$$F_0(u)(x,t) \equiv f(x,t,u(x,t),Du(x,t))$$

is a bounded and continuous operator from $W_p^{2,1}(Q_T)$ to $L_p(Q_T)$.

Proof. Let us estimate $\|F_0(u)\|_{p;Q_T}$ with the help of conditions A.1) - A.3). From inequality (3) for Du, we have

$$\|Du\|_{z_x,z_t;Q_T} \leq C_1 \cdot \|u\|^{1-\theta}_{W_p^{2,1}(Q_T)} \cdot \|u\|^{1-\theta}_{\infty,\infty;Q_T} + C_2 \cdot \|u\|_{\infty,\infty;Q_T}\,. \tag{5}$$

From condition A.2) it follows that

$$\|F_0(u)\|_{p;Q_T} \leq \left\|\hat{b}_\delta\right\|_{p;Q_T} + C_3 \cdot \left\|\hat{b}_{1,\delta}\right\|_{q,r;Q_T} \cdot \|Du\|^{\mu}{}_{z_x,z_t;Q_T}$$

with $\delta = \|u\|_{\infty;Q_T}$ and a positive constant C_3, independent of the function $u(x,t)$ from $W_p^{2,1}(Q_T)$. Here $\|u\|_{\infty;Q_T} = \|u\|_{\infty,\infty;Q_T}$. Then, based on (4) and inequalities (5), we obtain

$$\|F_0(u)\|_{p;Q_T} \leq \left\|\hat{b}_\delta\right\|_{p;Q_T} + \|u\|_{W_p^{2,1}(Q_T)} \cdot \Phi_1\left(\|u\|_{\infty;Q_T}\right) \cdot \left\|\hat{b}_{1,\delta}\right\|_{q,r;Q_T} + \Phi_2\left(\|u\|_{\infty;Q_T}\right), \tag{6}$$

where $\Phi_1, \Phi_2 : R_+ \to R_+$ are the increasing functions defined by the known data. (6) proves the boundedness of the operator $F_0(u)$.

On the other hand, $p > n+2$. Then, by virtue of the Sobolev embedding theorem [9], it follows that the embedding operator $I : W_p^{2,1}(Q_T) \to L_{z_x,z_t}(Q_T)$ is completely continuous. By virtue of (6), the operator $F_0 : L_{z_x,z_t}(Q_T) \to L_p(Q_T)$ is bounded and, by the general property of the superposition operator, is continuous.

Thus the operator $F_0 : W_p^{2,1}(Q_T) \to L_p(Q_T)$ is completely continuous as a composition of completely continuous and continuous operators. Lemma 2 is proved. ◀

We have

$$\Delta u - \frac{\partial u}{\partial t} = f(x,t,u,Du) = \frac{f(x,t,u,Du)}{1+|Du|^{\mu}} \cdot (1+|Du|^{\mu}).$$

Hence, for the function $u(x,t)$

$$\begin{cases} \Delta u - c(x,t)\,u(x,t) - \frac{\partial u}{\partial t} = f_0(x,t) + f_1(x,t)\cdot |Du|^{\mu}\ ,\ (x,t)\in Q_T\,, \\ u|_{\partial Q_T} = \varphi(x,t)\,,\ x\in\partial\Omega,\ t\in(0,T)\,, \\ u(x,0) = \psi(x)\,,\ \ x\in\Omega, \end{cases} \tag{7}$$

where

$$f_1(x,t) = \frac{f(x,t,u,Du)}{1+|Du|^{\mu}},\ \ f_0(x,t) = f_1(x,t) - c(x,t)\,u(x,t)\,,$$

$$c(x,t) = \hat{b}_{\delta}(x,t) \geq 0 \ \text{ with } \delta = \|u\|_{\infty;Q_T}\,.$$

Consider in the space $W_p^{2,1}(Q_T)$ with $p > n+2$ the boundary value problem

$$\begin{cases} \Delta\vartheta - c(x,t)\,\vartheta - \frac{\partial\vartheta}{\partial t} = f_1(x,t)\cdot|D\vartheta|^{\mu} + \tau\cdot f_0(x,t)\,,\ (x,t)\in Q_T, \\ \vartheta|_{\partial Q_T} = \tau\cdot\varphi(x,t)\,,\ x\in\partial\Omega,\ t\in(0,T)\,, \\ \vartheta(x,0) = \tau\cdot\psi(x)\,,\ \ x\in\Omega, \end{cases} \tag{8}$$

with the parameter $\tau\in[0,1]$ and the functions $c(x,t)\,, f_1(x,t)$ and $f_0(x,t)$ defined above.

For problem (8), the following lemma on the uniqueness of the solution is valid.

Lemma 3. *For any fixed τ, the problem* (8) *has at most one solution $u(x,t)$ from $W_p^{2,1}(Q_T)\ , p > n+2$.*

Proof. Assume the contrary. Then for the difference $\omega = \vartheta - z$ of two possible solutions ϑ and z, we have

$$\begin{cases} \Delta\omega - c(x,t)\,\omega - \frac{\partial\omega}{\partial t} = f_1(x,t)\cdot\sum_{i=1}^{n} h_i(x,t)\cdot D_i\omega,\ (x,t)\in Q_T\,, \\ \omega|_{\partial Q} = 0,\ x\in\partial\Omega,\ t\in(0,T)\,, \\ \omega(x,0) = 0,\ \ x\in\Omega, \end{cases}$$

where

$$h_i(x,t) = \int_0^1 H_i(x,t,\nu)\,d\nu,$$

$$H_i(x,t,\nu) = \mu \cdot \left[\sum_{k=1}^{n} (\nu \cdot D_k\omega + D_k z)^2\right]^{\frac{\mu}{2}-1} \cdot (\nu \cdot D_i\omega + D_i z)(x,t)$$

for $\sum_{k=1}^{n} (\nu \cdot D_k\omega + D_k z)^2 (x,t) \neq 0$ and

$$H_i(x,t,\nu) = 0$$

for $\sum_{k=1}^{n} (\nu \cdot D_k\omega + D_k z)^2 (x,t) = 0, i = 1, ..., n.$

Since $c(x,t) \geq 0, c(x,t) \in L_p(Q_T)$, $f_1(x,t) \cdot h_i(x,t) \in L_{q,r}(Q_T) \quad (i = 1, ..., n)$ and $p > n+2$, from the results of [7, pp. 58-71] it follows that $\omega(x,t) \equiv 0$ in Q_T. Lemma 3 is proved. ◄

Now let ϑ_1 and ϑ_2 be solutions to problem (8) corresponding to the values τ_1 and τ_2 $(\tau_2 > \tau_1)$ of the parameter τ, respectively. Then for the difference $\tilde{\vartheta} = \vartheta_2 - \vartheta_1$, we have

$$\begin{cases} \Delta\tilde{\vartheta} - c(x,t)\,\tilde{\vartheta} - \frac{\partial\tilde{\vartheta}}{\partial t} = f_1(x,t) \sum_{i=1}^{n} \tilde{h}_i(x,t)\, D_i\tilde{\vartheta} + (\tau_2 - \tau_1)\, f_0\ (x,t), (x,t) \in Q_T; \\ \tilde{\vartheta}|_{\partial Q_T} = (\tau_2 - \tau_1)\,\varphi(x,t),\ x \in \partial\Omega,\ t \in (0,T), \\ \tilde{\vartheta}(x,0) = (\tau_2 - \tau_1) \cdot \psi(x),\ \ x \in \Omega, \end{cases}$$

where

$$\tilde{h}_i(x,t) = \int_0^1 \tilde{H}_i(x,t,\nu)\,d\nu, \tilde{H}_i(x,t,\nu) =$$

$$= \mu \cdot \left[\sum_{k=1}^{n} \left(\nu \cdot D_k\tilde{\vartheta} + D_k\vartheta_1\right)^2\right]^{\frac{\mu}{2}-1} \cdot \left(\nu \cdot D_i\tilde{\vartheta} + D_i\vartheta\right)(x,t)$$

for $\sum_{k=1}^{n} \left(\nu \cdot D_k\tilde{\vartheta} + D_k\vartheta_1\right)^2 (x,t) \neq 0$ and

$$\tilde{H}_i(x,t,\nu) = 0$$

for $\sum_{k=1}^{n} \left(\nu \cdot D_k\tilde{\vartheta} + D_k\vartheta_1\right)^2 (x,t) = 0.$

Assume

$$\mathrm{K} = (\tau_2 - \tau_1) \cdot (1 + \|u\|_\infty), \quad \tau_2 > \tau_1.$$

Lemma 4. $\|\vartheta_2 - \vartheta_1\|_\infty \leq (\tau_2 - \tau_1) \cdot \left(1 + \|u\|_{\infty;Q_T}\right).$

Proof. For the function $\left(\tilde{\vartheta} - \mathrm{K}\right)$, we have

$$\begin{cases} \Delta\left(\tilde{\vartheta} - \mathrm{K}\right) - c(x,t)\left(\tilde{\vartheta} - \mathrm{K}\right) - \frac{\partial}{\partial t}\left(\tilde{\vartheta} - \mathrm{K}\right) = \\ f_1(x,t)\sum_{i=1}^{n}\tilde{h}_i(x,t)\cdot D_i\left(\tilde{\vartheta} - \mathrm{K}\right) + (\tau_2 - \tau_1) f_0\ (x,t) + c(x,t)\,\mathrm{K}, (x,t) \in Q_T; \\ \left(\tilde{\vartheta} - \mathrm{K}\right)\Big|_{\partial Q_T} = (\tau_2 - \tau_1)\,\varphi - \mathrm{K},\ x \in \partial\Omega,\ t \in (0,T), \\ \left(\tilde{\vartheta} - \mathrm{K}\right)\Big|_{t=0} = (\tau_2 - \tau_1)\cdot\psi - K,\ \ x \in \Omega. \end{cases}$$

Further

$$c(x,t)\,\mathrm{K} + (\tau_2 - \tau_1) f_0(x,t) = (\tau_2 - \tau_1)\cdot\left[f_0(x,t) + c(x,t)\left(1 + \|u\|_{\infty;Q_T}\right)\right] =$$
$$= (\tau_2 - \tau_1)\left[c(x,t)\left(\|u\|_{\infty;Q_T} - u(x,t)\right) + (f_1(x,t) + c(x,t))\right] \geq 0$$

in Q_T and

$$\begin{aligned} \left(\tilde{\vartheta} - \mathrm{K}\right)\Big|_{\partial Q_T} &= (\tau_2 - \tau_1)\,\varphi - (\tau_2 - \tau_1)\left(1 + \|u\|_{\infty;Q_T}\right) = \\ &= (\tau_2 - \tau_1)\cdot\left[\varphi - 1 - \|u\|_{\infty;Q_T}\right] \leq \\ &\leq (\tau_2 - \tau_1)\cdot\left[\varphi - \|u\|_{\infty;Q_T}\right] \leq 0, \end{aligned}$$

$$\begin{aligned} \left(\tilde{\vartheta} - \mathrm{K}\right)\Big|_{t=0} &= (\tau_2 - \tau_1)\,\psi(x) - (\tau_2 - \tau_1)\left(1 + \|u\|_{\infty;Q_T}\right) = \\ &= (\tau_2 - \tau_1)\cdot\left(\varphi - 1 - \|u\|_{\infty;Q_T}\right) \leq \\ &\leq (\tau_2 - \tau_1)\cdot\left[\psi - \|u\|_{\infty;Q_T}\right] \leq 0. \end{aligned}$$

Then from [7, pp. 58-71] it follows that $\tilde{\vartheta} \leq$ Kin Q_T. Similarly, inequality $\tilde{\vartheta} \geq -\mathrm{K}$ is proved in $\bar{Q}_T$. Lemma 4 is proved. ◀

Theorem 1. *Let conditions A.1) and A.2) be satisfied. Then there exists a function $\Phi : R_+^3 \to R_+$ increasing in each argument such that for any possible solution $u \in W_p^{2,1}(Q_T)$ of problem* (1) *the a priori estimate*

$$\|u\|_{W_p^{2,1}(Q_T)} \leq \Phi\left(\|u\|_{\infty;Q_T}, \|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)}, \|\psi\|_{W_p^{2-2/p}(\Omega)}\right) \tag{9}$$

holds.

The function Φ depends only on the known data included in the conditions of the theorem (including $\left\|\hat{b}_\delta\right\|_{p;Q_T}, \left\|\hat{b}_{1,\delta}\right\|_{q,r;Q_T}$ with $\delta = \|u\|_{\infty;Q_T}$).

Proof. Consider a parametric family of problems (8). It should be noted that, by Lemma 3, the solution of problem (8) for $\tau = 1$ coincides with the solution of problem (7) and (in view of the adopted notation) with the solution of problem (1).

Let ϑ_1 and ϑ_2 be solutions to problem (8), corresponding to the values τ_1 and τ_2 $(\tau_2 > \tau_1)$ of the parameter τ, respectively. Then, by Lemma 4, we have

$$\|\vartheta_2 - \vartheta_1\|_{\infty;Q} \le (\tau_2 - \tau_1) \cdot \left(1 + \|u\|_{\infty;Q_T}\right) \ . \tag{10}$$

On the other hand, this function is a solution to the problem

$$\begin{cases} \Delta\tilde{\vartheta} - c\,(x,t)\,\tilde{\vartheta} - \frac{\partial\tilde{\vartheta}}{\partial t} = f_1\,(x,t)\,(|D\vartheta_2|^{\mu} - |D\vartheta_1|^{\mu}) + (\tau_2 - \tau_1)\,f_0\ \ (x,t)\,,(x,t) \in Q_T; \\ \tilde{\vartheta}\Big|_{\partial Q_T} = (\tau_2 - \tau_1)\,\varphi\,(x,t)\,,\ x \in \partial\Omega,\ \ t \in (0,T)\,, \\ \tilde{\vartheta}\Big|_{t=0} = (\tau_2 - \tau_1) \cdot \psi\,(x)\,,\ \ x \in \Omega. \end{cases}$$

Consequently

$$\begin{aligned} &\left\|\Delta\tilde{\vartheta} - c\,(x,t)\,\tilde{\vartheta} - \tfrac{\partial\tilde{\vartheta}}{\partial t}\right\|_{p;Q_T} = \\ &= \|f_1 \cdot |D\vartheta_2|^{\mu} - f_1 \cdot |D\vartheta_1|^{\mu} + (\tau_2 - \tau_1)\,f_0\ \|_{p;Q_T} \le \\ &\le \left\|f_1 \cdot \left|D\tilde{\vartheta} + D\vartheta_1\right|^{\mu} + f_1 \cdot |D\vartheta_1|^{\mu} + (\tau_2 - \tau_1)\,f_0\right\|_{p;Q_T} \le \\ &\le \left\|f_1 \cdot \left[2^{\mu-1}\left(\left|D\tilde{\vartheta}\right|^{\mu} + |D\vartheta_1|^{\mu}\right) + |D\vartheta_1|^{\mu}\right] + (\tau_2 - \tau_1)\,f_0\right\|_{p;Q_T} \le \\ &\le 2^{\mu-1} \cdot \|f_1\|_{q,r;Q_T} \cdot \left\|D\tilde{\vartheta}\right\|^{\mu}_{\infty;Q_T} + \\ &+2^{\mu} \cdot \|f_1\|_{q,r;Q_T} \cdot \|D\vartheta_1\|^{\mu}_{\infty;Q_T} + (\tau_2 - \tau_1) \cdot \|f_0\|_{p;Q_T}. \end{aligned} \tag{11}$$

On the other hand, it follows from (5) that

$$\left\|D\tilde{\vartheta}\right\|^{\mu}_{\infty;Q_T} \le 2^{\mu-1} \cdot \left(C_1^{\mu} \cdot \left\|\tilde{\vartheta}\right\|_{W_p^{2,1}(Q_T)} \cdot \left\|\tilde{\vartheta}\right\|^{\mu-1}_{\infty;Q} + C_2^{\mu} \cdot \left\|\tilde{\vartheta}\right\|^{\mu-1}_{\infty;Q_T}\right). \tag{12}$$

By virtue of the well-known linear theory of parabolic problems, the inequality

$$\left\|\tilde{\vartheta}\right\|_{W_p^{2,1}(Q_T)} \le A \cdot \left(\left\|\Delta\tilde{\vartheta} - c \cdot \tilde{\vartheta} - \frac{\partial\tilde{\vartheta}}{\partial t}\right\|_{p;Q_T} + \right.$$

$$\left. + (\tau_2 - \tau_1) \cdot \|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)} + (\tau_2 - \tau_1) \cdot \|\psi\|_{W_p^{2-2/p}(\Omega)}\right)$$

is fulfilled. Here $A = A\left(Q, n, p, q, \|c\|_{p;Q_T}\right)$.
Then, using inequalities (10)-(12), we obtain

$$\left\|\tilde{\vartheta}\right\|_{W_p^{2,1}(Q_T)} \le A \cdot 2^{\mu-1} \cdot \|f_1\|_{q,r;Q_T} \times$$

$$\times \left(C_1^{\mu} \cdot 2^{\mu-1} \cdot (\tau_2 - \tau_1)^{\mu-1} \cdot \left(1 + \|u\|_{\infty;Q_T}\right)^{\mu-1} \cdot \left\|\tilde{\vartheta}\right\|_{W_p^{2,1}(Q_T)} + \right.$$

$$\left. + C_2^{\mu} \cdot 2^{\mu-1} \cdot (\tau_2 - \tau_1)^{\ \mu} \cdot \left(1 + \|u\|_{\infty;Q_T}\right)^{\ \mu} \right) +$$

$$+ 2^{\mu} \cdot A \cdot \|f_1\|_{q,r;Q_T} \cdot \|D\vartheta_1\|_{\infty,Q_T}^{\mu} + A \cdot (\tau_2 - \tau_1) \cdot \|f_0\|_{q;Q_T} +$$

$$+ A \cdot (\tau_2 - \tau_1) \cdot \left(\ \|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)} + \|\psi\|_{W_p^{2-2/p}(\Omega)} \ \right) =$$

$$= A \cdot 2^{2\mu-2} \cdot C_1^{\mu} \cdot \|f_1\|_{q,r;Q_T} \cdot (\tau_2 - \tau_1)^{\mu-1} \cdot \left(1 + \|u\|_{\infty;Q_T}\right)^{\mu-1} \cdot \left\|\tilde{\vartheta}\right\|_{W_p^{2,1}(Q_T)} +$$

$$+ A \cdot C_2^{\mu} \cdot 2^{2\mu-2} \cdot \|f_1\|_{q,r;Q_T} \cdot (\tau_2 - \tau_1)^{\mu} \cdot \left(1 + \|u\|_{\infty;Q_T}\right)^{\ \mu} +$$

$$+ 2^{\mu} \cdot A \cdot \|f_1\|_{q,r;Q_T} \cdot \|D\vartheta_1\|_{\infty,Q_T}^{\mu} +$$

$$+ A \cdot (\tau_2 - \tau_1) \cdot \left(\|f_0\|_{p;Q_T} + \|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)} + \|\psi\|_{W_p^{2-2/p}(\Omega)} \right).$$

Hence it follows that

$$\left\|\tilde{\vartheta}\right\|_{W_p^{2,1}(Q_T)} \le 2A \cdot \left(C_2^{\mu} \cdot 2^{2\mu-2} \cdot \|f_1\|_{q,r;Q_T} + \|f_0\|_{p;Q_T} + \right.$$

$$\left. + \|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)} + \|\psi\|_{W_p^{2-2/p}(\Omega)} \right) +$$

$$+ A \cdot 2^{\mu+1} \cdot \|f_1\|_{q,r;Q_T} \cdot \|D\vartheta_1\|_{\infty,Q_T}^{\mu}, \tag{13}$$

for

$$0 < \tau_2 - \tau_1 \le h, \tag{14}$$

where

$$h = \left(A C_1^{\mu} 2^{2\mu-1}\right)^{-\frac{1}{\mu-1}} \cdot \|f_1\|_{q,r;Q_T}^{-\frac{1}{\mu-1}} \cdot \left(1 + \|u\|_{\infty;Q_T}\right)^{-1},$$

τ_1 and τ_2 are any numbers from the interval $[0,1]$, satisfying the inequality (14), and ϑ_1, ϑ_2 are the corresponding solutions of the problem (8) from $W_p^{2,1}(Q_T)$.

So the inequality (13) becomes

$$\left\|\tilde{\vartheta}\right\|_{W_p^{2,1}(Q_T)} \le A^0 + A \cdot 2^{\mu+1} \cdot \|f_1\|_{q,r;Q_T} \cdot \|D\vartheta_1\|_{\infty,Q_T}^{\mu}, \tag{15}$$

where the constant A^0 is defined from (13):

$$A^0 = 2A\left(C_2^{\mu}2^{2\mu-2}\cdot\|f_1\|_{q,r;Q_T}+\|f_0\|_{p;Q_T}+\|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)}+\|\psi\|_{W_p^{2-2/p}(\Omega)}\right).$$

Denote $\tau^{(k-1)}=\tau_1,\ \tau^{(k)}=\tau_2,\ \vartheta^{(k-1)}=\vartheta_1,\ \vartheta^{(k)}=\vartheta_2$. From the inequality (15), we obtain

$$\left\|\vartheta^{(k)}\right\|_{W_p^{2,1}(Q_T)}\le A^0+\left\|\vartheta^{(k-1)}\right\|_{W_p^{2,1}(Q_T)}+A\cdot 2^{\mu+1}\cdot\|f_1\|_{q,r;Q_T}\cdot\left\|D\vartheta^{(k-1)}\right\|_{\infty;Q_T}^{\mu}$$

for $0<\tau^{(k)}-\tau^{(k-1)}\le h,\ \tau^{(k-1)},\tau^{(k)}\in[0,1]$. Hence, by virtue of the independence of h of the indicated values $\tau^{(k-1)}$, $\tau^{(k)}$ and the embedding inequality $W_p^{2,1}(Q_T)\to C^{1,0}\left(\bar Q_T\right)$ with $p>n+2$

$$\left\|D\vartheta^{(k-1)}\right\|_{\infty;Q_T}\le const\cdot\left\|\vartheta^{(k-1)}\right\|_{W_p^{2,1}(Q_T)},$$

after a finite number of iterations, we obtain the assertion of Theorem 1. In this case, the first iteration ($k=1$) corresponds to $t^{(0)}=0$ and $\vartheta^{(0)}=0$. Theorem 1 is proved. ◀

Remark 1. *Theorem 1 remains valid for equations of the form* (1) *under other boundary conditions, where the maximum principle is valid for the corresponding linear boundary value problem.*

3. Non-improvability of the index $\mu=2-\frac{n}{q}-\frac{2}{r}$

In this section, we give an example of a boundary value problem of the form (1) for which all the conditions of Theorem 1 are satisfied, except condition A.3), i.e., equalities (2). For this counterexample, the corresponding inequality holds and it is shown that the assertion of Theorem 1 is not true.

Let $n=1$ and $Q_T=(-1,1)\times(0,1)$,

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=b_\varepsilon(x,t)\cdot\left|\frac{\partial u}{\partial x}\right|^{\mu},\ \ (x,t)\in Q_T, \tag{16}$$

$$u(-1,t)=\frac{t-2}{(2-t+\varepsilon)^{\delta}}\equiv\varphi_0(t),\quad u(1,t)=\frac{t}{(2-t+\varepsilon)^{\delta}}\equiv\varphi_1(t),t\in(0,1);$$

$$u(x,0)=\frac{x-1}{(x^2+1+\varepsilon)^{\delta}}\equiv\psi(x),\quad x\in(-1,1), \tag{17}$$

with $b_\varepsilon(x,t)=\frac{\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}}{\left|\frac{\partial u}{\partial x}\right|^{\mu}}$.

Here $0 < \varepsilon \leq 1,\ 0 < \delta < \frac{1}{2},\ \mu > 2 - \frac{n}{q} - \frac{2}{r} = 2 - \frac{1}{q} - \frac{2}{r},\ q > p,\ \ r > p,\ p > 1$ and $p > 3$.

This problem with the given parameters has a unique solution

$$u(x,t) = \frac{x+t-1}{(x^2+1-t+\varepsilon)^\delta}.$$

Assume that

$$\sum_{i=0}^{1} \|\varphi_i\|_{W_p^{2-1/p,1}(\partial(0,1))} + \|\psi\|_{W_p^{2-2/p}(-1,1)} \leq c_1, \tag{18}$$

where c_1 is a positive constant independent of $\varepsilon,\ \varepsilon \in (0,1]$.

Let

$$u(x,t) = \varepsilon^k \cdot \vartheta(y,\tau),\ y = \frac{x}{\varepsilon},\ \tau = \frac{t}{\varepsilon^2},\ k = 1, \tag{19}$$

where $\vartheta \in C^{\infty,1}\left(R^2\right)$ and

$$\sup_{R^2} |\vartheta(y,\tau)| < \infty,\ \sup_{R^2} |D\vartheta(y,\tau)| < \infty. \tag{20}$$

This choice of function $u(x,t)$ provides the boundedness of the norm

$$\|u\|_{\infty;Q_T} \leq \sup_{Q_T} |u| \leq c,$$

where the constant c is independent of $\varepsilon,\ \varepsilon \in (0,1]$. On the other hand, we have

$$\|u\|_{W_p^{2,1}(Q_T)} = \left(\sum_{|\alpha|\leq 2} \int_{Q_T} |D^\alpha u|^p \, dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^p dxdt \right)^{1/p} \geq$$

$$\geq \varepsilon^{k-2+\frac{3}{p}} \cdot \left(\sum_{|\alpha|=2} \int_{Q_T^\varepsilon} |D^\alpha \vartheta|^p \, dyd\tau + \int_{Q_T^\varepsilon} \left|\frac{\partial \vartheta}{\partial t}\right|^p dyd\tau \right)^{1/p},$$

where $Q_T^\varepsilon = \left\{(y,\tau)\,|\varepsilon y \in (-1,1),\ \varepsilon^2\tau \in (0,1)\right\}$. Hence, for a domain Q_T such that $Q_T \in Q_T^\varepsilon$ for $\varepsilon \in (0,1]$, we have

$$\|u\|_{W_p^{2,1}(Q_T)} \leq \varepsilon^{-1+\frac{3}{p}} \cdot \left(\sum_{|\alpha|=2} \int_{Q_T^\varepsilon} |D^\alpha \vartheta| \, dyd\tau + \int_{Q_T^\varepsilon} \left|\frac{\partial \vartheta}{\partial t}\right|^p dyd\tau \right)^{1/p}.$$

Then for the function $\vartheta(y,\tau)$ with

$$\sum_{|\alpha|=2}\int_{Q_T^{\varepsilon}}|D^{\alpha}\vartheta|^{p}\,dyd\tau+\int_{Q_T^{\varepsilon}}\left|\frac{\partial\vartheta}{\partial t}\right|^{p}dyd\tau>0$$

and for

$$p>3 \tag{21}$$

we obtain the following limit relation:
$\|u\|_{W_p^{2,1}(Q_T)}\to+\infty$ as $\varepsilon\to 0$ $(\varepsilon>0)$.

It should be noted that the inequality (21) implies the embedding

$$W_p^{2,1}(Q_T)\subset C\left(\bar{Q}_T\right).$$

For the norm $\|b_{\varepsilon}\|_{q,r;Q_T}$, we have

$$\|b_{\varepsilon}\|_{q,r;Q_T}=\left(\int_0^1\left(\int_{-1}^1|b_{\varepsilon}|^{q}\,dx\right)^{r/q}dt\right)^{1/r}\leq$$

$$\leq\varepsilon^{-2+\mu+\frac{1}{q}+\frac{1}{r}}\cdot\left\{\left\|\frac{\partial^2\vartheta/\partial y^2}{|\partial\vartheta/\partial y|^{\mu}}\right\|_{q,r;R^2}+\left\|\frac{\partial\vartheta/\partial\tau}{|\partial\vartheta/\partial y|^{\mu}}\right\|_{q,r;R^2}\right\}.$$

Hence for

$$\mu>2-\frac{1}{q}-\frac{2}{r} \tag{22}$$

we obtain

$$\|b_{\varepsilon}\|_{q,r;Q_T}\leq c_1(q,r),$$

where the constant c_1 depends only on q and r, and is independent of ε, $\varepsilon\in(0,1]$.

Thus, if $\mu>2-\frac{1}{q}-\frac{2}{r}$, then δ, $0<\delta<\frac{1}{2}$ can be chosen such that $u(x,t)$ is a solution of the problem (16)-(17) for any $\varepsilon>0$, $\|u\|_{\infty;Q_T}\leq const$, $\|b_{\varepsilon}\|_{q,r;Q_T}\leq c_1$ uniformly. However, $\|u\|_{W_p^{2,1}(Q_T)}\to+\infty$ as $\varepsilon\to 0$.

It follows that, under the conditions of Theorem 1, equalities (2) cannot be replaced by inequalities (without additional assumptions).

4. Resolvability theory

Consider the boundary value problem (1) with the conditions A.1) - A.3) and the following Lipschitz condition.

A.4) Let

$$|f(x,t,u,\eta)-f(x,t,u,\xi)|\leq b_2(x,t,u,\xi,\eta)\cdot|\eta-\xi|$$

for almost all $(x,t) \in Q_T$ and for all $(u,\xi,\eta) \in R \times R^n \times R^n$, where $b_2(x,t,u,\xi,\eta)$ is a measurable function in (x,t) for all $(u,\xi,\eta) \in R \times R^n \times R^n$, continuous in (u,ξ,η) for almost all $(x,t) \in Q_T$, and for any fixed $l > 0$

$$b_{2,l}(x,t) \equiv \sup \{ b_2(x,t,u,\xi,\eta) \, | \, |u| \le l, |\xi| \le l, \ |\eta| \le l \}$$

belongs to $L_{q,r}(Q_T)$, $p > n+2$, $q > p$, $r > p$.

Recall the well-known definitions.

Definition 1. *The function $u^+(x,t)$ from $W_p^{2,1}(Q_T)$ with $p > n+2$ is called the upper solution of problem* (1) *if*

$$\begin{cases} \Delta u^+ - \frac{\partial u^+}{\partial t} \le f(x,t,u^+,Du^+) \quad a.e. \ in \ Q_T, \\ u^+(x,t)|_{\partial Q_T} \ge \varphi(x,t), \ x \in \partial\Omega, \ t \in (0,T), \\ u^+(x,t) \ge \psi(x), \ \ x \in \Omega. \end{cases}$$

Definition 2. *The function $u^-(x,t)$ from $W_p^{2,1}(Q_T)$ with $p > n+2$ is called a lower solution to problem* (1) *if*

$$\begin{cases} \Delta u^- - \frac{\partial u^-}{\partial t} \ge f(x,t,u^-,Du^-) \quad a.e. \ in \ Q_T, \\ u^-|_{\partial Q_T} \le \varphi(x,t), \ x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) \le \psi(x), \ \ x \in \Omega. \end{cases}$$

Lemma 5. *Let a real-valued function $F_0(x,t,u,\xi)$, defined on $Q_T \times R \times R^n$, satisfy the Carathéodory condition A.1) with $f = F_0(x,t,u,\xi)$ and*

$$\sup_{(u,\xi) \in R \times R^n} |F_0(.,.,u,\xi)| \in L_p(Q_T) \quad with \ p > n+2. \tag{23}$$

Then the boundary value problem

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = F_0(x,t,u,\xi), \quad (x,t) \in Q_T, \\ u|_{\partial Q_T} = \varphi(x,t), \ x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) = \psi(x), \ \ x \in \Omega, \end{cases}$$

where $\varphi \in W_p^{2-1/p,1}(\partial Q_T)$, $\psi \in W_p^{2-2/p}(\Omega)$, has a solution $u(x,t) \in W_p^{2,1}(Q_T)$.

Proof. Consider the boundary value problem

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = F_0\left(x,t,\vartheta,D\vartheta\right), \quad (x,t) \in Q_T, \\ u|_{\partial Q_T} = \varphi\left(x,t\right), \, x \in \partial\Omega, \ t \in (0,T), \\ u|_{t=0} = \psi\left(x\right), \ \ x \in \Omega \end{cases}$$

for an arbitrary function $\vartheta\left(x,t\right) \in C^{1,0}\left(\bar{Q}_T\right)$. Then, according to the well-known linear theory of parabolic problems, this boundary value problem for any function $\vartheta \in C^{1,0}\left(\bar{Q}_T\right)$ has a solution $u\left(x,t\right)$ from $W_p^{2,1}\left(Q_T\right), \ u = A\vartheta$, which, by virtue of (23), satisfies the inequality

$$\|u\|_{W_p^{2,1}(Q_T)} \leq c_1, \tag{24}$$

holds, where the constant c_1 is independent of $\vartheta\left(x,t\right) \in C^{1,0}\left(\bar{Q}_T\right)$.

The operator $A : C^{1,0}\left(\bar{Q}_T\right) \to W_p^{2,1}\left(Q_T\right)$ $(p > n+2)$ is continuous and, due to the compact embedding of $W_p^{2,1}\left(Q_T\right) \to C^{1,0}\left(\bar{Q}_T\right)$, is a completely continuous operator from $C^{1,0}\left(\bar{Q}_T\right)$ to $C^{1,0}\left(\bar{Q}_T\right)$. By virtue of estimate (24), there exists a ball in the space $C^{1,0}\left(\bar{Q}_T\right)$, which the operator A transforms into itself. Then, according to the well-known Schauder theorem, the operator A has a fixed point $u\left(x,t\right)$ in $C^{1,0}\left(\bar{Q}_T\right)$, which, by the definition of the operator A, then belongs to the space $W_p^{2,1}\left(Q_T\right)$. Lemma 5 is proved. ◀

Let us now define for a function $u\left(x,t\right)$ from $W_p^{2,1}\left(Q_T\right)$ with $p > n+2$ **the truncation operator** σ by the relation

$$\sigma u\left(x,t\right) = \begin{cases} u^+\left(x,t\right) \ \ for \ \ u\left(x,t\right) > u^+\left(x,t\right), \\ u\left(x,t\right) \ \ for \ u^-\left(x,t\right) \leq u\left(x,t\right) \leq u^+\left(x,t\right), \\ u^-\left(x,t\right) \ \ for \ u\left(x,t\right) < u^-\left(x,t\right), \end{cases}$$

and consider the boundary value problem

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = f\left(x,t,\sigma u,Du\right), \quad (x,t) \in Q_T, \\ u|_{\partial Q_T} = \varphi\left(x,t\right), \, x \in \partial\Omega, \ t \in (0,T), \\ u\left(x,0\right) = \psi\left(x\right), \ \ x \in \Omega. \end{cases} \tag{25}$$

Lemma 6. *Let the function $f(x,t,\sigma u, Du)$ satisfy the conditions A.1) -A.4) with $p > n+2$, $\varphi \in W_p^{2-1/p,1}(\partial Q_T)$ and $\psi \in W_p^{2-2/p}(\Omega)$. Let $u(x,t)$ from $W_p^{2,1}(Q_T)$ be a solution of the problem* (25) *. Then*

$$u^-(x,t) \le u(x,t) \le u^+(x,t), \qquad (x,t) \in \bar{Q}_T. \tag{26}$$

Proof. For the function $\omega = u - u^+$, we have

$$\begin{cases} \Delta\omega - \frac{\partial \omega}{\partial t} \ge f(x,t,\sigma u, Du) - f(x,t,u^+,Du^+), & (x,t) \in Q_T, \\ \omega|_{\partial Q_T} \le 0,\ x \in \partial\Omega,\ t \in (0,T), \\ \omega(x,0) \le 0,\ \ x \in \Omega. \end{cases} \tag{27}$$

Let us now assume the converse of the lemma. Then the set

$$G = \{(x,t) \in Q_T |\, \omega(x,t) > 0\},$$

is not empty and

$$\Delta\omega - \frac{\partial\omega}{\partial t} \ge f\left(x,t,u^+,Du\right) - f\left(x,t,u^+,Du^+\right), \quad (x,t) \in Q_T.$$

Hence, due to condition A.4), we obtain

$$\Delta\omega - \frac{\partial\omega}{\partial t} \ge -\tilde{b}_{2,l}(x,t)\cdot|D\omega|, \quad (x,t) \in Q_T,$$

where $\tilde{b}_2(x,t) = b_2(x,t,u^+(x,t), Du^+(x,t), Du(x,t))$ and $\tilde{b}_2 \in L_{q,r}(Q_T)$ with $p > n+2,\ q > p, r > p$. Then from the results of [7, pp. 58-71] it follows that $\omega(x,t)$ reaches a strong positive maximum at the boundary (Q_T). This contradicts the boundary condition in (27). Thus, it is proved that $u(x,t) \le u^+(x,t)$ in the domain Q_T. The inequality $u^-(x,t) \le u(x,t)$ is proved similarly in Q_T. Lemma 6 is proved. ◀

Theorem 2. *Let conditions A.1) - A.4) be satisfied with some $p > n+2$, $\varphi \in W_p^{2-1/p,1}(\partial Q_T)$ and $\psi \in W_p^{2-2/p}(\Omega)$. Let there exist upper $u^+(x,t)$ and lower $u^-(x,t)$ solutions of the problem* (1) *from $W_p^{2,1}(Q_T)$ such that $u^+(x,t) \ge u^-(x,t)$ in $\bar{Q}_T$. Then there exists a solution $u(x,t)$ of the problem* (1) *from $W_p^{2,1}(Q_T)$ and*

$$u^-(x,t) \le u(x,t) \le u^+(x,t), \qquad (x,t) \in \bar{Q}_T.$$

Proof. Let

$$M = \max_{Q_T} \left\{ \max_{Q_T} u^+ (x,t), -\min_{Q_T} u^- (x,t) \right\}.$$

Then $\|u\|_{\infty,Q_T} \leq M$ and by Theorem 1

$$\|u\|_{W_p^{2,1}(Q_T)} \leq \Phi \left(M, \|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)}, \|\psi\|_{W_p^{2-2/p}(\Omega)} \right).$$

By virtue of the Sobolev embedding theorem [9], we have

$$\|u\|_{C^{1,0}(\bar{Q}_T)} \leq C_2 \cdot \|u\|_{W_p^{2,1}(Q_T)} \, (p > n+2), \tag{28}$$

where the positive constant C_2 is independent of the function $u \in W_p^{2,1}(Q_T)$. Then we obtain

$$\max_{\bar{Q}_T} |Du(x,t)| \leq M_1, \ M_1 = C_2 \cdot \Phi \left(M, \ \|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)}, \|\psi\|_{W_p^{2-2/p}(\Omega)} \right).$$

Define the function

$$F_1(x,t,u,\xi) = \begin{cases} f(x,t,u,\xi) & for \ |\xi| \leq M_2, \\ f\left(x,t,u,M_2 \cdot \frac{\xi}{|\xi|}\right) & for \ |\xi| > M_2, \end{cases}$$

where

$$M_2 = \max \left\{ M_1, \max_{\bar{Q}_T} \left| Du^+ \right|, \max_{\bar{Q}_T} \left| Du^- \right| \right\}.$$

This function satisfies conditions A.1)-A.3) with the corresponding inequality

$$|F_1(x,t,u,\xi)| \leq \begin{cases} b(x,t,u) + b_1(x,t,u) \cdot |\xi|^\mu & for \ |\xi| \leq M_2, \\ b(x,t,u) + b_1(x,t,u) \cdot M_2^\mu & for \ |\xi| > M_2. \end{cases}$$

The function F_1 also satisfies condition A.4) with the corresponding function b_2.

Consider now the boundary value problem

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = F_1(x,t,\sigma u, Du), & (x,t) \in Q_T, \\ u|_{\partial Q_T} = \varphi(x,t), \ x \in \partial\Omega, \ t \in (0,T), \\ u|_{t=0} = \psi(x), \ x \in \Omega. \end{cases} \tag{29}$$

The function

$$\bar{F}_1(x,t,u,\xi) = \begin{cases} F_1(x,t,u^+(x,t),\xi) & for \ \ u > u^+(x,t), \\ F_1(x,t,u(x,t),\xi) & for \ \ u^-(x,t) \leq u(x,t) \leq u^+(x,t), \\ F_1(x,t,u^-(x,t),\xi) & for \ \ u < u^-(x,t) \end{cases}$$

satisfies the conditions of Lemma 5, and

$$\bar{F}_1(x,t,u(x,t),Du(x,t)) = F_1(x,t,\sigma u(x,t),Du(x,t)).$$

Therefore, Lemma 5 is applicable to problem (29), due to which there exists a solution $u(x,t)$ of this problem from the space $W_p^{2,1}(Q_T)$, $p > n+2$.

The function $u^+(x,t)$ is the upper solution and $u^-(x,t)$ is the lower solution of the problem

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = F_1(x,t,u,Du), \quad (x,t) \in Q_T, \\ u|_{\partial Q_T} = \varphi(x,t), \ x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) = \psi(x), \ \ x \in \Omega. \end{cases} \tag{30}$$

By Lemma 6 we have

$$u^-(x,t) \le u(x,t) \le u^+(x,t), \qquad (x,t) \in \bar{Q}_T$$

and consequently $\sigma u(x,t) = u(x,t)$. So the obtained solution $u(x,t)$ of the problem (29) is also a solution of the problem (30).

Now, applying Theorem 1 to the solution $u(x,t)$ of the problem (30), we obtain

$$\|u\|_{W_p^{2,1}(Q_T)} \le \Phi\left(M, \|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)}, \|\psi\|_{W_p^{2-2/p}(\Omega)}\right).$$

Then it follows from the embedding inequality (28) that

$$\max_{Q_T}|Du| \le M_1 \quad A \quad M_1 = C_2 \cdot \Phi\left(M, \ \|\varphi\|_{W_p^{2-1/p,1}(\partial Q_T)}, \|\psi\|_{W_p^{2-2/p}(\Omega)}\right)..$$

Since for $|\xi| \le M_1 \le M_2$ the function $F_1 = f$, we finally deduce that the solution $u(x,t)$ of the problem (30) is the solution of the problem (1). Theorem 2 is proved. ◀

5. Maximum principle and condition A.4)

Let $g(x,t,\xi)$ be a function defined on $Q_T \times R^n$ with values in R and satisfying Carathéodory conditions, i.e. let it be measurable in (x,t) for all $\xi \in R^n$, continuous in ξ for almost all$(x,t) \in Q_T$ and such that the function $\sup\limits_{|\xi| \le l} |g(x,t,\xi)|$ belongs to the space $L_p(Q_T)$ with some $p > n+2$ for any fixed $l > 0$.

For an arbitrarily fixed function $\vartheta(x,t)$ from $W_p^{2,1}(Q_T)$, consider the following inequalities for the function $\omega(x,t)$ from $W_p^{2,1}(Q_T)$:

$$\begin{cases} \Delta\omega - \frac{\partial\omega}{\partial t} \geq g(x,t,D\vartheta + D\omega) - g(x,t,D\vartheta), \quad (x,t) \in Q_T, \\ \omega|_{\partial Q_T} \leq 0, \quad x \in \partial\Omega, \ t \in (0,T), \\ \omega|_{t=0} \leq 0, \ \ x \in \Omega. \end{cases} \tag{31}$$

If the function g satisfies only the above conditions, then condition (31) does not imply the inequality

$$\omega(x,t) \leq 0, \quad (x,t) \in Q_T \tag{32}$$

in general.

Moreover, if the Lipschitz condition A.4) is replaced by the Hölder condition of the form

A.5) $|f(x,t,u,\eta) - f(x,t,u,\xi)| \leq b_2(x,t,u,\xi,\eta) \cdot |\eta - \xi|^{\lambda}, \quad 0 < \lambda < 1$ with the same function b_2 from the condition A.4), then, with any index $0 < \lambda < 1$, for the function g the inequality (32) does not follow from (31) in general case.

Remark. For an arbitrarily fixed function $\vartheta(x,t)$ from $W_p^{2,1}(Q_T)$ $(p > n+2)$, consider the following inequalities for the function $\omega(x,t)$ from $W_p^{2,1}(Q_T)$:

$$\begin{cases} \Delta\omega - \frac{\partial\omega}{\partial t} - c(x,t)\,\omega \geq g(x,t,D\vartheta + D\omega) - g(x,t,D\vartheta), \quad (x,t) \in Q_T, \\ \omega|_{\partial Q_T} \leq 0, \quad x \in \partial\Omega, \ t \in (0,T), \\ \omega|_{t=0} \leq 0, \ \ x \in \Omega, \end{cases} \tag{33}$$

with the function $c(x,t)$ from $L_p(Q_T)$ $(p > n+2)$ and $c(x,t) \geq 1$ in Q_T. Then inequalities (33) and the Hölder condition A.5) with λ, $0 < \lambda < \frac{n+1}{p}$, do not imply inequality (32) for the function g in the general case.

6. Some applications

Choosing functions from different "test" sets of functions as upper and lower solutions, one can obtain various existence theorems for solutions of boundary value problems of the form (1).

Let us consider as a "test" set the functions of the form

$$u(x,t) = \tau,$$

where τ is a real number.

Theorem 3. *Let conditions A.1)-A.4) be satisfied with some* $p > n+2$, $\varphi \in W_p^{2-1/p,1}(\partial Q_T)$ *and* $\psi \in W_p^{2-p/2}(\Omega)$. *Let the functions* f, φ *and* ψ *be such that there are two numbers* τ^+ *and* τ^-, $\tau^+ \geq \tau^-$ *which satisfy*

$$\begin{cases} f(x,t,\tau^-,0) \leq 0 \leq f(x,t,\tau^+,0) \ , \ (x,t) \in Q_T, \\ \tau^- \leq \varphi(x,t) \leq \tau^+, \ x \in \partial\Omega, \ t \in (0,T), \\ \tau^- \leq \psi(x,t) \leq \tau^+, \ x \in \Omega. \end{cases}$$

Then there exists a solution $\bar{u}(x,t)$ *of the boundary value problem* (1) *in* $W_p^{2,1}(Q_T)$ *and*

$$\tau^- \leq \bar{u}(x,t) \leq \tau^+, \ (x,t) \in \bar{Q}_T.$$

To prove this theorem, it suffices to apply Theorem 2 with $u^+ = \tau^+$ and $u^- = \tau^-$.

Let us now take the functions of the form

$$u(x,t) = \tau \cdot \frac{|x|^2 + t}{2}$$

as a "test" set, where τ is a real number.

Theorem 4. *Let conditions A.1) -A.4) be satisfied with some* $p > n+2$, $\varphi \in W_p^{2-1/p,1}(\partial Q_T)$ *and* $\psi \in W_p^{2-2/p}(\Omega)$. *Let the functions* f, φ *and* ψ *be such that there are two numbers* τ^+ *and* τ^-, $\tau^+ \geq \tau^-$ *which satisfy*

$$\begin{cases} f\left(x,t,\tau^+ \cdot \frac{|x|^2+t}{2}, \tau^+ \cdot x\right) \geq n\tau^+ - \frac{\tau^+}{2}, & (x,t) \in Q_T, \\ f\left(x,t,\tau^- \cdot \frac{|x|^2+t}{2}, \tau^- \cdot x\right) \leq n\tau^- - \frac{\tau^-}{2}, & (x,t) \in Q_T, \\ \tau^- \cdot \frac{|x|^2+t}{2} \leq \varphi(x,t) \leq \tau^+ \cdot \frac{|x|^2+t}{2}, & (x,t) \in \partial Q_T, \\ \tau^- \cdot \frac{|x|^2}{2} \leq \psi(x) \leq \tau^+ \cdot \frac{|x|^2}{2}, & x \in \Omega. \end{cases}$$

Then there exists a solution $\bar{u}(x,t)$ *of the boundary value problem* (1) *in* $W_p^{2,1}(Q_T)$ *and*

$$\tau^- \cdot \frac{|x|^2 + t}{2} \leq \bar{u}(x,t) \leq \tau^+ \cdot \frac{|x|^2 + t}{2}, \ (x,t) \in Q_T.$$

To prove this theorem, it suffices to apply Theorem 2 with $u^+ = \tau^+ \cdot \frac{|x|^2+t}{2}$ and $u^- = \tau^- \cdot \frac{|x|^2+t}{2}$.

Remark 2. *Similarly, we obtain a theorem on the solvability of the boundary value problem* (1) *if, as a "test" set of functions, we take functions of the form*

$$u(x,t) = \tau \cdot \frac{|x - x_0|^2 + |t - t_0|}{2}$$

with corresponding τ^-, $\tau^+ \in R$ *and* $x_0 \in R^n, t_0 \in R$.

Let us now take the functions of the form

$$u(x,t) = u_0(x,t) + \tau \cdot \vartheta_0(x,t)$$

as a "test" set, where $u_0(x,t)$ is a solution of the problem

$$\begin{cases} \Delta u_0 - \frac{\partial u_0}{\partial t} = 0, \quad (x,t) \in Q_T, \\ u_0|_{\partial Q_T} = \varphi(x,t), \; x \in \partial\Omega, \; t \in (0,T), \\ u|_{t=0} = \psi(x), \;\; x \in \Omega \end{cases}$$

and $\vartheta_0(x,t)$ is some first eigenfunction of the boundary value problem

$$\begin{cases} \Delta \vartheta_0 - \frac{\partial \vartheta_0}{\partial t} + \lambda_1 \vartheta_0 = 0, \quad (x,t) \in Q_T, \\ \vartheta_0|_{\partial Q_T} = 0, \;\; (x,t) \in \partial\Omega \times (0,T), \\ \vartheta_0|_{t=0} = 0, \;\; x \in \Omega \end{cases}$$

with $\vartheta_0(x,t) > 0$ in Q_T, τ is a real number, and λ_1 is an eigenvalue.

Theorem 5. *Let conditions A.1) -A.4) be satisfied with some* $p > n+2$, $\varphi \in W_p^{2-1/p,1}(\partial Q_T)$ *and* $\psi \in W_p^{2-2/p}(\Omega)$. *Let the functions* f, φ *and* ψ *be such that there are two numbers* τ^+ *and* τ^-, $\tau^+ \geq \tau^-$ *which satisfy*

$$f(x,t,u_0 + \tau^+\vartheta_0, Du_0 + \tau^+ D\vartheta_0) + \lambda_1 \tau^+ \cdot \vartheta_0 \geq 0, \quad (x,t) \in Q_T,$$

$$f(x,t,u_0 + \tau^-\vartheta_0, Du_0 + \tau^- D\vartheta_0) + \lambda_1 \tau^- \cdot \vartheta_0 \leq 0, \quad (x,t) \in Q_T.$$

Then there exists a solution $\bar{u}(x,t)$ *of the boundary value problem* (1) *from the space* $W_p^{2,1}(Q_T)$ *and*

$$u_0 + \tau^- \cdot \vartheta_0 \leq \bar{u}(x,t) \leq u_0 + \tau^+ \cdot \vartheta_0, \quad (x,t) \in Q_T.$$

To prove this theorem, it suffices to apply Theorem 2 with $u^+ = u_0 + \tau^+ \cdot \vartheta_0$ and $u^- = u_0 + \tau^- \cdot \vartheta_0$.

Remark 3. *Using the change $\vartheta(x,t) \to u(x,t)$, defined by the relation $u = S(x,t,\vartheta(x,t))$ with a smooth function S, we can reduce finding the upper u^+ and the lower u^- solutions of the boundary value problem* (1) *to finding the upper ϑ^+ and the lower ϑ^- solutions of another boundary value problem induced by the function S.*

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