Azerbaijan Journal of Mathematics
V. 13, No 2, 2023, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2023.2.159

Asymptotic Solutions of MHD Boundary Layer Equations Leading the Flow of Viscous Fluid Due to a Stretching Surface

U.M. Muchale, I. M. Chandarki*, B.B. Singh

Abstract. Asymptotic solutions of the boundary layer equations leading the flow of viscous fluid caused by stretching surface; outcomes are based on the asymptotic integration of second order linear differential equations. Also proved that the asymptotic formulae will exhibit non-oscillatory behaviour. These asymptotic solutions will always be bounded in nature.

Key Words and Phrases: boundary layer, MHD flow, asymptotic integration, asymptotic solutions.

2010 Mathematics Subject Classifications: 35B40,76D03, 76D10, 34B15

1. Introduction

Flows with viscous property on a sheet with stretching has vital industrial applications, for example, in metallurgical processes.

Also many applications are found in the field of engineering processes for example polymer extrusion and continuous casting. In view of these applications the works done by Anderson et al. [3] and Pop and Na [13] can be referred. As far as industries are concerned the magneto hydrodynamic (MHD) flow problems have become more popular in recent years.

The primary efforts to investigate MHD flow over a wall with stretching in fluid with electricity is reported by Pavlov [12]. Following Pavlov [12] the work in this regard are due to Chakrabarti and Gupta [5], Vajravelu [22], Takhar et al. [20], Kumari et al.[10], Anderson et al. [8] Vajravelu and Rollins [23], Anderson [1, 2], Watanabe and Pop [24], Lawrence and Rao [11], etc. Effect of the transfer of mass for the MHD flow over a stretching permeable surface subject

*Corresponding author.

to suction/injection has slight acknowledgment so far [5, 22, 23]. These studies are limited only to relative low values of mass transfer parameter, K (say) and the extension of the problem to include large values of K remains almost incomplete. This forms the subject matter of the present paper.

2. Basic equations

Let us consider the flow with electricity conducting incompressible fluid (with electrical conductivity σ) over a penetrable wall overlapping with the plane $y = 0$, the flow being restricted to $y > 0$. Two forces which are equal and opposite are introduced along the X-axis so that the wall is stretched keeping the origin fixed, and a uniform magnetic field B_0 is imposed along the Y-axis. The basic boundary layer equations for the stretching flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \vartheta\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho}, \tag{2}$$

where u is a velocity along X-axis, whereas v is the velocity component along Y-axis, and ρ and ϑ are the density and kinematic viscosity of the fluid, respectively. The boundary conditions to the problem are

$$u = \alpha x, \quad v = -V_0 \quad at \quad y = 0; \;\; u \to 0 \;\; at \;\; y \to \infty, \tag{3}$$

where $\alpha > 0$, and V_0 is the velocity of suction ($V_0 > 0$) or injection ($V_0 < 0$), respectively.

Defining the variables

$$u = \alpha x f'(x), \quad v = -(\alpha\vartheta)^{\frac{1}{2}} f(t)\,, \quad t = \left(\frac{\alpha}{\vartheta}\right)^{\frac{1}{2}} y \tag{4}$$

and substituting them into equation (2), we get the following ordinary differential equation:

$$f''' + ff'' - f'^2 - Mf' = 0, \tag{5}$$

subject to the boundary conditions

$$f(0) = S, \quad f'(0) = 1, \;\; f'(\infty) = 0, \tag{6}$$

where the prime denotes differentiation w. r. t. t. Here $M = \frac{\sigma B_0^2}{a\rho}$ is the magnetic parameter and $S = \frac{V_0}{(a\vartheta)^{\frac{1}{2}}}$ is the suction ($S > 0$) or injection ($K < 0$) parameter.

Aim in this work is to discover the asymptotic solutions of equations (1)-(2) as the independent variable 't' tends to infinity. The asymptotic formulae will be calculated based on the asymptotic integration of second order linear differential equations. It has also been shown that the asymptotic formulae will be non oscillatory and will be bounded in nature. The study of the asymptotic behaviours of boundary layer equations have been the subject of considerable research and hence have been dealt with by authors like Hartman [7], Serrin [14], Singh [15, 16], Singh and Singh [18], Singh and Kumar [17], Kumar and Singh [9], Chinquing et el. [6], Singh and Verma [19], Tiryaki and Yaman [21], etc.

3. Asymptotic integrations

The asymptotic nature of equations (5)-(6) can be studied with the help of following lemmas of Brighi [4].

Lemma 1. *Let $f = f(t)$ be a solution of the equations (5)-(6). Then we have $f''' \geq -ff''$ in such a way that if $f < 0$ at infinity, we deduce that there exists $t_1 \geq t_0$ such that necessarily $f'' < 0$ and $f''' > 0$ on $[t_1, \infty)$. So, f is bounded.*

Proof. If now $f > 0$ at infinity and unbounded, then $f(t) \to \infty$ as $t \to \infty$, and there exists $t_1 \geq t_0$ such that $f''' < 0$ and $f > 1$ on $[t_1, \infty)$.

Therefore, $f'''(t) \geq -f''(t)$ for $t \geq t_1$, and by integrating between $s \geq t_1$ and ∞, we obtain $-f''(s) \geq f'(s)$. Integrating next between t_1 and $t \geq t_1$, we get $f'(t_1) - f'(t) \geq f(t) - f(t_1)$ and a contradiction by passing to the limit as $t \to \infty$. ◀

Lemma 2. $f''(\infty) = 0\ \forall\ M \in \mathbb{R}+$ *and there exists a sequence* $\{t_n\}_0^\infty$ *such that* $\lim_{n\to\infty} f'''(t_n) = \lim_{n\to\infty} f(t_n)f''(t_n) = 0$.

Proof. $\exists\ \{x_n\}_0^\infty$ since $f'(\infty) = 0$ satisfying $f''(x_n) \to 0$ (here x_n is such that $f''(x_n) = f'(n+1) - f'(n)$). Now, multiplying equation (5) by f'' and then integrating, we obtain

$$\frac{1}{2}f''(t)^2 - \frac{1}{2}f''(0)^2 - \frac{M}{2}\left\{f'(t)^2 - f'(0)^2\right\} + \frac{1}{3}f'(t)^3 = -\int_0^t f(\xi)f''(\xi)d\xi \quad (7)$$

$\forall\ t$.

However, f remains positive or negative for large t, the function $t \to \int_0^t f(\xi)f''(\xi)d\xi$ has a limit as $t \to \infty$ and we deduce from (7) that $f''(\infty)$ exists. Then $f''(\infty) = 0$ holds. ◀

From equations (5)

$$h'' + fh' - (M + f')h = 0, \tag{8}$$

where

$$f' = h. \tag{9}$$

To eliminate the middle term in (8), let us put

$$h = X \exp\left(-\int_0^t f(s)ds\right) \tag{10}$$

in (8), to get

$$X'' - qX = 0, \tag{11}$$

where

$$q = M + \frac{3}{2}f' + \frac{1}{4}f^2. \tag{12}$$

Let us assume that $\lambda = f(\infty) > 0$. Hence, from (12),

$$q \sim \frac{1}{4}\lambda^2 \quad (as\ \ t \to \infty) \tag{13}$$

and it is easy to verify that the integrals

$$\int^{\infty} q''(s)q(s)^{\frac{-3}{2}}ds \quad and \quad \int^{\infty} q'(s)^2 q(s)^{\frac{-3}{2}}ds$$

converge.

Therefore (see [14]), equation (11) has the fundamental system of solutions $\{X_1, X_2\}$ such that

$$\begin{aligned} X_1(\eta) &\sim q(t)^{\frac{-1}{4}} \exp\left(-\textstyle\int_0^t \sqrt{q(s)}ds\right) \\ X_2(\eta) &\sim q(t)^{\frac{-1}{4}} \exp\left(\textstyle\int_0^t \sqrt{q(s)}ds\right) \qquad (t \to \infty). \end{aligned}$$

Then, in view of (13), we get

$$\begin{aligned} X_1(t) &\sim \exp\left(-\tfrac{\lambda}{2}t\right) \\ X_2(t) &\sim \exp\left(\tfrac{\lambda}{2}t\right) \qquad (t \to \infty). \end{aligned} \tag{14}$$

From (10),

$$\begin{aligned} h &\sim C_1 \exp\left(-\tfrac{\lambda}{2}t\right) \\ h &\sim C_2 \exp\left(o(t)\right) \qquad (t \to \infty). \end{aligned} \tag{15}$$

Using (9), (14) in (15), we have

$$f'(t) \sim C_1 \exp(-\lambda t) \qquad (t \to \infty), \tag{16}$$

$$f'(t) \sim C_2 \exp(ot) \qquad (t \to \infty) \tag{17}$$

for some $C_1 \in R^+$.

In addition, integrating (5) between t and ∞, we get

$$f''(t) + f(t)f'(t) = -2\int_t^\infty f'^2 ds - M[\lambda - f(t)],$$

which implies that

$$f''(t) \sim -\lambda C_1 \exp(-\lambda t) \qquad (t \to \infty). \tag{18}$$

4. Conclusion

From the above discussions, it is obvious that the solution (16) will exhibit asymptotic behaviour as $t \to \infty$, where as (17) will not. Similarly, (18) will also show asymptotic behaviour as $t \to \infty$.

Also, the solution (18) will be concave at infinity. And the solutions (16), (18) will show non-oscillating nature. This can be proved in the following manner:

we have

$$f(t) \sim \lambda\ , \quad f'(t) \sim -C_1 \exp(-\lambda t), \quad f''(t) \sim\ C_2 \exp(-\lambda t) \qquad (t \to \infty),$$

where $C_1, C_2 > 0$.

From (5) we get

$$f''' \sim -C_3 \exp(-\lambda t) \ \ as \qquad (t \to \infty)\,,$$

and easily, by induction, we see that for all $k \geq 1$, there is a constant $C_k > 0$ such that

$$f^{(k)}(t) \sim (-1)^k\, C_k \exp(-\lambda t) \ \ as \qquad (t \to \infty)\,.$$

Therefore, f is non-oscillating.

Acknowledgment

This research work has been carried out of the grants received from Punyashlok Ahilyadevi Holkar Solapur University, Solapur under Seed Money for Research grant (Ref. No PAHSUS/ARAD-TA-2/2021-22/9045 dated December 13th, 2021). The corresponding author is thankful to the funding Institution.

References

[1] H.I. Andersson, *MHD flow of a viscoelastic fluid past a stretching surface*, Acta Mechanica, **95(1)**, 1992, 227–230.

[2] H.I. Andersson, *An exact solution of the Navier-Stokes equations for magnetohydrodynamic flow*, Acta Mechanica, **113(1)**, 1995, 241–244.

[3] H.I. Andersson, O.R. Hansen, B. Holmedal, *Diffusion of a chemically reactive species from a stretching sheet*, International Journal of Heat and Mass Transfer, **37(4)**, 1994, 659–664.

[4] B. Brighi, *On a similarity boundary layer equation*, Zeitschrift fur Analysis und ihre Anwendungen, **21(4)**, 2002, 931–948.

[5] A. Chakrabarti, A.S. Gupta, *Hydromagnetic flow and heat transfer over a stretching sheet*, Quart. Appl. Math., **37**, 1979, 73–78.

[6] L. Chunquing, A.D. Macgillvray, S.P. Hastings, *Asymptotic behaviour of solution of a similarity equation for laminar flows in channels with porous walls*, IMA J. Appl. Math., **49**, 1992, 139–162.

[7] P. Hartman, *On the asymptotic behaviours of solutions of a differential equations in boundary layer theory*, Z. Angew. Math. Mech.(ZAMM), **44**, 1964, 123–128.

[8] H.I. Andersson, K.H. Bech, B.S. Dandapat, *Magneto-hydrodynamic flow of a power-low fluid over a stretching sheet*, Int. J. Non-linear Mech., **27(1)**, 1992, 929–936.

[9] A. Kumar, B.B. Singh, *Some aspects related to asymptotic solutions of the laminar boundary layer equations with heat flux*, Astrophysics and Space Science, **165**, 1990, 41–49.

[10] M. Kumari, H.S. Takhar, G. Nath, *MHD flow and heat transfer over a stretching surface with prescribed wall temperature or heat flux*, Heat and mass transfer, **25(6)**, 1990, 331–336.

[11] P.S. Lawrence, B.N. Rao, *Magnetohydrodynamic flow past a semi-infinite moving plate*, Acta Mechanica, **117(1)**, 1996, 159–164.

[12] K.B. Pavlov, *Magnetohydrodynamic flow of an incompressible viscous fluid caused by deformation of a plane surface*, Magnitnaya Gidrodinamika, **4**, 1974, 146–147.

[13] I. Pop, T.-Y. Na, *A note on MHD flow over a stretching permeable surface*, Mechanics Research Communications, **25(3)**, 1998, 263–269.

[14] J. Serrin, *Asymptotic behaviour of velocity profiles in the Prandtl boundary layer theory*, Proc. Roy. Soc., Ser. A, **299**, 1967, 491–507.

[15] B.B. Singh, *Asymptotic behaviour of similar profiles*, Indian J. Pure Appl. Math., **17**, 1986, 1067–1072.

[16] B.B. Singh, *Asymptotic behaviour and uniqueness of solutions of the boundary layer equations in an incompressible flow*, J. Sci. Res. (BHU), **37**, 1987, 37–44.

[17] B.B. Singh, A. Kumar, *Asymptotic behaviours of solutions of MFD boundary layer flow on a moving continuous surface with heat flux*, Hunan Daxue Xuebao, **16**, 1989, 12–16.

[18] B.B. Singh, A.K. Singh, *Asymptotic behaviour, existence and uniqueness of the solutions of boundary layer flow equations at a three-dimensional stagnation point with strong blowing*, Indian J. Techno., **26**, 1988, 27–33.

[19] B.B. Singh, S.K. Verma, *Asymptotic behaviour of the laminar boundary layer flow of an incompressible fluid past a flat plate*, Indian J. Engrg. Materials Sci., **6(6)**, 1999, 346–348.

[20] H.S. Takhar, C.D.S. Devi, G. Nath, *MHD flow with heat and mass transfer due to a point sink*, Indian J. Pure and Appl. Math., **17(10)**, 1986, 1242–1247.

[21] A. Tiryaki, S. Yaman, *Asymptotic behaviour of a class on non-linear functional differential equations of third order*, Appl. Math. Lett., **14**, 2001, 327–332.

[22] K. Vajravelu, *Hydromagnetic flow and heat transfer over a continuous, moving, porous, flat surface*, Acta Mechanica, **64(3)**, 1986, 179–185.

[23] K. Vajravelu, K. Rollins, *Hydromagnetic flow in an oscillatory channel*, J. Math. Phys. Sci., **31**, 1997, 11–24.

[24] T. Watanabe, I. Pop, *Hall effects on magnetohydrodynamic boundary layer flow over a continuous moving flat plate*, Acta Mechanica, **108(1)**, 1995, 35–47.

Uzma M. Muchale
Department of Mathematics, PAH Solapur University, Solapur-413255, Solapur (M.S), India.
E-mail: ummuchale15@sus.ac.in

Imran M. Chandarki
Department of Mathematics, N B Navale Sinhgad CoE,Solapur-413255(M.S), India.
E-mail: imchandarki.nbnscoe@sinhgad.edu

Brijbhan B. Singh
Department of Mathematics, Dr. Babasaheb Ambedkar Technological University, Lonere-402103, Raigad (M.S)India.
E-mail: bbsingh@dbatu.ac.in

Received 19 February 2023
Accepted 18 April 2023