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Parabolic Nonsingular Integral Operator and Its Commutators on Weighted Orlicz Spaces

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Abstract. We obtain the sufficient conditions for the boundedness of the parabolic nonsingular integral operator $\mathcal{R}$ and its commutators $[b, \mathcal{R}]$ on weighted Orlicz spaces $L^{\Phi}_{w}(\mathbb{D}^{n+1}_{+})$ with the weight function w belonging to Muckenhoupt class A_{i_Φ}.

Key Words and Phrases: weighted Orlicz space, parabolic nonsingular integral, commutator, BMO.

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1. Introduction

Sobolev spaces, which are sets of functions with a certain degree of smoothness, are commonly used and studied in a wide variety of fields of mathematics, and have turned out to be one of the most powerful tools in analysis created in the 20th Century. Since the 1960s, the need to use wider spaces of functions than Sobolev spaces came from various practical problems. Orlicz spaces have been studied as the generalization of Sobolev spaces since they were introduced by Orlicz [17, 18] (see [10, 11, 20]). The theory of Orlicz spaces plays a crucial role in many fields of mathematics including geometric, probability, stochastic, Fourier analyses and PDE (see [20]).

Throughout this paper the following notations will be used:

$$x = (x', t), y = (y', \tau) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R},\ \mathbb{R}^{n+1}_{+} = \mathbb{R}^n \times \mathbb{R}_{+};$$

$$x = (x'', x_n, t) \in \mathbb{D}^{n+1}_{+} = \mathbb{R}^{n-1} \times \mathbb{R}_{+} \times \mathbb{R}_{+},\ \mathbb{D}^{n+1}_{-} = \mathbb{R}^{n-1} \times \mathbb{R}_{-} \times \mathbb{R}_{+};$$

for any $f \in L^p_w(A)$, $A \subset \mathbb{R}^{n+1}$

$$\|f\|_{L^p_w(A)} = \left(\int_A |f(y)|^p w(y) dy\right)^{1/p}.$$

In the sequel, along with the standard parabolic metric $\varrho(x) = \max(|x'|, |t|^{1/2})$, we will use the equivalent one $\rho(x) = \left(\frac{|x'|^2+\sqrt{|x'|^4+4t^2}}{2}\right)^{1/2}$ introduced by Fabes and Riviére in [3]. The topology induced by this metric consists of ellipsoids

$$\mathcal{E}_r(x) = \left\{y \in \mathbb{R}^{n+1} : \frac{|x'-y'|^2}{r^2} + \frac{|t-\tau|^2}{r^4} < 1\right\}, \ |\mathcal{E}_r| = Cr^{n+2}.$$

Let $x = (x', t) = (x'', x_n, t) \in \mathbb{D}_+^{n+1}$. For any $x \in \mathbb{D}_+^{n+1}$ define $\widetilde{x} = (x'', -x_n, t) \in \mathbb{D}_-^{n+1}$ and let $x^0 = (x'', 0, 0) \in \mathbb{R}^{n-1}$. Consider the semi-ellipsoids $\mathcal{E}_r^+(x^0) = \mathcal{E}_r(x^0) \cap \mathbb{D}_+^{n+1}$. Now we define the parabolic nonsingular integral (see [2]) by

$$\mathcal{R}f(x) = \int_{\mathbb{D}_+^{n+1}} \frac{|f(y)|}{\rho(\widetilde{x}-y)^{n+2}}\, dy. \tag{1}$$

The commutators generated by $b \in L^1_{\rm loc}(\mathbb{D}_+^{n+1})$ and the operator $\mathcal{R}$ are defined by

$$[b, \mathcal{R}]f(x) = \int_{\mathbb{D}_+^{n+1}} \frac{b(x)-b(y)}{\rho(\widetilde{x}-y)^{n+2}}\, f(y)\, dy.$$

In [14, 15] we have studied the boundedness of the nonsingular integral operator on weighted Orlicz spaces. Quite recently, we have also studied [16] the boundedness of the parabolic nonsingular integral operator in Orlicz spaces.

In this work we deal with the boundedness of parabolic nonsingular integral operator $\mathcal{R}$ (Theorem 2) and its commutator $[b, \mathcal{R}]$ (Theorem 8) in weighted Orlicz spaces $L_w^\Phi(\mathbb{D}_+^{n+1})$.

The standard summation convention on repeated upper and lower indices is adopted. The letter C is used for various positive constants which can be different in different occasions may change from one occurrence to another.

2. Definitions and preliminary results

Even though the A_p class is well known, for completeness, we state here the definition of A_p weight functions. Hereinafter, $\mathcal{E}(x, r)$ is the ellipsoid in $\mathbb{R}^n$ of radius r centered at x and $|\mathcal{E}(x, r)| = v_n r^{n+2}$ will be its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let $\mathbb{E} = \{\mathcal{E}(x, r) : x \in \mathbb{R}^n, \ r > 0\}$.

Definition 1. *For $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^{n+1} \to [0, \infty)$ is said to be an A_p weight if*

$$\sup_{\mathcal{E} \in \mathbb{E}} \left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} w(x)dx\right) \left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} w(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty.$$

A locally integrable function $w:\mathbb{R}^n\to[0,\infty)$ *is said to be an* A_1 *weight if*

$$\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} w(y)dy \le Cw(x), \qquad a.e.\ \ x\in\mathcal{E}$$

for some constant $C>0$ *. We define* $A_\infty=\bigcup_{p\ge1}A_p$ *.*

For any $w\in A_\infty$ and any Lebesgue measurable set A, we write $w(A)=\int_A w(x)dx$.

We recall the definition of Young functions.

Definition 2. *A function* $\Phi:[0,\infty)\to[0,\infty]$ *is called a Young function if* Φ *is convex, left-continuous,* $\lim\limits_{r\to0^+}\Phi(r)=\Phi(0)=0$ *and* $\lim\limits_{r\to\infty}\Phi(r)=\infty$ *.*

The convexity and the condition $\Phi(0)=0$ provide that any Young function is increasing. In particular, if there exists $s\in(0,\infty)$ such that $\Phi(s)=\infty$, then it follows that $\Phi(r)=\infty$ for $r\ge s$.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0<\Phi(r)<\infty \qquad \text{for} \qquad 0<r<\infty.$$

If $\Phi\in\mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0\le s\le\infty$, let

$$\Phi^{-1}(s)\equiv\inf\{r\ge0:\Phi(r)>s\} \qquad (\inf\emptyset=\infty).$$

A Young function Φ is said to satisfy the Δ_2 condition, denoted by $\Phi\in\Delta_2$, if

$$\Phi(2r)\le k\Phi(r), \qquad r>0$$

for some $k>1$. If $\Phi\in\Delta_2$, then $\Phi\in\mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2 condition, denoted by $\Phi\in\nabla_2$, if

$$\Phi(r)\le\frac{1}{2k}\Phi(kr), \qquad r\ge0$$

for some $k>1$.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r)\equiv\begin{cases}\sup\{rs-\Phi(s):s\in[0,\infty)\} & \text{if} \quad r\in[0,\infty),\\ \infty & \text{if} \quad r=\infty.\end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and it satisfies $\widetilde{\widetilde{\Phi}}=\Phi$. Note that $\Phi\in\nabla_2$ if and only if $\widetilde{\Phi}\in\Delta_2$.

It is also known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r, \qquad r \geq 0. \tag{2}$$

We recall an important pair of indices used for Young functions. For any Young function Φ , write

$$h_\Phi(t) = \sup_{s>0} \frac{\Phi(st)}{\Phi(s)}, \quad t > 0.$$

The lower and upper dilation indices of Φ are defined by

$$i_\Phi = \lim_{t\to 0^+} \frac{\log h_\Phi(t)}{\log t} \text{ and } I_\Phi = \lim_{t\to\infty} \frac{\log h_\Phi(t)}{\log t},$$

respectively.

Lemma 1. *[10, Lemma 1.3.2] Let* $\Phi \in \Delta_2$ *. Then there exist* $p > 1$ *and* $b > 1$ *such that*

$$\frac{\Phi(t_2)}{t_2^p} \leq \frac{b\Phi(t_1)}{t_1^p}$$

for $0 < t_1 < t_2$ *.*

Lemma 2. *[22, Proposition 62.20] Let* Φ *be a Young function with canonical representation*

$$\Phi(t) = \int_0^t \varphi(s)ds, \quad t \geq 0.$$

(1) Assume that $\Phi \in \Delta_2$ *. More precisely* $\Phi(2t) \leq A\Phi(t)$ *for some* $A \geq 2$ *. If* $p > 1 + \log_2 A$ *, then*

$$\int_t^\infty \frac{\varphi(s)}{s^p} ds \lesssim \frac{\Phi(t)}{t^p}, \quad t > 0.$$

(2) Assume that $\Phi \in \nabla_2$ *. Then*

$$\int_0^t \frac{\varphi(s)}{s} ds \lesssim \frac{\Phi(t)}{t}, \quad t > 0.$$

Definition 3. *For a Young function* Φ *and* $w \in A_\infty$ *, the set*

$$L_w^\Phi(\mathbb{R}^{n+1}) \equiv \left\{ f - \textit{measurable}: \int_{\mathbb{R}^{n+1}} \Phi(k|f(x)|)w(x)dx < \infty \textit{ for some } k > 0 \right\}$$

is called the weighted Orlicz space. The local weighted Orlicz space $L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^{n+1})$ *is defined as the set of all functions* f *such that* $f\chi_{\mathcal{E}} \in L_w^\Phi(\mathbb{R}^{n+1})$ *for all ellipsoids* $\mathcal{E} \subset \mathbb{R}^{n+1}$ *.*

Note that $L_w^\Phi(\mathbb{R}^{n+1})$ is a Banach space with respect to the norm

$$\|f\|_{L_w^\Phi(\mathbb{R}^{n+1})} \equiv \|f\|_{L_w^\Phi} = \inf\left\{\lambda>0: \int_{\mathbb{R}^{n+1}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x)dx \le 1\right\}$$

and

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_w^\Phi}}\Big) w(x)dx \le 1. \tag{3}$$

The following analogue of the Hölder inequality is known:

$$\left|\int_{\mathbb{R}^{n+1}} f(x)g(x)w(x)dx\right| \le 2\|f\|_{L_w^\Phi}\|g\|_{L_w^{\tilde{\Phi}}}. \tag{4}$$

For the proofs of (2) and (4), see, for example, [20].

For a weight w, a measurable function f and $t>0$, let

$$m(w,\ f,\ t) = w(\{x\in\mathbb{R}^{n+1}: |f(x)|>t\}).$$

Definition 4. *The weak weighted Orlicz space*

$$WL_w^\Phi(\mathbb{R}^{n+1}) = \{f - \textit{measurable}: \|f\|_{WL_w^\Phi} < \infty\}$$

is defined by the norm

$$\|f\|_{WL_w^\Phi(\mathbb{R}^{n+1})} \equiv \|f\|_{WL_w^\Phi} = \inf\left\{\lambda>0\ :\ \sup_{t>0}\Phi(t)m\Big(w,\ \frac{f}{\lambda},\ t\Big)\ \le 1\right\}.$$

We can prove the following by a direct calculation:

$$\|\chi_{\mathcal{E}}\|_{L_w^\Phi} = \|\chi_{\mathcal{E}}\|_{WL_w^\Phi} = \frac{1}{\Phi^{-1}\left(w(\mathcal{E})^{-1}\right)}, \quad \mathcal{E}\in\mathbb{E}, \tag{5}$$

where $\chi_{\mathcal{E}}$ denotes the characteristic function of $\mathcal{E}$.

3. Parabolic nonsingular integral operator in the weighted Orlicz space $L_w^\Phi(\mathbb{D}_+^{n+1})$

The following theorem is valid (see, for example, [10, 19]).

Theorem 1. *Let $\mathcal{R}$ be a parabolic nonsingular integral operator, defined by* (1), *$f\in L_w^p(\mathbb{D}_+^{n+1})$, $1\le p<\infty$ and $w\in A_p$. Then there exists a constant C independent of f such that*

$$\|\mathcal{R}f\|_{L_w^p(\mathbb{D}_+^{n+1})} \le C_p\|f\|_{L_w^p(\mathbb{D}_+^{n+1})}, \quad 1<p<\infty$$

and

$$\|\mathcal{R}f\|_{WL_w^1(\mathbb{D}_+^{n+1})} \le C_1\|f\|_{L_w^1(\mathbb{D}_+^{n+1})}.$$

The boundedness result for the parabolic nonsingular integral operator on weighted Orlicz spaces is given in the following theorem.

Theorem 2. *Let* Φ *be a Young function,* $w \in A_{i_\Phi}$ *and* $\mathcal{R}$ *be a parabolic nonsingular integral operator, defined by* (1)*. If* $\Phi \in \Delta_2 \bigcap \nabla_2$ *, then the operator* $\mathcal{R}$ *is bounded on* $L^\Phi_w(\mathbb{D}^{n+1}_+)$ *and if* $\Phi \in \Delta_2$ *, then the operator* $\mathcal{R}$ *is bounded from* $L^\Phi_w(\mathbb{D}^{n+1}_+)$ *to* $WL^\Phi_w(\mathbb{D}^{n+1}_+)$ *.*

Proof. Let us first prove that for $\Phi \in \Delta_2$ the parabolic nonsingular integral operator $\mathcal{R}$ is bounded from $L^\Phi_w(\mathbb{D}^{n+1}_+)$ to $WL^\Phi_w(\mathbb{D}^{n+1}_+)$.

We take $f \in L^\Phi_w(\mathbb{D}^{n+1}_+)$ satisfying $\|f\|_{L^\Phi_w} = 1$. Fix $\lambda > 0$ and define $f_1 = \chi_{\{|f|>\lambda\}} \cdot f$ and $f_2 = \chi_{\{|f|\le\lambda\}} \cdot f$. Then $f = f_1 + f_2$. We have

$$w\big(\{x \in \mathbb{D}^{n+1}_+ : \big|\mathcal{R}f(x)\big| > \lambda\}\big) \le w\big(\big\{x \in \mathbb{D}^{n+1}_+ : \big|\mathcal{R}f_1(x)\big| > \frac{\lambda}{2}\}\big) \\ + w\big(\big\{x \in \mathbb{D}^{n+1}_+ : \big|\mathcal{R}f_2(x)\big| > \frac{\lambda}{2}\}\big)$$

and

$$\Phi(\lambda) w\big(\{x \in \mathbb{D}^{n+1}_+ : \big|\mathcal{R}f(x)\big| > \lambda\}\big) \\ \le \Phi(\lambda) w\big(\big\{x \in \mathbb{D}^{n+1}_+ : \big|\mathcal{R}f_1(x)\big| > \frac{\lambda}{2}\}\big) + \Phi(\lambda) w\big(\big\{x \in \mathbb{D}^{n+1}_+ : \big|\mathcal{R}f_2(x)\big| > \frac{\lambda}{2}\}\big).$$

We know that from the weighted weak (1,1) boundedness and the weighted L^p , $p \in (1,\infty)$ boundedness of $\mathcal{R}$ it follows

$$w\big(\{x \in \mathbb{D}^{n+1}_+ : \big|\mathcal{R}(\chi_{\{|f|>\lambda\}} \cdot f)(x)\big| > \lambda\}\big) \lesssim \frac{1}{\lambda} \int_{\{x \in \mathbb{D}^{n+1}_+ : |f(x)|>\lambda\}} |f(x)| w(x) dx$$

and

$$w\big(\{x \in \mathbb{D}^{n+1}_+ : \big|\mathcal{R}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)\big| > \lambda\}\big) \lesssim \frac{1}{\lambda^p} \int_{\{x \in \mathbb{D}^{n+1}_+ : |f(x)|\le\lambda\}} |f(x)|^p w(x) dx.$$

Since $f_1 \in WL^1_w(\mathbb{D}^{n+1}_+)$ and $\frac{\Phi(\lambda)}{\lambda}$ is increasing, we have

$$\Phi(\lambda) w\big(\big\{x \in \mathbb{D}^{n+1}_+ : |\mathcal{R}f_1(x)| > \frac{\lambda}{2}\}\big) \lesssim \frac{\Phi(\lambda)}{\lambda} \int_{\mathbb{D}^{n+1}_+} |f_1(x)| w(x) dx \\ = \frac{\Phi(\lambda)}{\lambda} \int_{\{x \in \mathbb{D}^{n+1}_+ : |f(x)|>\lambda\}} |f(x)| w(x) dx \\ \lesssim \int_{\mathbb{D}^{n+1}_+} |f(x)| \frac{\Phi(|f(x)|)}{|f(x)|} w(x) dx$$

$$= \int_{\mathbb{D}_+^{n+1}} \Phi(|f(x)|)w(x)dx.$$

By Lemma 1 and $f_2 \in L_w^p(\mathbb{D}_+^{n+1})$ we have

$$\Phi(\lambda)\, w\big(\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}f_2(x)| > \frac{\lambda}{2}\}\big) \lesssim \frac{\Phi(\lambda)}{\lambda^p} \int_{\mathbb{D}_+^{n+1}} |f_2(x)|^p w(x)dx$$

$$= \frac{\Phi(\lambda)}{\lambda^p} \int_{\{x\in\mathbb{D}_+^{n+1}:|f(x)|\le\lambda\}} |f(x)|^p w(x)dx$$

$$\lesssim \int_{\mathbb{D}_+^{n+1}} |f(x)|^p \frac{\Phi(|f(x)|)}{|f(x)|^p} w(x)dx = \int_{\mathbb{D}_+^{n+1}} \Phi(|f(x)|)w(x)dx.$$

Thus we get

$$w\big(\{x \in \mathbb{D}_+^{n+1} : \big|\mathcal{R}f(x)\big| > \lambda\}\big) \le \frac{C}{\Phi(\lambda)} \int_{\mathbb{D}_+^{n+1}} \Phi(|f(x)|)w(x)dx$$

$$\le \frac{1}{\Phi\left(\frac{\lambda}{C\|f\|_{L_w^\Phi}}\right)}.$$

Since $\|\cdot\|_{L_w^\Phi}$ norm is homogeneous, this inequality is true for every $f \in L_w^\Phi(\mathbb{D}_+^{n+1})$.

Now let us prove that for $\Phi \in \Delta_2 \bigcap \nabla_2$ the nonsingular integral operator $\mathcal{R}$ is bounded in $L_w^\Phi(\mathbb{D}_+^{n+1})$.

Using the distribution functions, we have

$$\int_{\mathbb{D}_+^{n+1}} \Phi\left(\frac{|\mathcal{R}f(x)|}{\Lambda}\right) w(x)dx = \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{\lambda}{\Lambda}\right) w\big(\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}f(x)| > \lambda\}\big)d\lambda$$

$$= \frac{2}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right) w\big(\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}f(x)| > 2\lambda\}\big)d\lambda.$$

The following inequality is valid:

$$w\big(\{x \in \mathbb{D}_+^{n+1} : \big|\mathcal{R}f(x)\big| > 2\lambda\}\big) \le w\big(\{x \in \mathbb{D}_+^{n+1} : \big|\mathcal{R}(\chi_{\{|f|>\lambda\}} \cdot f)\big| > \lambda\}\big)$$

$$+\, w\big(\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)| > \lambda\}\big).$$

Let $p > 1$ be sufficiently large. The weighted weak $(1,1)$ boundedness and the weighted L^p -boundedness of $\mathcal{R}$ (see Theorem 1) give us

$$w\big(\{x \in \mathbb{D}_+^{n+1} : \big|\mathcal{R}(\chi_{\{|f|>\lambda\}} \cdot f)(x)\big| > \lambda\}\big) \lesssim \frac{1}{\lambda} \int_{\{x\in\mathbb{D}_+^{n+1}:|f(x)|>\lambda\}} |f(x)|w(x)dx$$

and

$$w\big(\{x\in\mathbb{D}_+^{n+1} : \big|\mathcal{R}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)\big)>\lambda\}\big|\lesssim\frac{1}{\lambda^p}\int_{\{x\in\mathbb{D}_+^{n+1}:|f(x)|\le\lambda\}}|f(x)|^p w(x)dx.$$

The same calculation as we used for the maximal operator works for the first term to obtain

$$\frac{1}{\Lambda}\int_0^\infty\varphi\left(\frac{2\lambda}{\Lambda}\right)w\big(\{x\in\mathbb{D}_+^{n+1} : |\mathcal{R}(\chi_{\{|f|>\lambda\}}\cdot f)(x)|>\lambda\}\big)d\lambda$$
$$\le\int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{c|f(x)|}{\Lambda}\right)w(x)dx. \tag{6}$$

As for the second term, a similar computation still works, but we use the fact that $\Phi\in\Delta_2$:

$$\frac{1}{\Lambda}\int_0^\infty\varphi\left(\frac{2\lambda}{\Lambda}\right)w\big(\{x\in\mathbb{D}_+^{n+1} : \big|\mathcal{R}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)\big|>\lambda\}\big)d\lambda$$
$$\lesssim\frac{1}{\Lambda}\int_0^\infty\varphi\left(\frac{2\lambda}{\Lambda}\right)\left(\int_{\{x\in\mathbb{D}_+^{n+1}:|f(x)|\le\lambda\}}|f(x)|^p w(x)dx\right)\frac{d\lambda}{\lambda^p}$$
$$\lesssim\frac{1}{\Lambda}\int_{\mathbb{D}_+^{n+1}}|f(x)|^p\left(\int_{|f(x)|}^\infty\varphi\left(\frac{2\lambda}{\Lambda}\right)\frac{d\lambda}{\lambda^p}\right)w(x)dx.$$

Using Lemma 2 (1), we have

$$\frac{1}{\Lambda}\int_0^\infty\varphi\left(\frac{2\lambda}{\Lambda}\right)w\big(\{x\in\mathbb{D}_+^{n+1} : \big|\mathcal{R}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)\big|>\lambda\}\big)d\lambda$$
$$\lesssim\int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{2|f(x)|}{\Lambda}\right)w(x)dx\le\int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{c|f(x)|}{\Lambda}\right)w(x)dx. \tag{7}$$

Thus, putting together (6) and (7), we obtain

$$\int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{|\mathcal{R}f(x)|}{\Lambda}\right)w(x)dx\le\int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{c_0|f(x)|}{\Lambda}\right)w(x)dx.$$

Again we shall label the constant we want to distinguish from other less important constants. As before, if we set $\Lambda=c_2\|f\|_{L_w^\Phi(\mathbb{D}_+^{n+1})}$, then we obtain

$$\int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{|\mathcal{R}f(x)|}{\Lambda}\right)w(x)dx\le 1.$$

Hence the operator norm of $\mathcal{R}$ is less than c_2 :

$$\|\mathcal{R}f\|_{L_w^\Phi(\mathbb{D}_+^{n+1})}\le\Lambda=c_2\|f\|_{L_w^\Phi(\mathbb{D}_+^{n+1})}.$$

◄

4. Commutator of parabolic nonsingular integral operators in the weighted Orlicz space

We recall the definition of the space $BMO(\mathbb{D}_+^{n+1})$.

Definition 5. *Suppose that* $f \in L^1_{\mathrm{loc}}(\mathbb{D}_+^{n+1})$. *Let*

$$\|f\|_* = \sup_{x \in \mathbb{D}_+^{n+1}, r>0} \frac{1}{|\mathcal{E}^+(x,r)|} \int_{\mathcal{E}^+(x,r)} |f(y) - f_{\mathcal{E}^+(x,r)}|dy,$$

where

$$f_{\mathcal{E}^+(x,r)} = \frac{1}{|\mathcal{E}^+(x,r)|} \int_{\mathcal{E}^+(x,r)} f(y)dy.$$

Define

$$BMO(\mathbb{D}_+^{n+1}) = \{f \in L^1_{\mathrm{loc}}(\mathbb{D}_+^{n+1}) \ : \ \|f\|_* < \infty\}.$$

The space $BMO(\mathbb{D}_+^{n+1})$ is a Banach space with respect to the norm $\|\cdot\|_*$. The parabolic maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|\mathcal{E}^+(x,r)|} \int_{\mathcal{E}^+(x,r)} |f(y)|dy, \qquad x \in \mathbb{R}^n$$

for a locally integrable function f on $\mathbb{R}^n$.

Let $M^\sharp$ be the sharp parabolic maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |\mathcal{E}^+(x,r)|^{-1} \int_{\mathcal{E}^+(x,r)} |f(y) - f_{\mathcal{E}^+(x,r)}|dy.$$

Theorem 3. *[13] Let* $1 < p < \infty$. *Then* $M : L^p_w(\mathbb{R}^n) \to L^p_w(\mathbb{R}^n)$ *if and only if* $w \in A_p(\mathbb{R}^n)$.

Theorem 4. *[9, Theorem 1] Let* Φ *be a Young function with* $\Phi \in \Delta_2 \cap \nabla_2$. *Assume in addition* $w \in A_{i_\Phi}(\mathbb{R}^n)$. *Then, there is a constant* $C \geq 1$ *such that*

$$\int_{\mathbb{R}^n} \Phi\left(Mf(x)\right) w(x)dx \leq C \int_{\mathbb{R}^n} \Phi\left(|f(x)|\right) w(x)dx \tag{8}$$

for any locally integrable function f .

From [4, Remark 2.5] and [5, Remark 6.1.3] we get the boundedness result which was proved in [6]:

Theorem 5. *[6] Let* Φ *be a Young function with* $\Phi \in \nabla_2$. *Assume in addition* $w \in A_{i_\Phi}(\mathbb{R}^n)$. *Then the modular inequality* (8) *holds.*

Remark 1. *Note that the strong modular inequality* (8) *implies the corresponding norm inequality. Indeed, let* (8) *hold. Then, using the sublinearity of* M *, convexity of* Φ *and* (3) *we have*

$$\int_{\mathbb{R}^n} \Phi\left(\frac{Mf(x)}{C\|f\|_{L^\Phi_w}}\right) w(x)dx = \int_{\mathbb{R}^n} \Phi\left(M\Big(\frac{f}{C\|f\|_{L^\Phi_w}}\Big)(x)\right) w(x)dx$$
$$\leq C\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{C\|f\|_{L^\Phi_w}}\right) w(x)dx \leq 1,$$

where C *is the constant in* (8)*. This implies* $\|Mf\|_{L^\Phi_w} \lesssim \|f\|_{L^\Phi_w}$ *.*

Lemma 3. *[8] Let* $b \in BMO(\mathbb{D}^{n+1}_+)$ *. Then there is a constant* $C > 0$ *such that*

$$\left|b_{\mathcal{E}^+(x,r)} - b_{\mathcal{E}^+(x,t)}\right| \leq C\|b\|_* \ln\frac{t}{r} \quad \textit{for} \quad 0 < 2r < t, \tag{9}$$

where C *is independent of* b *,* x *,* r *and* t *.*

Lemma 4. *[7] Let* $w \in A_\infty$ *,* $b \in BMO(\mathbb{D}^{n+1}_+)$ *and* Φ *be a Young function with* $\Phi \in \Delta_2$ *. Then,*

$$\sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(w(\mathcal{E}^+(x,r))^{-1}\big) \left\|b - b_{\mathcal{E}^+(x,r)}\right\|_{L^\Phi_w(\mathcal{E}^+(x,r))} \lesssim \|b\|_*. \tag{10}$$

Theorem 6. *[1, Theorem 1.13] Let* $b \in BMO(\mathbb{R}^{n+1})$ *. Suppose that* X *is a Banach space of measurable functions defined on* $\mathbb{R}^{n+1}$ *. Moreover, assume that* X *satisfies the lattice property, that is*

$$0 \leq g \leq f \quad \Rightarrow \quad \|g\|_X \lesssim \|f\|_X.$$

Assume that M *is bounded on* X *. Then the operator* M_b *is bounded on* X *, and the inequality*

$$\|M_b f\|_X \leq C\|b\|_*\|f\|_X$$

holds with the constant C *independent of* f *.*

Combining Theorems 5 and 6, we obtain the following statement.

Corollary 1. *Let* Φ *be a Young function with* $\Phi \in \nabla_2$ *and* $b \in BMO(\mathbb{D}^{n+1}_+)$ *. Assume in addition* $w \in A_{i_\Phi}(\mathbb{D}^{n+1}_+)$ *. Then* M_b *is bounded on* $L^\Phi_w(\mathbb{D}^{n+1}_+)$ *.*

The space $L^p_w(\mathbb{D}^{n+1}_+)$ coincides with the space

$$\left\{f(x) : \left|\int_{\mathbb{D}^{n+1}_+} f(y)g(y)dy\right| < \infty \quad \textit{for all} \quad g \in L^{p'}_{w^{-1}}(\mathbb{D}^{n+1}_+)\right\} \tag{11}$$

up to the equivalence of norms

$$\|f\|_{L^p_w(\mathbb{D}^{n+1}_+)} \approx \sup_{\|g\|_{L^{p'}_{w^{-1}}(\mathbb{D}^{n+1}_+)} \le 1} \left| \int_{\mathbb{D}^{n+1}_+} f(y)g(y)dy \right|. \tag{12}$$

The following statement holds:

Lemma 5. *Let* $1 \le p < \infty$ *. Then, for all* $f \in L^p_w(\mathbb{D}^{n+1}_+)$ *and* $g \in L^{p'}_{w^{-1}}(\mathbb{D}^{n+1}_+)$ *,*

$$\left| \int_{\mathbb{D}^{n+1}_+} f(y)g(y)dy \right| \le C \int_{\mathbb{D}^{n+1}_+} M^\sharp f(y) Mg(y)dy$$

with a constant $C > 0$ *independent of* f *.*

Lemma 6. *Let* $1 \le p < \infty$ *,* $w \in A_p$ *. Then*

$$\|f\|_{L^p_w(\mathbb{D}^{n+1}_+)} \le C \|M^\sharp f\|_{L^p_w(\mathbb{D}^{n+1}_+)}$$

with a constant $C > 0$ *independent of* f *.*

Proof. By (12) we have

$$\|f\|_{L^p_w(\mathbb{D}^{n+1}_+)} \le C \sup_{\|g\|_{L^{p'}_{w^{-1}}(\mathbb{D}^{n+1}_+)} \le 1} \left| \int_{\mathbb{D}^{n+1}_+} f(y)g(y)dy \right|.$$

According to Lemma 5,

$$\|f\|_{L^p_w(\mathbb{D}^{n+1}_+)} \le C \sup_{\|g\|_{L^{p'}_{w^{-1}}(\mathbb{D}^{n+1}_+)} \le 1} \int_{\mathbb{D}^{n+1}_+} M^\sharp f(y)\, Mg(y)dy.$$

By the Hölder inequality and Theorem 3, we derive

$$\|f\|_{L^p_w(\mathbb{D}^{n+1}_+)} \le C \sup_{\|g\|_{L^{p'}(\mathbb{D}^{n+1}_+)} \le 1} \|M^\sharp f\|_{L^p_w(\mathbb{D}^{n+1}_+)} \|Mg\|_{L^{p'}_{w^{-1}}(\mathbb{D}^{n+1}_+)}$$

$$\le C \sup_{\|g\|_{L^{p'}_{w^{-1}}(\mathbb{D}^{n+1}_+)} \le 1} \|M^\sharp f\|_{L^p_w(\mathbb{D}^{n+1}_+)} \|g\|_{L^{p'}_{w^{-1}}(\mathbb{D}^{n+1}_+)} \le C \|M^\sharp f\|_{L^p_w(\mathbb{D}^{n+1}_+)}.$$

◄

The boundedness result for the commutator of parabolic nonsingular integral operator on weighted Lebesgue spaces is given in the following theorem.

Theorem 7. *Let* $\mathcal{R}$ *be a parabolic nonsingular integral operator,* $b \in BMO$ *,* $1 < p < \infty$ *and* $w \in A_p$ *. Then the commutator operator* $[b, \mathcal{R}]$ *is bounded on the space* $L^p_w(\mathbb{D}^{n+1}_+)$ *.*

Proof. We are going to adapt an idea of Stromberg (see [21, pp. 417-418]). Observe that it is enough to prove

$$M^\sharp([b,\mathcal{R}]f)(x) \le C\|b\|_* \Big((M|\mathcal{R}f|^r)^{\frac{1}{r}}(x) + (M|f|^r)^{\frac{1}{r}}(x) \Big) \tag{13}$$

for all $r > 1$, $x \in \mathbb{R}^n$.

To see this, choose $1 < r < p$. Then (13), combined with Lemmas 3 and 6 and the L^p_w estimate on $\mathcal{R}$ (Theorem 1), implies

$$\begin{aligned}
\|[b,\mathcal{R}]f\|_{L^p_w} &\le \|M^\sharp[b,\mathcal{R}]f\|_{L^p_w} \le C\|b\|_* \Big(\|\big(M|\mathcal{R}f|^r\big)^{\frac{1}{r}}\|_{L^p_w} + \|\big(M|f|^r\big)^{\frac{1}{r}}\|_{L^p_w} \Big) \\
&= C\|b\|_* \Big(\|M|\mathcal{R}f|^r|\|_{L^{\frac{p}{r}}_w} + \|M|f|^r\|_{L^{\frac{p}{r}}_w} \Big) \\
&\le C\|b\|_* \Big(\|\mathcal{R}f|^r|\|_{L^{\frac{p}{r}}_w} + \||f|^r\|_{L^{\frac{p}{r}}_w} \Big) \\
&= C\|b\|_* \Big(\|\mathcal{R}f\|_{L^p_w} + \|f\|_{L^p_w} \Big) \le C\|b\|_* \|f\|_{L^p_w}.
\end{aligned}$$

◀

From this result and [12, Theorem 2.7], we have the following boundedness of $[b, \mathcal{R}]$ on $L^\Phi_w(\mathbb{D}^{n+1}_+)$.

Theorem 8. *Let* Φ *be a Young function,* $w \in A_{i_\Phi}$ *and* $\mathcal{R}$ *be a parabolic nonsingular integral operator, defined by* (1)*. If* $\Phi \in \Delta_2 \cap \nabla_2$ *and* $b \in BMO$ *, then the commutator operator* $[b, \mathcal{R}]$ *is bounded on* $L^\Phi_w(\mathbb{D}^{n+1}_+)$ *.*

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