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An Analysis of the Convergence Problem of a Function in Lipschitz spaces

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Abstract. In the present work, the degrees of convergence of a function of conjugate Fourier series in Lip φ and weighted Lipschitz classes using product Hausdorff operator are obtained. Some interesting applications of our results are also discussed in order to observe the nature of convergence of the function in both the classes. From our applications, we find that the rate of convergence obtained from our results are much better than those of earlier results and rate of convergence will be faster as α increases.

Key Words and Phrases: degree of convergence, Lip φ class of function, weighted Lipschitz class of function, product Hausdorff operator, conjugate Fourier series.

2010 Mathematics Subject Classifications: 41A10, 42B05

1. Introduction

In recent past, different studies on the degree of approximation of a function in Lipschitz, Hölder spaces using single means has been made by [2, 5, 6, 11, 12] etc. Recently, [4, 7, 13, 14] have studied the degree of approximation of conjugate function in Lipschitz, Hölder spaces using product means. The degree of approximation of function in Besov spaces of its derived Fourier series and in Sobolev spaces of its double Fourier series has been studied by [8] and [9] respectively. One can see [10] for details on approximation properties of operators.

In this paper, we investigate the degree of convergence of a function of conjugate Fourier series in Lip φ class and weighted Lipschitz class using product Hausdorff means, which gives a wider class of summability matrices. The Cesàro, Hölder, Euler and their product matrices Hausdorff means. Moreover, the degree of approximation of a function of conjugate Fourier series in Lipschitz spaces only gives the degree of a polynomial with respect to the function, but the degree of

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convergence of a function of conjugate Fourier series gives the convergence of the polynomial with respect to the function.

In this paper, we also study the rate of convergence of the function in question by means of some applications and observe that the degree of convergence of a function of conjugate Fourier series in Lip φ and weighted Lipschitz classes using product Hausdorff means gives much better results than those of earlier known results. We also observe that our results obtained in (9), (16) and (23) converge to 0 as $\alpha \to \infty$.

2. Preliminaries

2.1. Fourier and allied series

The Fourier series (F.S.) of a function h is given by

$$h(y) := \frac{1}{2}a_0 + \sum_{\alpha=1}^{\infty}(a_\alpha \cos \alpha y + b_\alpha \sin \alpha y). \tag{1}$$

We can denote the α^{th} partial sums of (1) by $s_\alpha(h; y)$.
The series

$$\tilde{h}(y) := \sum_{\alpha=1}^{\infty}(a_\alpha \sin \alpha y - b_\alpha \cos \alpha y) \tag{2}$$

is said to be the conjugate series of (1) which is also said to be conjugate Fourier series (C.F.S.).
We can denote the α^{th} partial sums of (2) by $\tilde{s}_\alpha(\tilde{h}; y)$, which is given by

$$\tilde{s}_\alpha(\tilde{h}; y) - \tilde{h}(y) = \frac{1}{2\pi}\int_0^\pi \psi_{(y)}(l)\frac{\cos(\alpha + \frac{1}{2})l}{\sin\frac{l}{2}}dl,$$

where

$$\psi_{(y)}(l) = \tilde{h}(y + l) - \tilde{h}(y - l)$$

and

$$\tilde{h}(y) = -\frac{1}{2\pi}\int_0^\pi \gamma_{(y)}(l)\cot\frac{l}{2}dl. \tag{3}$$

2.2. Product Hausdorff operator

If

$$m_{\alpha,\beta} \equiv \begin{cases} {}^{\alpha}C_\beta \Delta^{\alpha-\beta}\mu_\beta, & 0 \le \beta \le \alpha; \\ 0, & \beta > \alpha, \end{cases}$$

where $M \equiv (m_{\alpha,\beta})$ is an infinite lower triangular matrix and Δ is a forward operator defined as $\Delta\mu_\alpha \equiv \mu_\alpha - \mu_{\alpha+1}$ and $\Delta^{\beta+1}\mu_\alpha \equiv \Delta^\beta(\Delta\mu_\alpha)$, then $M \equiv (m_{\alpha,\beta})$ is said to be a Hausdorff matrix ([3]).
$M \equiv (m_{\alpha,\beta})$ is said to be regular if and only if

$$\int_0^1 |d\gamma(t)| \leq \infty. \tag{4}$$

Eq. (4) means $M \equiv (m_{\alpha,\beta})$ preserves the limit of each convergent sequence.
In eq. (4), $\gamma(t)$ denotes the mass function, which is continuous at $t = 0$ and $\gamma(t)$ is a function of bounded variation in the interval $[0, 1]$ such that $\gamma(0+) = 0;\ \gamma(1) = 1$ and $\gamma(t) = \frac{\gamma(t-0)+\gamma(t+0)}{2}$ for $0 < t < 1$.
The moment sequence (denoted by μ_α), is given by

$$\mu_\alpha = \int_0^1 t^\alpha d\gamma(t).$$

The product Hausdorff operator of the function $\tilde{h}$ is given by

$$MN_{(mn)}(\tilde{h}; y) := \sum_{\beta=0}^{\alpha} m_{\alpha,\beta} \sum_{\upsilon=0}^{\beta} n_{\beta,\upsilon} \tilde{s}_\upsilon(\tilde{h}; y), \quad \alpha, \beta = 0, 1, 2, \ldots \tag{5}$$

The C.F.S. is said to be summable to s by product Hausdorff method if $MN_{(mn)}(\tilde{h}; y) \to s$ as $\alpha, \beta \to \infty$.

Remark 1. *Eq. (5) reduces to*

(i) (C, p) *method if the mass function is* $\gamma(t) = p\int_0^t (1-l)^{p-1} dl$.

(ii) (E, q) *method if the mass function is*

$$\gamma(t) = \begin{cases} 0 & \text{if } t \in [0, b], \\ 1 & \text{if } t \in [b, 1], \end{cases}$$

where $b = \frac{1}{1+q}, q > 0$.

2.3. Lipschitz spaces

The norm of $h \in L^\vartheta[0, 2\pi]$ is given by

$$||h||_\vartheta := \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |h(y)|^\vartheta dy \right\}^{\frac{1}{\vartheta}}, & \text{for } 1 \leq \vartheta < \infty; \\ \operatorname*{ess\,sup}_{0<y<2\pi} |h(y)|, & \text{for } \vartheta = \infty. \end{cases}$$

Lip φ class of functions
A function $h \in Lip\ \varphi$ if

$$|h(y+l) - h(y)| = O(l^{\varphi}), \text{ for } 0 < \varphi \leq 1.$$

Lip (φ, ϑ) class of functions
A function $h \in Lip(\varphi, \vartheta)$ if

$$\left(\int_0^{2\pi} |h(y+l) - h(y)|^{\vartheta} dy \right)^{\frac{1}{\vartheta}} = O(l^{\varphi}),\ 0 < \varphi \leq 1, \vartheta \geq 1.$$

Lip $(\zeta(l), \vartheta)$ class of functions
A function $h \in Lip(\zeta(l), \vartheta)$ with $\zeta(l)$ being positive increasing function of l, if

$$\left(\int_0^{2\pi} |h(y+l) - h(y)|^{\vartheta} dy \right)^{\frac{1}{\vartheta}} = O(\zeta(l)).$$

$W(L^{\vartheta}, \zeta(l))$ class of functions
A function $h \in W(L^{\vartheta}, \zeta(l))$ with $\zeta(l)$ being positive increasing function of l, if

$$\left(\int_0^{2\pi} \left| (h(y+l) - h(y)) \sin^{\sigma}\left(\frac{y}{2}\right) \right|^{\vartheta} dy \right)^{\frac{1}{\vartheta}} = O(\zeta(l)), \quad \text{for } \sigma \geq 0, \vartheta \geq 1.$$

$$\text{If } \vartheta \to \infty, \text{ then } Lip\ \varphi \subseteq Lip(\varphi, \vartheta).$$
$$\text{If } \zeta(l) = l^{\varphi}\ (0 < \varphi \leq 1), \text{ then } Lip(\varphi, \vartheta) \subseteq Lip(\zeta(l), \vartheta).$$
$$\text{If } \sigma = 0, \text{ then } Lip(\zeta(l), \vartheta) \subseteq W(L^{\vartheta}, \zeta(l)).$$

2.4. Degree of convergence

The degree of convergence of a function $\tilde{h}$ gives how speedily $\tilde{p}_n$ converges to the function $\tilde{h}$, i.e.

$$||\tilde{h}(y) - \tilde{p}_n|| = O\left(\frac{1}{\lambda_n}\right),$$

where $\lambda_n \to \infty$ as $n \to \infty$ ([1]).

2.5. Notation

$$\begin{aligned}
\tilde{X}_{\upsilon} &= \int_0^1 \tilde{h}(t, l) d\gamma(t) \\
&= \int_0^1 \sum_{\beta=0}^{\alpha} {}^{\alpha}C_{\beta} t^{\beta} (1-t)^{\alpha-\beta} d\gamma_1(t) \int_0^1 \sum_{\upsilon=0}^{\beta} {}^{\beta}C_{\upsilon} t^{\upsilon} (1-t)^{\beta-\upsilon} \frac{\cos(\upsilon + \frac{1}{2})l}{\sin \frac{l}{2}} d\gamma_2(t).
\end{aligned}$$

3. Results and their proofs

In this section, we establish our theorems and related lemmas.

First we proof the following lemmas, which are useful for the proofs of our main theorems.

Lemma 1. $\left|\tilde{X}_v\right| = O\left(\frac{1}{l}\right)$, *for* $0 < l \leq \frac{1}{\alpha+1}$.

Proof. For $0 < t < 1$, $0 < l \leq \frac{1}{\alpha+1}$, $|\cos(vl)| \leq 1$, $\sup\limits_{0\leq t\leq 1} \frac{d\gamma_1(t)}{dt} = D$, $\sup\limits_{0\leq t\leq 1} \frac{d\gamma_2(t)}{dt} = C$ and $|\sin(\frac{l}{2})| \geq \frac{l}{\pi}$, we get

$$\begin{aligned}
\left|\tilde{X}_v\right| &= \left| \int_0^1 \sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} d\gamma_1(t) \right. \\
&\quad \left. \cdot \int_0^1 \sum_{v=0}^{\beta} {}^{\beta}C_v t^v (1-t)^{\beta-v} \frac{\cos(v+\frac{1}{2})l}{\sin\frac{l}{2}} d\gamma_2(t) \right| \\
&\leq \left| \left(\int_0^1 \sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} d\gamma_1(t) \right) \right. \\
&\quad \left. \cdot \left(\int_0^1 \sum_{v=0}^{\beta} {}^{\beta}C_v t^v (1-t)^{\beta-v} \frac{1}{\frac{l}{\pi}} d\gamma_2(t) \right) \right| \\
&= \frac{\pi CD}{l} \left| \int_0^1 \left(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} dt \right) \cdot \int_0^1 \left((1-t+t)^\beta \right) dt \right| \\
&= \frac{\pi CD}{l} \left| \int_0^1 (1-t+t)^\alpha dt \right| \\
&= \frac{\pi CD}{l} \left| \int_0^1 dt \right| \\
&= O\left(\frac{1}{l}\right).
\end{aligned}$$

◀

Lemma 2. $\left|\tilde{X}_v\right| = O\left(\frac{1}{(\alpha+1)l^2}\right)$, *for* $\frac{1}{\alpha+1} < l \leq \pi$.

Proof. For $0 < t < 1$, $\frac{1}{\alpha+1} < l \leq \pi$, $|\sin(\beta+1)l| \leq 1 \ \forall \ l$, $\sup\limits_{0\leq t\leq 1} \frac{d\gamma_1(t)}{dt} =$

D, $\sup\limits_{0\leq t\leq 1} \frac{d\gamma_2(t)}{dt} = C$ and $|\sin(\frac{l}{2})| \geq \frac{l}{\pi}$, we get

$$\begin{aligned}
\left|\tilde{X}_\upsilon\right| &= \Bigg| \int_0^1 \left(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} d\gamma_1(t) \right) \\
&\quad \cdot \int_0^1 \left(\sum_{\upsilon=0}^{\beta} {}^{\beta}C_\upsilon t^\upsilon (1-t)^{\beta-\upsilon} \frac{\cos(\upsilon+\frac{1}{2})l}{\sin\frac{l}{2}} d\gamma_2(t) \Bigg| \right) \\
&\leq \Bigg| \int_0^1 \left(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} d\gamma_1(t) \right) \\
&\quad \cdot \mathrm{Re}\Bigg(\int_0^1 \sum_{\upsilon=0}^{\beta} {}^{\beta}C_\upsilon t^\upsilon (1-t)^{\beta-\upsilon} \frac{e^{i(\upsilon+\frac{1}{2})l}}{\frac{l}{\pi}} \Bigg) d\gamma_2(t) \Bigg| \\
&= \frac{\pi}{l} \Bigg| \int_0^1 \left(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} d\gamma_1(t) \right) \\
&\quad \cdot (1-t)^\beta \mathrm{Re}\Bigg(\int_0^1 \sum_{\upsilon=0}^{\beta} {}^{\beta}C_\upsilon \left(\frac{t}{1-t}\right)^\upsilon e^{i\upsilon l} e^{\frac{il}{2}} \Bigg) d\gamma_2(t) \Bigg| \\
&\leq \frac{\pi C}{l} \Bigg| \int_0^1 \left(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} d\gamma_1(t) \right) \\
&\quad \cdot \mathrm{Re}\Bigg(e^{\frac{il}{2}} \int_0^1 (1-t+te^{il})^\beta \Bigg) dt \Bigg| \\
&\leq \frac{\pi CD}{l} \Bigg| \int_0^1 \left(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} dt \right) \Bigg| \\
&\quad \cdot \left| e^{\frac{il}{2}} \right| \mathrm{Re} \Bigg| \left(\int_0^1 (1+t(e^{il}-1))^\beta \right) dt \Bigg| \\
&= \frac{\pi CD}{l} \Bigg| \int_0^1 \left(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} dt \right) \Bigg| \left| \frac{\sin(\beta+1)l}{2(\beta+1)\sin\frac{l}{2}} \right| \\
&\leq \frac{\pi^2 CD}{2l^2} \cdot \Bigg| \int_0^1 \left(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} \frac{1}{\beta+1} dt \right) \Bigg|.
\end{aligned}$$

Since $\frac{1}{\beta+1}$ is non-increasing function,

$$\left|\tilde{X}_\upsilon\right| \leq \frac{\pi^2 CD}{(\alpha+1)l^2} \Bigg| \int_0^1 \left(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta (1-t)^{\alpha-\beta} dt \right) \Bigg|$$

$$= \frac{\pi^2 CD}{(\alpha+1)l^2}\Big|\int_0^1 (1-t+t)^\alpha dt\Big|$$
$$= O\Big(\frac{1}{(\alpha+1)l^2}\Big).$$

◀

Now, we establish our main theorems.

Theorem 1. *If $\tilde{h}$ is a function, conjugate to a function h (2π periodic and Lebesgue integrable), then the degree of convergence of $\tilde{h}$ in $Lip\varphi$ ($0<\varphi\le 1$) class using product Hausdorff operator, is given by*

$$T_\alpha(y) = ||MN_{(mn)} - \tilde{h}(y)||_\infty = \begin{cases} (\alpha+1)^{-\varphi}, & 0<\varphi<1 \\ \frac{\log(e\pi(\alpha+1))}{\alpha+1}, & \varphi = 1 \end{cases} \tag{6}$$

for $\alpha = 0,1,2,3,...$

Proof. We write

$$\tilde{s}_\alpha(\tilde{h};y) - \tilde{h}(y) = \frac{1}{2\pi}\int_0^\pi \psi_y(l)\Big(\frac{\cos(\alpha+\frac{1}{2})l}{\sin\frac{l}{2}}\Big)dl.$$

Now,

$$\begin{aligned} T_\alpha(y) &= MN_{(mn)} - \tilde{h}(y) \\ &= \sum_{\beta=0}^{\alpha} m_{\alpha,\beta}\sum_{\upsilon=0}^{\beta} n_{\beta,\upsilon}(\tilde{s}_\alpha(\tilde{h};y)-\tilde{h}(y)) \\ &= \frac{1}{2\pi}\int_0^\pi \Big[\psi_y(l)\int_0^1\Big(\sum_{\beta=0}^{\alpha} {}^{\alpha}C_\beta t^\beta(1-t)^{\alpha-\beta}d\gamma_1(t)\Big) \\ &\quad \cdot \int_0^1\Big(\sum_{\upsilon=0}^{\beta} {}^{\beta}C_\upsilon t^\upsilon(1-t)^{\beta-\upsilon}\frac{\cos(\upsilon+\frac{1}{2})l}{\sin\frac{l}{2}}d\gamma_2(t)\Big)\Big]dl \\ &= \frac{1}{2\pi}\int_0^\pi \psi_y(l)\tilde{X}_\upsilon dl \\ &= \Big(\int_0^{\frac{1}{\alpha+1}} + \int_{\frac{1}{\alpha+1}}^{\pi}\Big)\psi_y(l)\tilde{X}_\upsilon dl \\ &= A + B \quad (say). \end{aligned} \tag{7}$$

Considering $\psi_y(l) \in Lip\ \varphi$ and using Lemmas 1 and 2 give

$$\begin{aligned}
\left|MN_{(mn)} - \tilde{h}(y)\right| = |A+B| &\leq \left(\int_0^{\frac{1}{\alpha+1}} \frac{|\psi_y(l)|}{l} dl\right) + \left(\int_{\frac{1}{\alpha+1}}^{\pi} \frac{|\psi_y(l)|}{(\alpha+1)l^2} dl\right) \\
&= O\left[\left(\int_0^{\frac{1}{\alpha+1}} \frac{l^{\varphi}}{l} dl\right) + \left(\int_{\frac{1}{\alpha+1}}^{\pi} \frac{l^{\varphi}}{(\alpha+1)l^2} dl\right)\right] \\
&= O\left[\left(\frac{l^{\varphi}}{\varphi}\right)_0^{\frac{1}{\alpha+1}} + \frac{1}{\alpha+1}\left(\frac{l^{\varphi-1}}{\varphi-1}\right)_{\frac{1}{\alpha+1}}^{\pi}\right] \\
&= O\begin{cases} \left(\frac{1}{\alpha+1}\right)^{\varphi} + \frac{1}{\alpha+1}\left(\frac{\pi^{\varphi-1} - (\frac{1}{\alpha+1})^{\varphi-1}}{\varphi-1}\right), & 0<\varphi<1 \\ \frac{\log(e\pi(\alpha+1))}{\alpha+1}, & \varphi = 1 \end{cases} \\
&= O\begin{cases} (\alpha+1)^{-\varphi} + (\alpha+1)^{-\varphi+1-1}, & 0<\varphi<1 \\ \frac{\log(e\pi(\alpha+1))}{\alpha+1}, & \varphi = 1 \end{cases} \\
&= O\begin{cases} (\alpha+1)^{-\varphi}, & 0<\varphi<1 \\ \frac{\log(e\pi(\alpha+1))}{\alpha+1}, & \varphi = 1. \end{cases} \qquad (8)
\end{aligned}$$

Combining (7) and (8)

$$T_{\alpha}(y) = ||MN_{(mn)} - \tilde{h}(y)||_{\infty} = O\begin{cases} (\alpha+1)^{-\varphi}, & 0<\varphi<1 \\ \frac{\log(e\pi(\alpha+1))}{\alpha+1}, & \varphi = 1 \end{cases} \qquad (9)$$

for $\alpha = 0, 1, 2, 3, ...$ ◄

Theorem 2. *If $\tilde{h}$ is a function, conjugate to a function h (2π periodic and Lebesgue integrable), then the degree of convergence of $\tilde{h}$ in $W(L^{\vartheta}, \zeta(l))$ class ($\vartheta > 1$ and $0 \leq \sigma \leq 1 - \frac{1}{\vartheta}$) using product Hausdorff operator, is given by*

$$T_{\alpha}(y) = ||MN_{(mn)} - \tilde{h}(y)||_{\vartheta} = O\left((\alpha+1)^{\sigma+\frac{1}{\vartheta}} \zeta\left(\frac{1}{\alpha+1}\right)\right),$$

with $\zeta(l)$ satisfying the following conditions:

$$\left\{\frac{\zeta(l)}{l}\right\} \text{ is non-increasing,} \qquad (10)$$

$$\left(\int_0^{\frac{1}{\vartheta+1}} \left(\frac{|\psi_y(l)|}{\zeta(l)} \sin^{\sigma}\left(\frac{l}{2}\right)\right)^{\vartheta} dl\right)^{\frac{1}{\vartheta}} = O\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\vartheta}}, \qquad (11)$$

$$\left(\int_{\epsilon}^{\frac{1}{n+1}}\left(\frac{\zeta(l)}{l\sin^{\sigma}(\frac{l}{2})}\right)^{\omega}dl\right)^{\frac{1}{\omega}}=O\left((\alpha+1)^{\sigma+\frac{1}{\vartheta}}\zeta\left(\frac{1}{\alpha+1}\right)\right),\tag{12}$$

$$\int_{\frac{1}{\alpha+1}}^{\pi}\left(\left(\frac{l^{-\delta}|\psi_y(l)|}{\zeta(l)}\right)^{\vartheta}dl\right)^{\frac{1}{\vartheta}}=O\left((\alpha+1)^{\delta}\right),\tag{13}$$

where δ *is an arbitrary number,* $0<\delta<\sigma+\frac{1}{\vartheta}$, $\frac{1}{\vartheta}+\frac{1}{\omega}=1$ *for* $\vartheta>1$*; (11) and (13) hold uniformly in* y.

Proof. Applying Hölder's inequality and the fact that $\psi_y(l)\in W(L_\vartheta,\zeta(l))$ in first integral of (7), we have

$$|A|\le\left(\int_0^{\frac{1}{\alpha+1}}\left(\frac{|\psi_y(l)|\sin^{\sigma}(\frac{l}{2})}{\zeta(l)}\right)^{\vartheta}dl\right)^{\frac{1}{\vartheta}}\left(\int_0^{\frac{1}{\alpha+1}}\left(\frac{\zeta(l)|\tilde{X}_v|}{\sin^{\sigma}(\frac{l}{2})}\right)^{\omega}dl\right)^{\frac{1}{\omega}}.$$

Using Lemma 1, conditions (11) and (12), we get

$$\begin{aligned}|A|&=O\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\vartheta}}\zeta\left(\frac{1}{\alpha+1}\right)\left(\int_0^{\frac{1}{\alpha+1}}\left(\frac{1}{l^{1+\sigma}}\right)^{\omega}dl\right)^{\frac{1}{\omega}}\\
&=O\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\vartheta}}\zeta\left(\frac{1}{\alpha+1}\right)\left(\lim_{\epsilon\to0}\int_{\epsilon}^{\frac{1}{\alpha+1}}\frac{dl}{l^{(\sigma+1)\omega}}\right)^{\frac{1}{\omega}}\\
&=O\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\vartheta}}\zeta\left(\frac{1}{\alpha+1}\right)\left(\left[\frac{l^{-(1+\sigma)\omega+1}}{-(1+\sigma)\omega+1}\right]_{\epsilon}^{\frac{1}{\alpha+1}}\right)^{\frac{1}{\omega}}\\
&=O\left[\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\vartheta}}\zeta\left(\frac{1}{\alpha+1}\right)(\alpha+1)^{1+\sigma-\frac{1}{\omega}}\right]\\
&=O\left[\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\vartheta}}\zeta\left(\frac{1}{\alpha+1}\right)(\alpha+1)^{\sigma+\frac{1}{\vartheta}}\right]\\
&=O\left[(\alpha+1)^{\sigma}\zeta\left(\frac{1}{\alpha+1}\right)\right].\end{aligned}\tag{14}$$

Again applying Hölder's inequality and the fact that $\psi_y(l)\in W(L_\vartheta,\zeta(l))$ in second integral of (7), we have

$$|B|\le\left(\int_{\frac{1}{\alpha+1}}^{\pi}\left(\frac{l^{-\delta}|\psi_y(l)|\sin^{\sigma}(\frac{l}{2})}{\zeta(l)}\right)^{\vartheta}dl\right)^{\frac{1}{\vartheta}}\left(\int_{\frac{1}{\alpha+1}}^{\pi}\left(\frac{\zeta(l)|\tilde{X}_v|}{l^{-\delta}\sin^{\sigma}(\frac{l}{2})}\right)^{\omega}dl\right)^{\frac{1}{\omega}}.$$

Using Lemma 2, conditions (10) and (13), and second mean value theorem for integrals, we have

$$=O\left((\alpha+1)^{\delta}\right)\left(\int_{\frac{1}{\alpha+1}}^{\pi}\left(\frac{\zeta(l)}{(\alpha+1)l^{2+\sigma-\delta}}\right)^{\omega}dl\right)^{\frac{1}{\omega}}$$

$$= O\Big((\alpha+1)^{\delta}\Big)\zeta\Big(\frac{1}{\alpha+1}\Big)\Big(\int_{\frac{1}{\pi}}^{\alpha+1}\Big(\frac{1}{(\alpha+1)\cdot x^{\delta-\sigma-2}}\Big)^{\omega}\frac{dx}{x^2}\Big)^{\frac{1}{\omega}}$$

$$= O\Big((\alpha+1)^{\delta}\Big)\zeta\Big(\frac{1}{\alpha+1}\Big)\cdot\frac{1}{(\alpha+1)}\Big(\Big(\frac{x^{(2+\sigma-\delta)\omega-1}}{(2+\sigma-\delta)\omega-1}\Big)_{\frac{1}{\pi}}^{\alpha+1}\Big)^{\frac{1}{\omega}}$$

$$= O\Big((\alpha+1)^{\delta}\Big)\zeta\Big(\frac{1}{\alpha+1}\Big)\cdot\frac{1}{(\alpha+1)}\Big((\alpha+1)^{2+\sigma-\delta-\frac{1}{\omega}})\Big)$$

$$= O\Big((\alpha+1)^{(\sigma+1-\frac{1}{\omega})}\zeta\Big(\frac{1}{\alpha+1}\Big)\Big)$$

$$= O\Big((\alpha+1)^{\sigma+\frac{1}{\vartheta}}\zeta\Big(\frac{1}{\alpha+1}\Big)\Big). \tag{15}$$

Combining (7), (14) and (15), we get

$$T_{\alpha}(y) = ||MN_{(mn)} - \tilde{h}(y)||_{\vartheta}$$
$$= O\Big((\alpha+1)^{\sigma}\zeta\Big(\frac{1}{\alpha+1}\Big)\Big) + O\Big((\alpha+1)^{\sigma+\frac{1}{\vartheta}}\zeta\Big(\frac{1}{\alpha+1}\Big)\Big)$$
$$= O\Big((\alpha+1)^{\sigma+\frac{1}{\vartheta}}\zeta\Big(\frac{1}{\alpha+1}\Big)\Big). \tag{16}$$

◀

Theorem 3. *If $\tilde{h}$ is a function, conjugate to a function h (2π periodic and Lebesgue integrable), then the degree of convergence $\tilde{h}$ of conjugate Fourier series in weighted Lipschitz class $W(L^1, \zeta(l))$, $0 \le \sigma < 1$ by product Hausdorff means, is given by*

$$T_{\alpha}(y) = ||MN_{(mn)} - \tilde{h}(y)||_1 = O\Big((\alpha+1)^{(\sigma+1)}\zeta\Big(\frac{1}{\alpha+1}\Big)\Big),$$

with $\zeta(l)$ satisfying (10) and the following conditions:

$$\Big\{\frac{\zeta(l)}{l^{\sigma+\tau}}\Big\} \text{ is non-decreasing,} \tag{17}$$

$$\int_0^{\frac{1}{\alpha+1}} \frac{l^{\tau-1}|\psi_y(l)|\sin^{\sigma}(\frac{l}{2})}{\zeta(l)}dl = O\Big(\frac{1}{\alpha+1}\Big)^{\tau}, \tag{18}$$

for some $\tau > 0, \sigma+\tau < 1$,

$$\int_{\frac{1}{\alpha+1}}^{\pi} \frac{l^{-\delta}|\psi_y(l)|}{\zeta(l)}dl = O(\alpha+1)^{\delta}, \tag{19}$$

$$\left\{\frac{\zeta(l)}{l^{\sigma-\delta+2}}\right\} \text{ is non-increasing,} \tag{20}$$

where $0 < \delta < \sigma + 1$ *provided (18) and (19) hold uniformly in* y.

Proof. Following the proof of Theorem 1 for $\vartheta = 1$ i.e. $\omega = \infty$, we have

$$|A| = \int_0^{\frac{1}{\alpha+1}} \left(\frac{l^{\tau-1}|\psi_y(l)|\sin^\sigma(\frac{l}{2})}{\zeta(l)}\right)dl \cdot \underset{0<l\leq\frac{1}{\alpha+1}}{\text{ess sup}}\left|\frac{\zeta(l)}{l^\tau \sin^\sigma(\frac{l}{2})}\right|.$$

Using the condition (18) and $|\sin\frac{l}{2}| \leq \frac{\pi}{l}$, we have

$$\begin{aligned}
|A| &= O\left(\frac{1}{\alpha+1}\right)^\tau \cdot \underset{0<l\leq\frac{1}{\alpha+1}}{\text{ess sup}}\left|\frac{\zeta(l)}{l^{\sigma+\tau}}\right| \\
&= O\left(\frac{1}{\alpha+1}\right)^\tau \zeta\left(\frac{1}{\alpha+1}\right)\left|(\alpha+1)^{\sigma+\tau}\right| \\
&= O\left((\alpha+1)^\sigma \zeta\left(\frac{1}{\alpha+1}\right)\right).
\end{aligned} \tag{21}$$

Following again the proof of Theorem 1 for $\vartheta = 1$ i.e. $\omega = \infty$, we have

$$|B| = \frac{1}{\alpha+1}\int_{\frac{1}{\alpha+1}}^\pi \frac{l^{-\delta}|\psi_y(l)|\sin^\sigma(\frac{l}{2})}{\zeta(l)}dl \cdot \underset{\frac{1}{\alpha+1}<l\leq\pi}{\text{ess sup}}\left|\frac{\zeta(l)}{l^{-\delta+\sigma+2}}\right|.$$

Using the condition (19), we have

$$\begin{aligned}
|B| &= O\left((\alpha+1)^{\delta-1}\right)\zeta\left(\frac{1}{\alpha+1}\right)\left((\alpha+1)^{\sigma-\delta+2}\right) \\
&= O\left((\alpha+1)^{\sigma+1}\zeta\left(\frac{1}{\alpha+1}\right)\right).
\end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}
T_\alpha(y) &= ||MN_{(mn)} - \tilde{h}(y)||_1 \\
&= O\left((\alpha+1)^\sigma\zeta\left(\frac{1}{\alpha+1}\right)\right) + O\left((\alpha+1)^{\sigma+1}\zeta\left(\frac{1}{\alpha+1}\right)\right) \\
&= O\left((\alpha+1)^{\sigma+1}\zeta\left(\frac{1}{\alpha+1}\right)\right).
\end{aligned} \tag{23}$$

◀

4. Corollaries

Corollary 1. *In view of Remark 1(i), the degree of convergence of the function* $\tilde{h}$ *in Lip*φ $(0 < \varphi \leq 1)$ *class using* C^p *means is given by*

$$T_\alpha(y) = ||MN^{C^p}_{(mn)} - \tilde{h}(y)||_\infty = O\begin{cases}(\alpha+1)^{-\varphi}, & 0 < \varphi < 1\\ \frac{\log(e\pi(\alpha+1))}{\alpha+1}, & \varphi = 1.\end{cases}$$

Proof. Proof will run along the same lines of the proof of Theorem 1. ◄

Corollary 2. *In view of Remark 1(ii), the degree of convergence of the function* $\tilde{h}$ *in Lip*φ $(0 < \varphi \leq 1)$ *class using* E^q *means is given by*

$$T_\alpha(y) = ||MN^{E^q}_{(mn)} - \tilde{h}(y)||_\infty = O\begin{cases}(\alpha+1)^{-\varphi}, & 0 < \varphi < 1\\ \frac{\log(e\pi(\alpha+1))}{\alpha+1}, & \varphi = 1.\end{cases}$$

Proof. Proof will run along the same lines of the proof of Theorem 1. ◄

Corollary 3. *In view of Remark 1(i), the degree of convergence of the function* $\tilde{h}$ *in weighted Lipschitz class using* C^p *means is given by*

$$T_\alpha(y) = ||MN^{C^p}_{(mn)} - \tilde{h}(y)||_\vartheta = O\left((\alpha+1)^{\sigma+\frac{1}{\vartheta}}\zeta\left(\frac{1}{\alpha+1}\right)\right),$$

where $\zeta(l)$ *satisfies the conditions (17) to (20).*

Proof. Proof will run along the same lines of the proof of Theorem 2. ◄

Corollary 4. *In view of Remark 1(ii) the degree of convergence of the function* $\tilde{h}$ *in weighted Lipschitz class using* E^q *means is given by*

$$T_\alpha(y) = ||MN^{E^q}_{(mn)} - \tilde{h}(y)||_\vartheta = O\left((\alpha+1)^{\sigma+\frac{1}{\vartheta}}\zeta\left(\frac{1}{\alpha+1}\right)\right),$$

where $\zeta(l)$ *satisfies the conditions (17) to (20).*

Proof. Proof will run along the same lines of the proof of Theorem 2. ◄

Remark 2. *Corollaries 3 and 4 can be further deduced for* $\vartheta = 1$.

5. Applications

1. Application of Theorem 1:

Consider $\varphi = \frac{1}{2}$,

$$||T_\alpha(y)||_\infty = ||MN_{(mn)}(\tilde{h}; y) - \tilde{h}(y)||_\infty = O\Big((\alpha+1)^{-\frac{1}{2}}\Big).$$

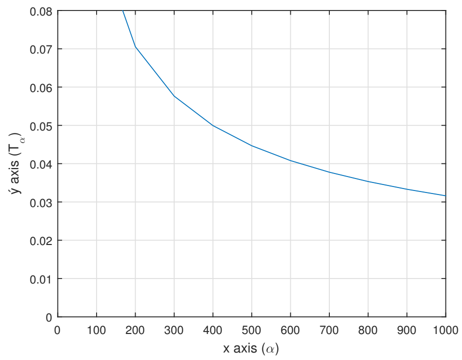
For $\varphi = 1$,

$$||T_\alpha(y)||_\infty = ||MN_{(mn)}(\tilde{h}; y) - \tilde{h}(y)||_\infty = O\Big(\frac{\log(e\pi(\alpha+1))}{\alpha+1}\Big).$$

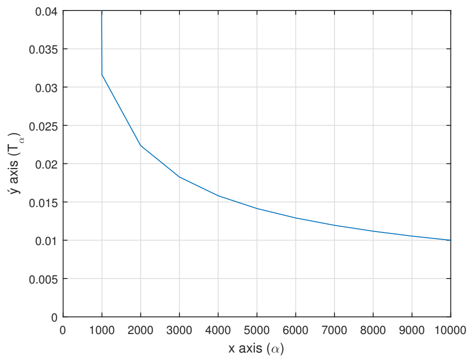
Table 1

α	$\|\|T_\alpha(y)\|\|_\vartheta = O\Big((\alpha+1)^{-\frac{1}{2}}\Big),\ \varphi = \frac{1}{2}$	α	$\|\|T_\alpha(y)\|\|_\vartheta = O\Big(\frac{\log(e\pi(\alpha+1))}{\alpha+1}\Big),\ \varphi = 1$
1000	0.031607	1000	0.0090444
10000	0.009999	10000	0.0011354
100000	0.003162	100000	0.0001366
...	...	...	...
...	...	...	...
...	...	...	...
∞	0.0	∞	0.0

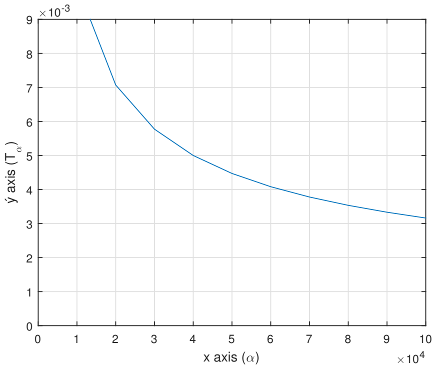
Now, we draw the following graphs of $T_\alpha(\cdot)$ for different values of α:



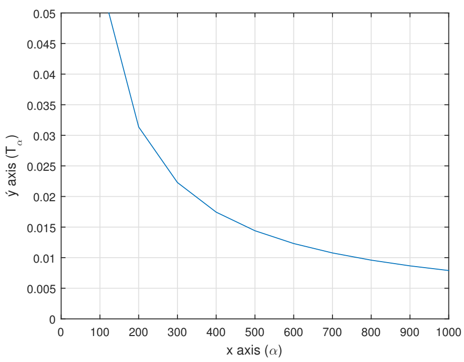
(a) For $\alpha = 1000, \varphi = \frac{1}{2}$



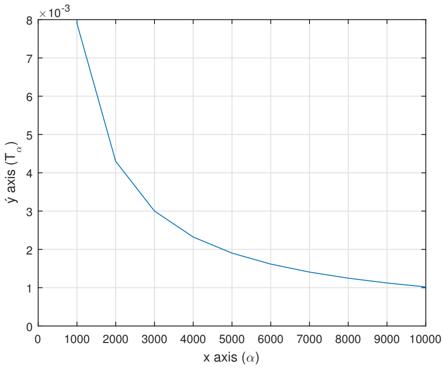
(b) For $\alpha = 10000, \varphi = \frac{1}{2}$



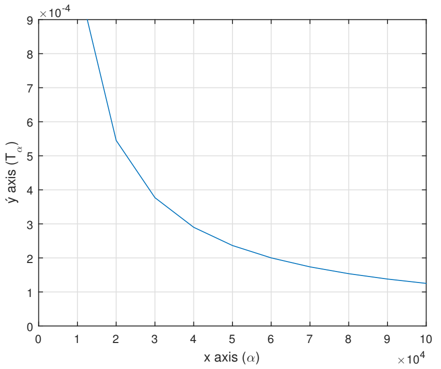
(c) For $\alpha = 100000, \varphi = \frac{1}{2}$



(d) For $\alpha = 1000, \varphi = 1$



(e) For $\alpha = 10000, \varphi = 1$



(f) For $\alpha = 100000, \varphi = 1$

Figure 1: The degree of convergence of the function $\tilde{h}$

Application of Theorem 2:

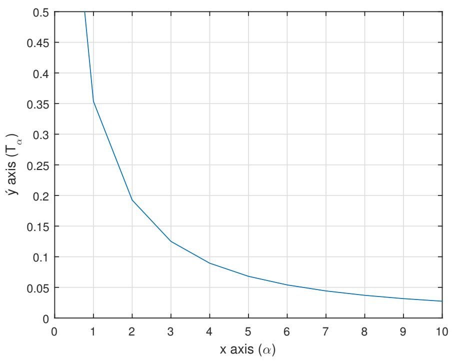
Consider $\zeta\left(\frac{1}{\alpha+1}\right) = \frac{1}{(\alpha+1)^2}, \ \sigma = 0, \ \vartheta = 2.$

$$||T_\alpha(y)||_\vartheta = ||MN_{(mn)}(\tilde{h}; y) - \tilde{h}(y)||_\vartheta = O\left(\frac{1}{(\alpha+1)^{\frac{3}{2}}}\right).$$

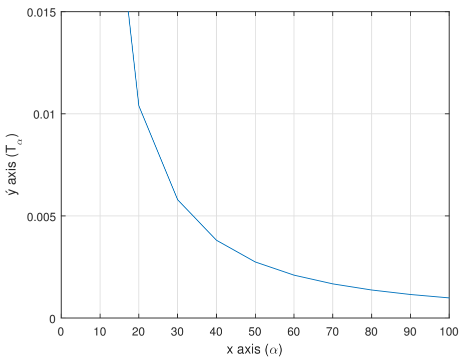
Table 2

α	$\|\|T_\alpha(y)\|\|_\vartheta = O\left(\frac{1}{(\alpha+1)^{\frac{3}{2}}}\right)$
10	0.027410
100	0.000985
1000	0.000032
10000	0.000001
100000	0.000000
.	.
∞	0.0

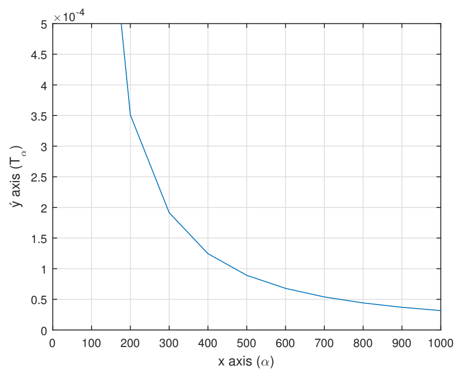
Now, we draw the following graphs of $T_\alpha(\cdot)$ for different values of α:



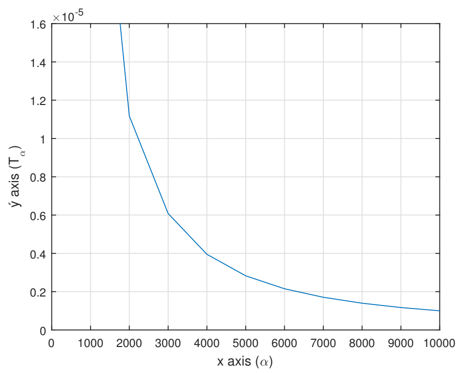
(a) For α=10



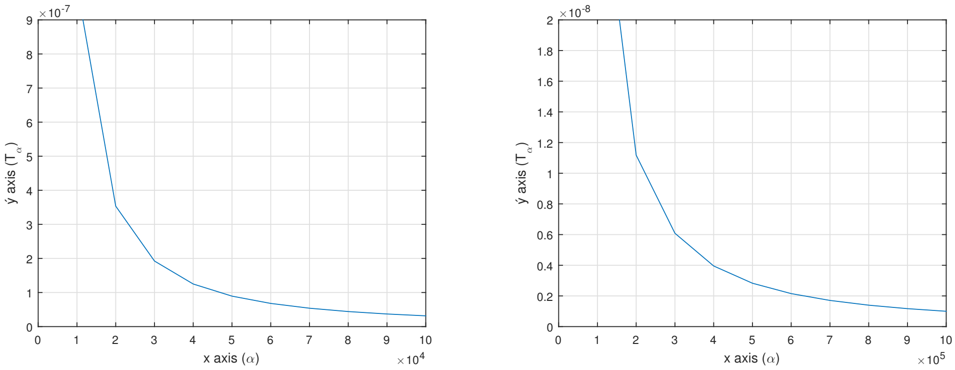
(b) For α=100



(c) For α=1000



(d) For α=10000



(e) For α=100000

(f) For α=1000000

Figure 2: The degree of convergence of the function $\tilde{h}$

Application of Theorem 3:

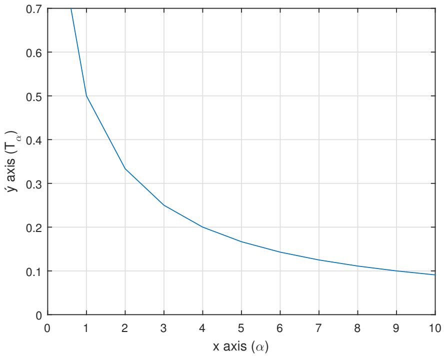
Consider $\zeta\left(\frac{1}{\alpha+1}\right) = \frac{1}{(\alpha+1)^2}, \ \sigma = 0, \ \vartheta = 1, \ \omega = \infty.$

$$||T_\alpha(y)||_\vartheta = ||MN_{(mn)}(\tilde{h}; y) - \tilde{h}(y)||_\vartheta = O\left(\frac{1}{\alpha+1}\right).$$

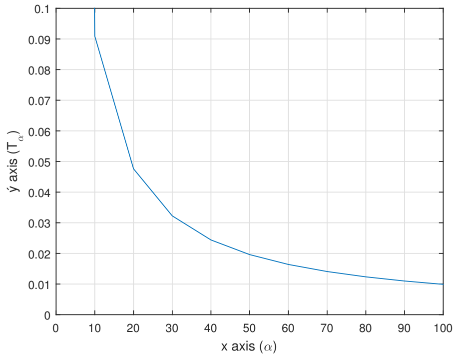
Table 3

α	$\|\|T_\alpha(y)\|\|_\vartheta = O\left(\frac{1}{\alpha+1}\right)$
10	0.09090
100	0.00990
1000	0.00099
10000	0.00009
100000	0.00001
1000000	0.00000
.	.
∞	0.0

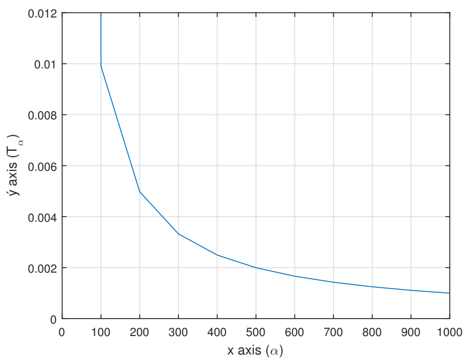
Now, we draw the following graphs of $T_\alpha(\cdot)$ for different values of α:



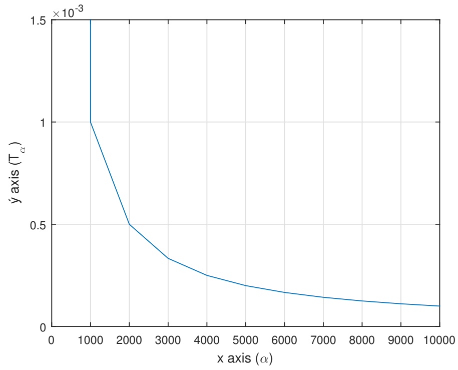
(a) For α=10



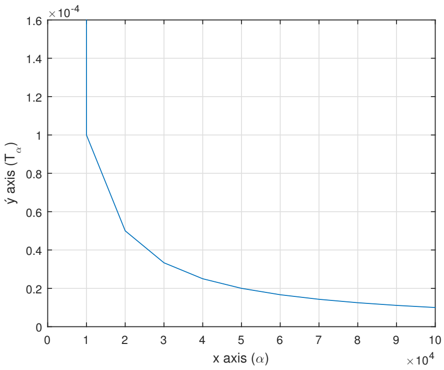
(b) For α=100



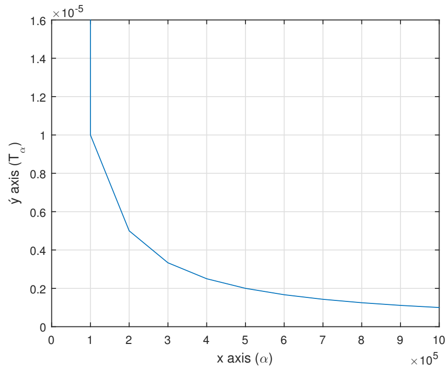
(c) For α=1000



(d) For α=10000



(e) For α=100000



(f) For α=1000000

Figure 3: The degree of convergence of the function $\tilde{h}$

Remark 3. *From above applications, we observe that the rate of convergence of $\tilde{h}$ will be faster as we increase α.*

6. Conclusion

- From Table 1 and Figures 1(a) to 1(f), we observe that the rate of convergence of $\tilde{h}$ is faster as α increases when $\varphi = \frac{1}{2}(0 < \varphi < 1)$, i.e. $||MN_{(mn)}(\tilde{h}; y) - \tilde{h}(y)||_\infty \to 0$ as $\alpha \to \infty$. Additionally, $\tilde{h}$ converges more speedily as α increases when $\varphi = 1$ i.e. $||MN_{(mn)}(\tilde{h}; y) - \tilde{h}(y)||_\infty \to 0$ as $\alpha \to \infty$.
- From Table 3 and Figures 3(a) to 3(f), we observe that the rate of convergence of $\tilde{h}$ is faster as α increases when $\vartheta = 1$, i.e. $||MN_{(mn)}(\tilde{h}; y) - \tilde{h}(y)||_1 \to 0$ as $\alpha \to \infty$. Additionally, from Table 2 and Figures 2(a) to 2(f), $\tilde{h}$ converges more speedily as α increases when $\vartheta = 2(\vartheta > 1)$, i.e. $||MN_{(mn)}(\tilde{h}; y) - \tilde{h}(y)||_\vartheta \to 0$ as $\alpha \to \infty$.
- The results obtained in this paper are much better than those obtained earlier.

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