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On Bihypernomials Related to Balancing and Chebyshev Polynomials

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Abstract. In this paper, we introduce and study balancing and Lucas-balancing bihypernomials as a generalization of bihyperbolic balancing and Lucas-balancing numbers. Moreover, we investigate properties of some types of Chebyshev bihypernomials and relations between them.
Key Words and Phrases: balancing numbers, bihyperbolic numbers, Chebyshev polynomials, balancing polynomials.
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1. Introduction and preliminary results

Let $n \geq 0$ be an integer. The Chebyshev polynomials of the first kind are defined by

$$\begin{aligned} T_n(x) &= \cos(n \arccos x),\ x \in [-1,1] \\ &= \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right),\ x \in \mathbb{C}. \end{aligned}$$

The Chebyshev polynomials of the second kind are defined by

$$\begin{aligned} U_n(x) &= \frac{\sin\left((n+1)\arccos x\right)}{\sin(\arccos x)},\ x \in [-1,1] \\ &= \frac{1}{2}\frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{\sqrt{x^2-1}},\ x \in \mathbb{C},\ |x| \neq 1. \end{aligned}$$

Chebyshev polynomials of the first and second kind may be also defined by recurrences. For $n \geq 2$ we have

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

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with

$$T_0(x) = 1,\ T_1(x) = x$$

and

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

with

$$U_0(x) = 1,\ U_1(x) = 2x.$$

More details about Chebyshev polynomials can be found for example in [4, 14, 19]. Two other families of polynomials

$$V_n = \frac{\cos\left((n+\frac{1}{2})\arccos x\right)}{\cos(\frac{1}{2}\arccos x)}$$

and

$$W_n = \frac{\sin\left((n+\frac{1}{2})\arccos x\right)}{\sin(\frac{1}{2}\arccos x)}$$

are named (see [10, 14]) as the third- and fourth-kind Chebyshev polynomials, respectively. Moreover, for $n \geq 2$ we have

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x)$$

with

$$V_0(x) = 1,\ V_1(x) = 2x - 1$$

and

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x)$$

with

$$W_0(x) = 1,\ W_1(x) = 2x + 1.$$

As we can see, polynomials $V_n(x)$ and $W_n(x)$ share the same recurrence relation as $T_n(x)$ and $U_n(x)$, and they differ only in the initial condition for $n = 1$.

In the literature, we can find many types of polynomials related to Chebyshev polynomials. In [11, 12], Li studied properties of Chebyshev polynomials and relationships between Chebyshev polynomials, Fibonacci polynomials, and their rth derivatives. The Pell and Pell-Lucas polynomials are modified Chebyshev polynomials with the complex variable, see [8]. Balancing and Lucas-balancing polynomials are ,,rescaled" Chebyshev polynomials, see [9]. In this paper, we will use Chebyshev, balancing and Lucas-balancing polynomials in the theory of bihypernomials.

The term bihypernomial was used for the first time in [21]. The authors introduced and studied the Fibonacci and Lucas bihypernomials as a generalization of bihyperbolic numbers. In [22], we can find properties of Pell and Pell-Lucas bihypernomials. Bihyperbolic numbers of the Fibonacci type (among others Fibonacci, Pell and Pell-Lucas bihyperbolic numbers) were examined in [6, 7]. The combinatorial properties of bihyperbolic balancing and Lucas-balancing numbers can be found in [5]. We will first recall the necessary definitions related to the bihyperbolic numbers, balancing numbers and polynomials and then define new bihypernomials.

Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + x_1 j_1 + x_2 j_2 + x_3 j_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $j_1,\ j_2,\ j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1. \tag{1}$$

The multiplication of bihyperbolic numbers can be performed analogously as the multiplication of algebraic expressions. The addition and the subtraction of bihyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients. Furthermore, $(\mathbb{H}_2, +, \cdot)$ is a commutative ring. For the algebraic properties of bihyperbolic numbers, see [3].

The sequence of balancing numbers, denoted by $\{B_n\}$, was introduced by Behera and Panda in [2]. A balancing number n with balancer r is the solution of the Diophantine equation $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$. In [2], it was proved that the balancing numbers satisfy the following recurrence relation:

$$B_n = 6B_{n-1} - B_{n-2} \quad \text{for} \quad n \geq 2 \tag{2}$$

with $B_0 = 0$, $B_1 = 1$. The sequence of balancing numbers is also given by Binet formula

$$B_n = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

where

$$r_1 = 3 + 2\sqrt{2}, \quad r_2 = 3 - 2\sqrt{2} \tag{3}$$

are the roots of the characteristic equation $r^2 - 6r + 1 = 0$, associated with the recurrence relation (2). In [15], the author introduced Lucas-balancing numbers, defined as follows: if B_n is a balancing number, then the number C_n for which $C_n^2 = 8B_n^2 + 1$ is called a Lucas-balancing number. The sequence $\{C_n\}$ of Lucas-balancing numbers is also defined by the recurrence relation

$$C_n = 6C_{n-1} - C_{n-2} \quad \text{for} \quad n \geq 2$$

with $C_0 = 1$, $C_1 = 3$.

The Binet type formula for the Lucas-balancing numbers has the following form:

$$C_n = \frac{1}{2}\left(r_1^n + r_2^n\right),$$

where r_1, r_2 are given by (3).

For nonnegative integer n, the nth bihyperbolic balancing number BhB_n and the nth bihyperbolic Lucas-balancing number BhC_n were defined as

$$BhB_n = B_n + B_{n+1}j_1 + B_{n+2}j_2 + B_{n+3}j_3,$$

$$BhC_n = C_n + C_{n+1}j_1 + C_{n+2}j_2 + C_{n+3}j_3,$$

where B_n is the nth balancing number, C_n is the nth Lucas-balancing number and j_1, j_2, j_3 are units which satisfy (1), see [5].

In the literature we can find many generalizations of balancing numbers, see [1, 13, 16, 17]. One of them is extension of numbers to polynomials, more precisely, defining for nonnegative integer n and complex x sequences of balancing polynomials $B_n(x)$ (see [20]) and Lucas-balancing polynomials $C_n(x)$ (see [18]).

Balancing polynomials are defined by the recurrence

$$B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x) \quad \text{for} \quad n \geq 2$$

with the initial terms $B_0(x) = 0$, $B_1(x) = 1$.

Lucas-balancing polynomials are defined by

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x) \quad \text{for} \quad n \geq 2$$

with $C_0(x) = 1$, $C_1(x) = 3x$.
For $x = 1$ we obtain $B_n(x) = B_n$, $C_n(x) = C_n$.

Binet type formulas for the balancing polynomials and Lucas-balancing polynomials have the following forms:

$$B_n(x) = \frac{\lambda^n(x) - \lambda^{-n}(x)}{\lambda(x) - \lambda^{-1}(x)}, \tag{4}$$

$$C_n(x) = \frac{1}{2}\left(\lambda^n(x) + \lambda^{-n}(x)\right),$$

where $\lambda(x) = 3x + \sqrt{9x^2 - 1}$ and $\lambda^{-1}(x) = 3x - \sqrt{9x^2 - 1}$.

Table 1 includes initial terms of balancing numbers, Lucas-balancing numbers, balancing type polynomials and Chebyshev type polynomials for $n = 0, 1, 2, 3, 4$.

Table 1: The balancing type numbers, balancing type polynomials and Chebyshev type polynomials.

n	0	1	2	3	4
B_n	0	1	6	35	204
C_n	1	3	17	99	577
$B_n(x)$	0	1	$6x$	$36x^2-1$	$216x^3-12x$
$C_n(x)$	1	$3x$	$18x^2-1$	$108x^3-9x$	$648x^4-72x^2+1$
$T_n(x)$	1	x	$2x^2-1$	$4x^3-3x$	$8x^4-8x^2+1$
$U_n(x)$	1	$2x$	$4x^2-1$	$8x^3-4x$	$16x^4-12x^2+1$
$V_n(x)$	1	$2x-1$	$4x^2-2x-1$	$8x^3-4x^2-4x+1$	$16x^4-8x^3-12x^2+4x+1$
$W_n(x)$	1	$2x+1$	$4x^2+2x-1$	$8x^3+4x^2-4x-1$	$16x^4+8x^3-12x^2-4x+1$

2. Bihypernomials of the balancing type

For nonnegative integer n and complex x, the nth balancing bihypernomial $BhB_n(x)$ and nth Lucas-balancing bihypernomial $BhC_n(x)$ are defined as

$$BhB_n(x) = B_n(x) + B_{n+1}(x)j_1 + B_{n+2}(x)j_2 + B_{n+3}(x)j_3, \tag{5}$$

$$BhC_n(x) = C_n(x) + C_{n+1}(x)j_1 + C_{n+2}(x)j_2 + C_{n+3}(x)j_3,$$

where $B_n(x)$ is the nth balancing polynomial, $C_n(x)$ is the nth Lucas-balancing polynomial and j_1, j_2, j_3 are units which satisfy (1).

For $x = 1$ we obtain $BhB_n(1) = BhB_n$ and $BhC_n(1) = BhC_n$.

Theorem 1. *For nonnegative integer n and complex x, we have*

$$BhB_n(x) = 6xBhB_{n-1}(x) - BhB_{n-2}(x) \ for \ n \geq 2 \tag{6}$$

with $BhB_0(x) = j_1 + 6xj_2 + (36x^2-1)j_3$
and $BhB_1(x) = 1 + 6xj_1 + (36x^2-1)j_2 + (216x^3-12x)j_3$.

Proof. For $n = 2$ we have

$$\begin{aligned} BhB_2(x) &= 6xBhB_1(x) - BhB_0(x) \\ &= 6x(1 + 6xj_1 + (36x^2-1)j_2 + (216x^3-12x)j_3) \\ &\quad - j_1 - 6xj_2 - (36x^2-1)j_3 \\ &= 6x + (36x^2-1)j_1 + (216x^3-12x)j_2 + (1296x^4-108x^2+1)j_3 \\ &= B_2(x) + B_3(x)j_1 + B_4(x)j_2 + B_5(x)j_3. \end{aligned}$$

Let $n \geq 3$. Then by the definition of balancing polynomials we have

$$\begin{aligned} BhB_n(x) &= B_n(x) + B_{n+1}(x)j_1 + B_{n+2}(x)j_2 + B_{n+3}(x)j_3 \\ &= (6xB_{n-1}(x) - B_{n-2}(x)) + (6xB_n(x) - B_{n-1}(x))j_1 \\ &\quad + (6xB_{n+1}(x) - B_n(x))j_2 + (6xB_{n+2}(x) - B_{n+1}(x))j_3 \\ &= 6x(B_{n-1}(x) + B_n(x)j_1 + B_{n+1}(x)j_2 + B_{n+2}(x)j_3) \\ &\quad - (B_{n-2}(x) + B_{n-1}(x)j_1 + B_n(x)j_2 + B_{n+1}(x)j_3) \\ &= 6xBH_{n-1}(x) - BH_{n-2}(x), \end{aligned}$$

which ends the proof. ◄

In the same way we can easily prove the next theorem.

Theorem 2. *For nonnegative integer n and complex x, we have*

$$BhC_n(x) = 6xBhC_{n-1}(x) - BhC_{n-2}(x) \text{ } for \text{ } n \geq 2$$

with $BhC_0(x) = 1 + 3xj_1 + (18x^2 - 1)j_2 + (108x^3 - 9x)j_3$
and $BhC_1(x) = 3x + (18x^2 - 1)j_1 + (108x^3 - 9x)j_2 + (648x^4 - 72x^2 + 1)j_3$.

Now, we will give Binet type formulas for balancing bihypernomials and Lucas-balancing bihypernomials.

Theorem 3. *For nonnegative integer n and complex x, $|x| \neq \frac{1}{3}$, we have*

$$\begin{aligned} BhB_n(x) = &\frac{\lambda^n(x)}{\lambda(x) - \gamma(x)} \left(1 + \lambda(x)j_1 + \lambda^2(x)j_2 + \lambda^3(x)j_3\right) \\ &- \frac{\gamma^n(x)}{\lambda(x) - \gamma(x)} \left(1 + \gamma(x)j_1 + \gamma^2(x)j_2 + \gamma^3(x)j_3\right), \end{aligned} \tag{7}$$

where $\lambda(x) = 3x + \sqrt{9x^2 - 1}$ *and* $\gamma(x) = \lambda^{-1}(x) = 3x - \sqrt{9x^2 - 1}$.

Proof. Using (5) and (4), we have

$$\begin{aligned} BhB_n(x) &= B_n(x) + B_{n+1}(x)j_1 + B_{n+2}(x)j_2 + B_{n+3}(x)j_3 \\ &= \frac{\lambda^n(x) - \gamma^n(x)}{\lambda(x) - \gamma(x)} + \frac{\lambda^{n+1}(x) - \gamma^{n+1}(x)}{\lambda(x) - \gamma(x)} j_1 \\ &\quad + \frac{\lambda^{n+2}(x) - \gamma^{n+2}(x)}{\lambda(x) - \gamma(x)} j_2 + \frac{\lambda^{n+3}(x) - \gamma^{n+3}(x)}{\lambda(x) - \gamma(x)} j_3 \\ &= \frac{\lambda^n(x)}{\lambda(x) - \gamma(x)} (1 + \lambda(x)j_1 + \lambda^2(x)j_2 + \lambda^3(x)j_3) \\ &\quad - \frac{\gamma^n(x)}{\lambda(x) - \gamma(x)} (1 + \gamma(x)j_1 + \gamma^2(x)j_2 + \gamma^3(x)j_3), \end{aligned}$$

which ends the proof. ◄

Theorem 4. *For nonnegative integer n and complex x, we have*

$$BhC_n(x) = \frac{1}{2}\left[\lambda^n(x)\left(1+\lambda(x)j_1+\lambda^2(x)j_2+\lambda^3(x)j_3\right) \right. \\ \left. +\gamma^n(x)\left(1+\gamma(x)j_1+\gamma^2(x)j_2+\gamma^3(x)j_3\right)\right], \tag{8}$$

where $\lambda(x) = 3x+\sqrt{9x^2-1}$ and $\gamma(x) = \lambda^{-1}(x) = 3x-\sqrt{9x^2-1}$.

Using Binet type formulas for balancing bihypernomials and Lucas-balancing bihypernomials we will obtain general bilinear index-reduction formulas for these bihypernomials.

For simplicity of notation let

$$\hat{\lambda}(x) = 1+\lambda(x)j_1+\lambda^2(x)j_2+\lambda^3(x)j_3, \tag{9}$$

$$\hat{\gamma}(x) = 1+\gamma(x)j_1+\gamma^2(x)j_2+\gamma^3(x)j_3. \tag{10}$$

Thus, we can write (7) and (8) as

$$BhB_n(x) = \frac{\lambda^n(x)\hat{\lambda}(x)-\gamma^n(x)\hat{\gamma}(x)}{2\sqrt{9x^2-1}}, \tag{11}$$

$$BhC_n(x) = \frac{\lambda^n(x)\hat{\lambda}(x)+\gamma^n(x)\hat{\gamma}(x)}{2}, \tag{12}$$

respectively.

Theorem 5. *(general bilinear index-reduction formula for balancing bihypernomials) Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b = c+d$. Then for complex x, $|x| \neq \frac{1}{3}$, we have*

$$BhB_a(x)\cdot BhB_b(x) - BhB_c(x)\cdot BhB_d(x) \\ = \frac{\hat{\lambda}(x)\hat{\gamma}(x)}{36x^2-4}\left(\lambda^c(x)\gamma^d(x)-\lambda^a(x)\gamma^b(x)+\gamma^c(x)\lambda^d(x)-\gamma^a(x)\lambda^b(x)\right),$$

where $\hat{\lambda}(x)$, $\hat{\gamma}(x)$ are given by (9), (10), respectively.

Proof. By formula (11) we get

$$BhB_a(x)\cdot BhB_b(x) - BhB_c(x)\cdot BhB_d(x) \\ = \frac{1}{36x^2-4}\left(\lambda^a(x)\hat{\lambda}(x)-\gamma^a(x)\hat{\gamma}(x)\right)\left(\lambda^b(x)\hat{\lambda}(x)-\gamma^b(x)\hat{\gamma}(x)\right) \\ \cdot\left(\lambda^c(x)\hat{\lambda}(x)-\gamma^c(x)\hat{\gamma}(x)\right)\left(\lambda^d(x)\hat{\lambda}(x)-\gamma^d(x)\hat{\gamma}(x)\right) \\ = \frac{1}{36x^2-4}\left[\lambda^{a+b}(x)(\hat{\lambda}(x))^2-\lambda^a(x)\gamma^b(x)\hat{\lambda}(x)\hat{\gamma}(x)-\gamma^a(x)\lambda^b(x)\hat{\gamma}(x)\hat{\lambda}(x)\right. \\ +\gamma^{a+b}(x)(\hat{\gamma}(x))^2-\lambda^{c+d}(x)(\hat{\lambda}(x))^2+\lambda^c(x)\gamma^d(x)\hat{\lambda}(x)\hat{\gamma}(x) \\ \left.+\gamma^c(x)\lambda^d(x)\hat{\gamma}(x)\hat{\lambda}(x)-\gamma^{c+d}(x)(\hat{\gamma}(x))^2\right].$$

Since $a+b=c+d$ and $\hat{\lambda}(x)\cdot\hat{\gamma}(x)=\hat{\gamma}(x)\cdot\hat{\lambda}(x)$, we get

$$\begin{aligned}&BhB_a(x)\cdot BhB_b(x)-BhB_c(x)\cdot BhB_d(x)\\&=\frac{\hat{\lambda}(x)\hat{\gamma}(x)}{36x^2-4}\left(\lambda^c(x)\gamma^d(x)-\lambda^a(x)\gamma^b(x)+\gamma^c(x)\lambda^d(x)-\gamma^a(x)\lambda^b(x)\right).\end{aligned}$$

◀

Theorem 6. *(general bilinear index-reduction formula for Lucas-balancing bihypernomials) Let $a\geq 0$, $b\geq 0$, $c\geq 0$, $d\geq 0$ be integers such that $a+b=c+d$. Then*

$$\begin{aligned}&BhC_a(x)\cdot BhC_b(x)-BhC_c(x)\cdot BhC_d(x)\\&=\frac{\hat{\lambda}(x)\hat{\gamma}(x)}{4}\left(\lambda^a(x)\gamma^b(x)-\lambda^c(x)\gamma^d(x)+\gamma^a(x)\lambda^b(x)-\gamma^c(x)\lambda^d(x)\right),\end{aligned}$$

where $\hat{\lambda}(x)$, $\hat{\gamma}(x)$ are given by (9), (10), respectively.

Proof. By (12) we get

$$\begin{aligned}&BhC_a(x)\cdot BhC_b(x)-BhC_c(x)\cdot BhC_d(x)\\&=\frac{1}{4}\Big[\lambda^{a+b}(x)(\hat{\lambda}(x))^2+\lambda^a(x)\gamma^b(x)\hat{\lambda}(x)\hat{\gamma}(x)+\lambda^b(x)\gamma^a(x)\hat{\gamma}(x)\hat{\lambda}(x)\\&+\gamma^{a+b}(x)(\hat{\gamma}(x))^2-\lambda^{c+d}(x)(\hat{\lambda}(x))^2-\lambda^c(x)\gamma^d(x)\hat{\lambda}(x)\hat{\gamma}(x)\\&-\lambda^d(x)\gamma^c(x)\hat{\gamma}(x)\hat{\lambda}(x)-\gamma^{c+d}(x)(\hat{\gamma}(x))^2\Big].\end{aligned}$$

Since $a+b=c+d$ and $\hat{\lambda}(x)\cdot\hat{\gamma}(x)=\hat{\gamma}(x)\cdot\hat{\lambda}(x)$, we get the result. ◀

For special values of a,b,c,d, by Theorems 5-6, we can obtain some identities for balancing and Lucas-balancing bihypernomials:

- d'Ocagne type identity – for $a=n$, $b=m+1$, $c=n+1$, $d=m$,
- Vajda type identity – for $a=m+r$, $b=n-r$, $c=m$, $d=n$,
- first Halton type identity – for $a=m+r$, $b=n$, $c=r$, $d=m+n$,
- second Halton type identity – for $a=n+k$, $b=n-k$, $c=n+s$, $d=n-s$,
- Catalan type identity – for $a=n+r$, $b=n-r$, $c=d=n$,
- Cassini type identity – for $a=n+1$, $b=n-1$, $c=d=n$.

Putting $x = 1$ we can obtain general bilinear index-reduction formulas and d'Ocagne, Vajda, Halton, Catalan, Cassini type identities for balancing and Lucas-balancing bihyperbolic numbers.

Now, we will give the generating functions for balancing and Lucas-balancing bihypernomials.

Theorem 7. *The generating function for balancing bihypernomial sequence* $\{BhB_n(x)\}$ *is*

$$G(t) = \frac{BhB_0(x) + (BhB_1(x) - 6xBhB_0(x))t}{1 - 6xt + t^2},$$

where $BhB_0(x) = j_1 + 6xj_2 + (36x^2 - 1)j_3$
and $BhB_1(x) - 6xBhB_0(x) = 1 - j_2 - 6xj_3$.

Proof. Assume that the generating function of the balancing bihypernomial sequence $\{BhB_n(x)\}$ has the form $G(t) = \sum\limits_{n=0}^{\infty} BhB_n(x)t^n$. Then

$$G(t) = BhB_0(x) + BhB_1(x)t + BhB_2(x)t^2 + \dots$$

Hence we get

$$-6xt \cdot G(t) = -6xBhB_0(x)t - 6xBhB_1(x)t^2 - 6xBhB_2(x)t^3 - \dots$$
$$t^2 \cdot G(t) = BhB_0(x)t^2 + BhB_1(x)t^3 + BhB_2(x)t^4 + \dots$$

By adding these three equalities above, we get

$$G(t)(1 - 6xt + t^2) = BhB_0(x) + (BhB_1(x) - 6xBhB_0(x))t$$

since $BhB_n(x) = 6BhB_{n-1}(x) - BhB_{n-2}(x)$ (see (6)) and the coefficients of t^n for $n \geq 2$ are equal to zero. Moreover, by simple calculations we have

$$BhB_1(x) - 6xBhB_0(x) = 1 - j_2 - 6xj_3.$$

◀

In the same way we can prove the next result.

Theorem 8. *The generating function for Lucas-balancing bihypernomial sequence* $\{BhC_n(x)\}$ *is*

$$g(t) = \frac{BhC_0(x) + (BhC_1(x) - 6xBhC_0(x))t}{1 - 6xt + t^2},$$

where $BhC_0(x) = 1 + 3xj_1 + (18x^2 - 1)j_2 + (108x^3 - 9x)j_3$
and $BhC_1(x) - 6xBhC_0(x) = -3x - j_1 - 3xj_2 + (1 - 18x^2)j_3$.

Next, we will give the matrix representation of balancing and Lucas-balancing bihypernomials.

Theorem 9. *For positive integer n and complex x we have*

$$\begin{bmatrix} BhB_{n+1}(x) & -BhB_n(x) \\ BhB_n(x) & -BhB_{n-1}(x) \end{bmatrix} = \begin{bmatrix} BhB_2(x) & -BhB_1(x) \\ BhB_1(x) & -BhB_0(x) \end{bmatrix} \cdot \begin{bmatrix} 6x & -1 \\ 1 & 0 \end{bmatrix}^{n-1}. \tag{13}$$

Proof. (by induction on n) If $n = 1$, then, assuming that the matrix to the power of 0 is the identity matrix, the result is obvious. Assuming that the formula (13) holds for $n \geq 1$, we shall prove it for $n + 1$. Using induction hypothesis and formula (6), we have

$$\begin{bmatrix} BhB_2(x) & -BhB_1(x) \\ BhB_1(x) & -BhB_0(x) \end{bmatrix} \cdot \begin{bmatrix} 6x & -1 \\ 1 & 0 \end{bmatrix}^{n}$$

$$= \begin{bmatrix} BhB_{n+1}(x) & -BhB_n(x) \\ BhB_n(x) & -BhB_{n-1}(x) \end{bmatrix} \cdot \begin{bmatrix} 6x & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6xBhB_{n+1}(x) - BhB_n(x) & -BhB_{n+1}(x) \\ 6xBhB_n(x) - BhB_{n-1}(x) & -BhB_n(x) \end{bmatrix}$$

$$= \begin{bmatrix} BhB_{n+2}(x) & -BhB_{n+1}(x) \\ BhB_{n+1}(x) & -BhB_n(x) \end{bmatrix},$$

which ends the proof. ◀

In the same way we can prove the next theorem.

Theorem 10. *For positive integer n and complex x we have*

$$\begin{bmatrix} BhC_{n+1}(x) & -BhC_n(x) \\ BhC_n(x) & -BhC_{n-1}(x) \end{bmatrix} = \begin{bmatrix} BhC_2(x) & -BhC_1(x) \\ BhC_1(x) & -BhC_0(x) \end{bmatrix} \cdot \begin{bmatrix} 6x & -1 \\ 1 & 0 \end{bmatrix}^{n-1}.$$

Note that multiplication of bihyperbolic numbers and bihypernomials is commutative and determinant properties can be used. For example, calculating determinants of matrices in Theorems 9-10, we can also obtain Cassini identities. The use of algebraic operations and matrix algebra could give many other interesting properties of these bihypernomials.

3. Bihypernomials of the Chebyshev type

For nonnegative integer n and complex x, the nth Chebyshev bihypernomial of the first, second, third and fourth kind is defined by

$$BhT_n(x) = T_n(x) + T_{n+1}(x)j_1 + T_{n+2}(x)j_2 + T_{n+3}(x)j_3, \tag{14}$$

$$BhU_n(x) = U_n(x) + U_{n+1}(x)j_1 + U_{n+2}(x)j_2 + U_{n+3}(x)j_3,$$

$$BhV_n(x) = V_n(x) + V_{n+1}(x)j_1 + V_{n+2}(x)j_2 + V_{n+3}(x)j_3,$$

$$BhW_n(x) = W_n(x) + W_{n+1}(x)j_1 + W_{n+2}(x)j_2 + W_{n+3}(x)j_3,$$

respectively, where $T_n(x)$ is the nth Chebyshev polynomial of the first kind, $U_n(x)$ is the nth Chebyshev polynomial of the second kind, $V_n(x)$ is the nth Chebyshev polynomial of the third kind, $W_n(x)$ is the nth Chebyshev polynomial of the fourth kind and j_1, j_2, j_3 are units which satisfy (1).

The use of trigonometric relationships makes it possible to obtain dependencies between Chebyshev polynomials

$$T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x)),\ n = 2, 3, \ldots$$

$$V_n(x) = U_n(x) - U_{n-1}(x),\ n = 1, 2, \ldots$$

$$W_n(x) = U_n(x) + U_{n-1}(x),\ n = 1, 2, \ldots$$

see [14]. Using these properties, it is easy to show relationships between Chebyshev bihypernomials

$$BhT_n(x) = \frac{1}{2}(BhU_n(x) - BhU_{n-2}(x)),\ n = 2, 3, \ldots$$

$$BhV_n(x) = BhU_n(x) - BhU_{n-1}(x),\ n = 1, 2, \ldots$$

$$BhW_n(x) = BhU_n(x) + BhU_{n-1}(x),\ n = 1, 2, \ldots$$

In the next part of this section, we will examine Chebyshev bihypernomials of the second kind. The proofs of the theorems are the same as for the balancing bihypernomials, so we omit them. For nonnegative integer n and complex x, $|x| \neq 1$, we have

$$U_n(x) = \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)},$$

where

$$\alpha(x) = x + \sqrt{x^2 - 1},\ \beta(x) = x - \sqrt{x^2 - 1}, \tag{15}$$

see [14].

Theorem 11. *(Binet formula for Chebyshev bihypernomials of the second kind) For nonnegative integer n and complex x, $|x| \neq 1$, we have*

$$\begin{aligned} BhU_n(x) = & \frac{\alpha^{n+1}(x)}{\alpha(x)-\beta(x)} \left(1+\alpha(x)j_1+\alpha^2(x)j_2+\alpha^3(x)j_3\right) + \\ & - \frac{\beta^{n+1}(x)}{\alpha(x)-\beta(x)} \left(1+\beta(x)j_1+\beta^2(x)j_2+\beta^3(x)j_3\right), \end{aligned} \tag{16}$$

where $\alpha(x)$, $\beta(x)$ are given by (15).

For simplicity of notation let

$$\begin{aligned} \hat{\alpha}(x) &= 1+\alpha(x)j_1+\alpha^2(x)j_2+\alpha^3(x)j_3, \\ \hat{\beta}(x) &= 1+\beta(x)j_1+\beta^2(x)j_2+\beta^3(x)j_3. \end{aligned} \tag{17}$$

Then we can write (16) as

$$BhU_n(x) = \frac{\alpha^{n+1}(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x) - \frac{\beta^{n+1}(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x).$$

Using facts that $\alpha(x)-\beta(x) = 2\sqrt{x^2-1}$ and $\alpha(x)\cdot\beta(x) = 1$, one can easily prove the next theorem.

Theorem 12. *(general bilinear index-reduction formula for Chebyshev bihypernomials of the second kind) Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b=c+d$. Then for complex x, $|x| \neq 1$, we have*

$$\begin{aligned} & BhU_a(x)\cdot BhU_b(x) - BhU_c(x)\cdot BhU_d(x) \\ & = \frac{\hat{\alpha}(x)\hat{\beta}(x)}{4x^2-4}\Big(\alpha^c(x)\beta^d(x) - \alpha^a(x)\beta^b(x) + \beta^c(x)\alpha^d(x) - \beta^a(x)\alpha^b(x)\Big), \end{aligned}$$

where $\hat{\alpha}(x)$, $\hat{\beta}(x)$ are given by (17).

Theorem 13. *The generating function for the sequence $\{BhU_n(x)\}$ is*

$$h(t) = \frac{BhU_0(x) + (BhU_1(x) - 2xBhU_0(x))t}{1-2xt+t^2},$$

where $BhU_0(x) = 1+2xj_1+(4x^2-1)j_2+(8x^3-4x)j_3$ and $BhU_1(x) - 2xBhU_0(x) = -j_1 - 2xj_2 + (-4x^2+1)j_3$.

Theorem 14. *For positive integer n and complex x we have*

$$\begin{aligned} & \begin{bmatrix} BhU_{n+1}(x) & -BhU_n(x) \\ BhU_n(x) & -BhU_{n-1}(x) \end{bmatrix} \\ & = \begin{bmatrix} BhU_2(x) & -BhU_1(x) \\ BhU_1(x) & -BhU_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix}^{n-1}. \end{aligned}$$

4. Some identities and relations between bihypernomials

In this section, we will present some identities and relations between the previously defined bihypernomials. First, we give the known properties of polynomials, and then their bihypernomials versions.

Theorem 15. [9] *Let n be an integer, $n \geq 1$. Then*

$$C_n(x) = B_{n+1}(x) - 3xB_n(x), \tag{18}$$

$$C_n(x) = \frac{1}{2}(B_{n+1}(x) - B_{n-1}(x)), \tag{19}$$

$$C_n(x) = 3xB_n(x) - B_{n-1}(x), \tag{20}$$

$$C_n(x) = 3xC_{n-1}(x) + (9x^2 - 1)B_{n-1}(x). \tag{21}$$

Theorem 16. *Let $n \geq 0$ be an integer. Then*

$$BhC_n(x) = BhB_{n+1}(x) - 3xBhB_n(x).$$

Proof. By formula (18) we have

$$\begin{aligned}
&BhB_{n+1}(x) - 3xBhB_n(x)\\
&= B_{n+1}(x) + B_{n+2}(x)j_1 + B_{n+3}(x)j_2 + B_{n+4}(x)j_3\\
&\ - 3x\left(B_n(x) + B_{n+1}(x)j_1 + B_{n+2}(x)j_2 + B_{n+3}(x)j_3\right)\\
&= B_{n+1}(x) - 3xB_n(x) + (B_{n+2}(x) - 3xB_{n+1}(x))j_1\\
&\ + (B_{n+3}(x) - 3xB_{n+2}(x))j_2 + (B_{n+4}(x) - 3xB_{n+3}(x))j_3\\
&= C_n(x) + C_{n+1}(x)j_1 + C_{n+2}(x)j_2 + C_{n+3}(x)j_3 = BhC_n(x).
\end{aligned}$$

◀

Using (19)-(21) we can prove the following results.

Theorem 17. *Let $n \geq 1$ be an integer. Then*

$$BhC_n(x) = \frac{1}{2}(BhB_{n+1}(x) - BhB_{n-1}(x)),$$

$$BhC_n(x) = 3xBhB_n(x) - BhB_{n-1}(x),$$

$$BhC_n(x) = 3xBhC_{n-1}(x) + (9x^2 - 1)BhB_{n-1}(x).$$

It is easy to prove the following results.

Lemma 1. *Let $n \geq 0$ be an integer. Then*

$$\sum_{l=0}^{n} B_l(x) = \frac{(6x-1)B_n(x) - B_{n-1}(x) - 1}{6x-2}, \tag{22}$$

$$\sum_{l=0}^{n} C_l(x) = \frac{C_{n+1}(x) - C_n(x)}{6x-2} + \frac{1}{2}. \tag{23}$$

Theorem 18. *Let $n \geq 0$ be an integer. Then*

$$\sum_{l=0}^{n} BhB_l(x)$$
$$= \frac{BhB_{n+1}(x) - BhB_n(x) - [1 + j_1 + (6x-1)j_2 + (36x^2 - 6x - 1)j_3]}{6x-2}.$$

Proof. By (5) we get

$$\begin{aligned}
&\sum_{l=0}^{n} BhB_l(x) = BhB_0(x) + BhB_1(x) + \ldots + BhB_n(x)\\
&= B_0(x) + B_1(x)j_1 + B_2(x)j_2 + B_3(x)j_3\\
&\ + B_1(x) + B_2(x)j_1 + B_3(x)j_2 + B_4(x)j_3 + \cdots\\
&\ + B_n(x) + B_{n+1}(x)j_1 + B_{n+2}(x)j_2 + B_{n+3}(x)j_3\\
&= B_0(x) + B_1(x) + \cdots + B_n(x)\\
&\ + (B_1(x) + B_2(x) + \cdots + B_{n+1}(x) + B_0(x) - B_0(x))j_1\\
&\ + (B_2(x) + B_3(x) + \cdots + B_{n+2}(x) + B_0(x) + B_1(x)\\
&\ - B_0(x) - B_1(x))j_2\\
&\ + (B_3(x) + B_4(x) + \cdots + B_{n+3}(x) + B_0(x) + B_1(x) + B_2(x)\\
&\ - B_0(x) - B_1(x) - B_2(x))j_3.
\end{aligned}$$

By (22) we have

$$\sum_{l=0}^{n} B_l(x) = \frac{B_{n+1}(x) - B_n(x) - 1}{6x-2}.$$

Hence we get

$$\sum_{l=0}^{n} BhB_l(x) = \frac{B_{n+1}(x) - B_n(x) - 1}{6x-2}$$
$$+ \left(\frac{B_{n+2}(x) - B_{n+1}(x) - 1}{6x-2} - B_0(x)\right) j_1$$
$$+ \left(\frac{B_{n+3}(x) - B_{n+2}(x) - 1}{6x-2} - B_0(x) - B_1(x)\right) j_2$$
$$+ \left(\frac{B_{n+4}(x) - B_{n+3}(x) - 1}{6x-2} - B_0(x) - B_1(x) - B_2(x)\right) j_3$$
$$= \frac{1}{6x-2}\big[B_{n+1}(x) + B_{n+2}(x)j_1 + B_{n+3}(x)j_2 + B_{n+4}(x)j_3)$$
$$- (B_n(x) + B_{n+1}(x)j_1 + B_{n+2}(x)j_2 + B_{n+3}(x)j_3)$$
$$- (1 + j_1 + (6x-1)j_2 + (36x^2 - 6x - 1)j_3)\big]$$
$$= \frac{1}{6x-2}[BhB_{n+1}(x) - BhB_n(x)$$
$$- (1 + j_1 + (6x-1)j_2 + (36x^2 - 6x - 1)j_3)].$$

◀

In the same way, using (23), we can prove the following result.

Theorem 19. *Let $n \geq 0$ be an integer. Then*

$$\sum_{l=0}^{n} BhC_l(x) = \frac{BhC_{n+1}(x) - BhC_n(x)}{6x-2}$$
$$+ \frac{1}{2}[1 - j_1 - (6x+1)j_2 - (36x^2 + 6x - 1)j_3].$$

We will use the following results, see [14].

Theorem 20. [14] *For nonnegative integer n and complex x we have*

(i) $V_n(x) + W_n(x) = 2U_n(x)$,

(ii) $V_n(x) + V_{n-1}(x) = 2T_n(x),\ n = 1, 2, \ldots$

(iii) $W_n(x) - W_{n-1}(x) = 2T_n(x),\ n = 1, 2, \ldots$

(iv) $T_n(x) + T_{n-1}(x) = (1+x)V_{n-1}(x),\ n = 1, 2, \ldots$

(v) $T_n(x) - T_{n-1}(x) = (x-1)W_{n-1}(x),\ n = 1, 2, \ldots$

(vi) $T_n(x) - T_{n-2}(x) = 2(x^2 - 1)U_{n-2}(x),\ n = 2, 3, \ldots$

Theorem 21. *For nonnegative integer n and complex x we have*

(i) $BhV_n(x) + BhW_n(x) = 2BhU_n(x)$,

(ii) $BhV_n(x) + BhV_{n-1}(x) = 2BhT_n(x),\ n = 1, 2, \ldots$

(iii) $BhW_n(x) - BhW_{n-1}(x) = 2BhT_n(x),\ n = 1, 2, \ldots$

(iv) $BhT_n(x) + BhT_{n-1}(x) = (1 + x)BhV_{n-1}(x),\ n = 1, 2, \ldots$

(v) $BhT_n(x) - BhT_{n-1}(x) = (x - 1)BhW_{n-1}(x),\ n = 1, 2, \ldots$

(vi) $BhT_n(x) - BhT_{n-2}(x) = 2(x^2 - 1)BhU_{n-2}(x),\ n = 2, 3, \ldots$

Proof. We will present proof of (iv), the rest of the proofs are similar. Let n be an integer, $n \geq 1$. Using (14) and equality (iv) of Theorem 20, we have

$$\begin{aligned}
&BhT_n(x) + BhT_{n-1}(x) = T_n(x) + T_{n+1}(x)j_1 + T_{n+2}(x)j_2 + T_{n+3}(x)j_3 \\
&\ + T_{n-1}(x) + T_n(x)j_1 + T_{n+1}(x)j_2 + T_{n+2}(x)j_3 \\
&= (1 + x)V_{n-1}(x) + (1 + x)V_n(x)j_1 + (1 + x)V_{n+1}(x)j_2 + (1 + x)V_{n+2}(x)j_3 \\
&= (1 + x)BhV_{n-1}(x),
\end{aligned}$$

which ends the proof of (iv). ◀

At the end, we give summation formulas for the Chebyshev bihypernomials of the first kind. We will use the following theorem.

Theorem 22. [14]

$$T_0(x) + T_2(x) + T_4(x) + \cdots + T_{2n}(x) = \frac{1}{2}U_{2n}(x) + \frac{1}{2}, \tag{24}$$

$$T_1(x) + T_3(x) + \cdots + T_{2n+1}(x) = \frac{1}{2}U_{2n+1}(x). \tag{25}$$

Theorem 23. *Let $n \geq 0$. Then*

$$\sum_{l=0}^{n} BhT_{2l}(x) = \frac{1}{2}BhU_{2n}(x) + \frac{1}{2} - \frac{1}{2}\left(j_2 + 2xj_3\right).$$

Proof. Using (14), (24) and (25) we have

$$\begin{aligned}
&\sum_{l=0}^{n} BhT_{2l}(x) = BhT_0(x) + BhT_2(x) + \cdots + BhT_{2n}(x) \\
&= (T_0(x) + T_1(x)j_1 + T_2(x)j_2 + T_3(x)j_3) \\
&\ + (T_2(x) + T_3(x)j_1 + T_4(x)j_2 + T_5(x)j_3) + \cdots \\
&\ + (T_{2n}(x) + T_{2n+1}(x)j_1 + T_{2n+2}(x)j_2 + T_{2n+3}(x)j_3) \\
&= (T_0(x) + T_2(x) + \cdots + T_{2n}(x)) \\
&\ + (T_1(x) + T_3(x) + \cdots + T_{2n+1}(x))j_1 \\
&\ + (T_2(x) + T_4(x) + \cdots + T_{2n+2}(x) + T_0(x) - T_0(x))j_2 \\
&\ + (T_3(x) + T_5(x) + \cdots + T_{2n+3}(x) + T_1(x) - T_1(x))j_3 \\
&= \frac{1}{2}U_{2n}(x) + \frac{1}{2} + \frac{1}{2}U_{2n+1}(x)j_1 + \left(\frac{1}{2}U_{2n+2}(x) + \frac{1}{2} - 1\right)j_2 + \left(\frac{1}{2}U_{2n+3}(x) - x\right)j_3 \\
&= \frac{1}{2}U_{2n}(x) + \frac{1}{2} + \frac{1}{2}U_{2n+1}(x)j_1 + \left(\frac{1}{2}U_{2n+2}(x) - \frac{1}{2}\right)j_2 + \left(\frac{1}{2}U_{2n+3}(x) - x\right)j_3 \\
&= \frac{1}{2}\left(U_{2n}(x) + U_{2n+1}(x)j_1 + \frac{1}{2}U_{2n+2}(x)j_2 + \frac{1}{2}U_{2n+3}(x)j_3\right) + \frac{1}{2} - \frac{1}{2}j_2 - xj_3 \\
&= \frac{1}{2}BhU_{2n}(x) + \frac{1}{2} - \frac{1}{2}(j_2 + 2xj_3),
\end{aligned}$$

which ends the proof. ◀

In the same way, we can prove the next theorem.

Theorem 24. *Let $n \geq 0$. Then*

$$\sum_{l=0}^{n} BhT_{2l+1}(x) = \frac{1}{2}BhU_{2n+1}(x) - \frac{1}{2}\left(j_1 + 2xj_2 + \left(4x^2 - 1\right)j_3\right).$$

5. Concluding remarks

As it was mentioned in Introduction, balancing and Lucas-balancing polynomials are ,,rescaled" Chebyshev polynomials, more precisely, for positive integer n we have the following relations: $B_n(x) = U_{n-1}(3x)$ and $C_n(x) = T_n(3x)$, see [9]. Using these and other relationships between Fibonacci type polynomials and Chebyshev type polynomials, we can obtain new properties of defined bihypernomials.

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