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Continuous Relay Fusion Frame in Hilbert Spaces

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Abstract. In this paper we introduced the concept of continuous relay fusion frames in Hilbert spaces and we define the dual frames for continuous relay fusion frames. Finally we study the perturbation problem of continuous relay fusion frames.

Key Words and Phrases: continuous frame, continuous fusion frame, continuous relay fusion frame, Hilbert space.

2010 Mathematics Subject Classifications: 41A58, 42C15

1. Introduction

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaffer [3] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [2] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames.

Continuous frames were proposed by G. Kaiser [7] and independently by Ali, Antoine, and Gazeau [1] to a family indexed by some locally compact space endowed with a Radon measure. Gabrado and Han [5] called these frames as the ones associated with measurable spaces.

Let H, L be separable Hilbert spaces and let $B(H, L)$ be the space of all the bounded linear operators from H to L (if $H = L$, we write $B(H)$). Let (Ω, μ) be a positive measure space.

If $W \subseteq H$ and $V \subseteq L$ are subspaces, then we let $\pi_W \in B(H)$ and $P_V \in B(L)$ denote the orthogonal projections onto the subspaces W and V, respectively.

In this section we briefly recall the definitions of continuous frames and continuous fusion frames in Hilbert space.

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Definition 1. *[8] Let H be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ. A mapping $F : \Omega \to H$ is called a continuous frame with respect to (Ω, μ), if*

(1) F is weakly-measurable, i.e., for all $f \in H$, $w \to \langle f, F(w)\rangle$ is a measurable function on Ω.

(2) there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(w)\rangle|^2 d\mu(w) \leq B\|f\|^2, \quad \forall f \in H. \tag{1}$$

The constants A and B are called continuous frame bounds. F is called a tight continuous frame if $A = B$. The mapping F is called Bessel if the second inequality in (1) holds. In this case, B is called the Bessel constant. If μ is a counting measure and $\Omega = \mathbb{N}$, F is called a discrete frame.

Definition 2. *[4] Let $\{W_w\}_{w\in\Omega}$ be a family of closed subspaces of Hilbert space H and (Ω, μ) be a measure space with positive measure μ and $\nu : \Omega \to \mathbb{R}^+$. Then $\{W_w, \nu_w\}_{w\in\Omega}$ is called a continuous fusion frame with respect to (Ω, μ) and ν, if*

(1) for each $f \in H$, $\{\pi_{W_w} f\}_{w\in\Omega}$ is strongly measurable and ν is a measurable function from Ω to $\mathbb{R}^+$.

(2) there are two constants $0 < C, D < \infty$ such that

$$C\|f\|^2 \leq \int_\Omega \nu_w^2 \|\pi_{W_w} f\|^2 d\mu(w) \leq D\|f\|^2, \quad \forall f \in H,$$

where π_{W_w} is the orthogonal projection onto the subspace W_w. We call C and D the lower and the upper continuous fusion frame bounds, respectively.

2. Continuous relay fusion frame in Hilbert spaces

Definition 3. *Let $\{K_w\}_{w\in\Omega}$ be a sequence of separable Hilbert spaces and $\{W_w\}_{w\in\Omega}$ be a family of closed subspaces in H for each $w \in \Omega$. Let $\{V_{w,v}\}_{v\in\Omega_w}$ be a family of closed subspaces in K_w. Let $\{\alpha_{w,v}\}_{w\in\Omega,v\in\Omega_w}$ be a family of weights, i.e. $\alpha_{w,v} > 0$ for each $w \in \Omega$, $v \in \Omega_w$, and let $\Lambda_w \in B(H, K_w)$ for each $w \in \Omega$. Then $\{W_w, V_{w,v}, \nu_{w,v}\}_{w\in\Omega,v\in\Omega_w}$ is said to be a continuous relay fusion frame, if*

(1) for each $f \in H$, $\{\Lambda_w f\}_{w\in\Omega}$ is strongly measurable.

(2) for each $f \in H$, $\{\pi_{W_w} f\}_{w\in\Omega}$ is strongly measurable.

(3) for each $f \in K_w$, $\{P_{V_{w,v}} f\}_{w\in\Omega, v\in\Omega_w}$ *is strongly measurable.*

(4) there exist constants $0 < A \leq B < \infty$ *such that*

$$A\|f\|^2 \leq \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} f\|^2 d\mu(v) d\mu(w) \leq B\|f\|^2, \quad \forall f \in H, \tag{2}$$

where $P_{V_{w,v}}$ *is the orthogonal projection onto the subspace* $V_{w,v}$*. We call* A *and* B *the lower and the upper continuous relay fusion frame bounds, respectively.*

Definition 4.

$$\left(\int_\Omega \int_{\Omega_w} \oplus V_{w,v}\right)_{l^2} = \left\{\{f_{w,v}\}_{w\in\Omega, v\in\Omega_w}, f_{w,v} \in V_{w,v}, \int_\Omega \int_{\Omega_w} \|f_{w,v}\|^2 < \infty\right\},$$

with inner product given by

$$\langle \{f_{w,v}\}_{w,v}, \{g_{w,v}\}_{w,v}\rangle = \int_\Omega \int_{\Omega_w} \langle f_{w,v}, g_{w,v}\rangle d\mu(w) d\mu(v),$$

is a Hilbert space with respect to the pointwise operations.

Lemma 1. *Let* $\mathcal{R}$ *be a Bessel relay-fusion sequence in* H *with Bessel bound* B*. Then, for each sequence* $\{f_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ *with* $f_{w,v} \in V_{w,v}$*,*

$$\int_\Omega \int_{\Omega_w} \alpha_{w,v} \pi_{W_w} \Lambda_w^* f_{w,v} d\mu(v) d\mu(w)$$

converges.

Proof. Let

$$f = \{f_{w,v}\}_{w\in\Omega_w, v\in\Omega_w} \in \left(\int_\Omega \int_{\Omega_w} \oplus V_{w,v}\right)_{l^2}$$

and

$$g = \int_\Omega \int_{\Omega_w} \alpha_{w,v} \pi_{W_w} \Lambda_w^* f_{w,v} d\mu(v) d\mu(w).$$

Then we have

$$\begin{aligned}
\|g\| &= \| \int_\Omega \int_{\Omega_w} \alpha_{w,v} \pi_{W_w} \Lambda_w^* f_{w,v} d\mu(w) d\mu(v)\| \\
&= \sup_{\|h\|=1} \left|\langle \int_\Omega \int_{\Omega_w} \alpha_{w,v} \pi_{W_w} \Lambda_w^* f_{w,v} d\mu(w) d\mu(v), h\rangle\right| \\
&= \sup_{\|h\|=1} \left| \int_\Omega \int_{\Omega_w} \langle f_{w,v}, \alpha_{w,v} P_{V_{w,v}} \Lambda_w \pi_{W_w} h\rangle d\mu(v) d\mu(w)\right|
\end{aligned}$$

$$\leq \sup_{\|h\|=1} \left(\int_{\Omega} \int_{\Omega_w} \|f_{w,v}\|^2 d\mu(v) d\mu(w)) \right)^{\frac{1}{2}}$$
$$\left(\int_{\Omega} \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} h\|^2 d\mu(v) d\mu(w) \right)^{\frac{1}{2}}$$
$$\leq \sqrt{B} \|f\|.$$

◀

Definition 5. *Let $\mathcal{R}$ be a relay fusion frame for H. Then the analysis operator for $\mathcal{R}$ is defined by*

$$T_{\mathcal{R}} : H \to \left(\int_{\Omega} \int_{\Omega_w} \oplus V_{w,v} \right)_{l^2}, \text{ with } T_{\mathcal{R}}(f) = \{\alpha_{w,v} P_{V_{w,v}} \Lambda_w \pi_{W_w} f\}_{w\in\Omega, v\in\Omega_w}, \forall f \in H.$$

We call the adjoint $T_{\mathcal{R}}^*$ of the analysis operator the synthesis operator of $\mathcal{R}$.

Proposition 1. *Let $\mathcal{R}$ be a relay fusion frame for H. Then*

$$T_{\mathcal{R}}^* f = \int_{\Omega} \int_{\Omega_w} \alpha_{w,v} \pi_{W_w} \Lambda_w^* f_{w,v} d\mu(v) d\mu(w), \quad \forall \{f_{w,v}\}_{w\in\Omega, v\in\Omega_w} \in \left(\int_{\Omega} \int_{\Omega_w} \oplus V_{w,v} \right)_{l^2}.$$

Proof. Let $g \in H$ and $f = \{f_{w,v}\}_{w\in\Omega, v\in\Omega_w} \in \left(\int_{\Omega} \int_{\Omega_w} \oplus V_{w,v} \right)_{l^2}$. Then

$$\begin{aligned}
\langle T_{\mathcal{R}}(g), f \rangle &= \langle \{\alpha_{w,v} P_{V_{w,v}} \Lambda_w \pi_{W_w} g\}_{w\in\Omega, v\in\Omega_w}, \{f_{w,v}\}_{w\in\Omega, v\in\Omega_w} \rangle \\
&= \int_{\Omega} \int_{\Omega_w} \langle g, \alpha_{w,v} \pi_{W_w} \Lambda_w^* f_{w,v} \rangle d\mu(v) d\mu(w) \\
&= \langle g, T_{\mathcal{R}}^*(f) \rangle.
\end{aligned}$$

◀

Theorem 1. *The following assertions are equivalent:*

(1) $\mathcal{R}$ is a relay fusion frame for H.

(2) $T_{\mathcal{R}}$ is injective and has a closed range.

Proof. (1) $\implies$ (2) We have for each $f \in H$:

$$\int_{\Omega} \int_{\Omega_w} \alpha_{w,v} \|P_{V_{w,v}} \Lambda_w \pi_{W_w} f\|^2 d\mu(v) d\mu(w) = \|T_{\mathcal{R}} f\|^2.$$

First we prove that $T_{\mathcal{R}}$ is injective. Let $f \in H$ be such that $T_{\mathcal{R}} f = 0$. Since

$$A\|f\|^2 \leq \|T_{\mathcal{R}} f\|^2, \quad \forall f \in H,$$

we have $f = 0$.

We now show that $T_{\mathcal{R}}$ has a closed range. Let $\{T_{\mathcal{R}}(x_n)\}_{n\in\mathbb{N}} \in Range(T_{\mathcal{R}})$ be such that $\lim_{n\to\infty} T_{\mathcal{R}}(x_n) = y$. For $n, m \in \mathbb{N}$, we have

$$A\|x_n - x_m\|^2 \leq \|T_{\mathcal{R}}(x_n - x_m)\|^2.$$

Since $\{T_{\mathcal{R}}(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H, we have $\|x_n - x_m\| \to 0$, as $n, m \to \infty$. Therefore the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and hence there exists $x \in H$ such that $x_n \to x$ as $n \to \infty$. And we have

$$\|T_{\mathcal{R}}(x_n - x)\|^2 \leq B\|x_n - x\|^2.$$

Thus $\|T_{\mathcal{R}} x_n - T_{\mathcal{R}} x\| \to 0$ as $n \to \infty$ implies that $T_{\mathcal{R}} x = y$. So the range of $T_{\mathcal{R}}$ is closed.

(2) $\Longrightarrow$ (1) This is obvious. ◀

By composing $T_{\mathcal{R}}$ and $T^*_{\mathcal{R}}$, we obtain the frame operator for $\mathcal{R}$.

Definition 6. *Let $\mathcal{R}$ be a relay fusion frame. A frame operator $S_{\mathcal{R}}$ for $\mathcal{R}$ is defined by*

$$S_{\mathcal{R}} f = T^*_{\mathcal{R}} T_{\mathcal{R}} f = \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}\pi_{W_w}\Lambda^*_w P_{V_{w,v}}\pi_{W_w} f d\mu(v)d\mu(w).$$

Theorem 2. *$\mathcal{R}$ is a Bessel relay fusion sequence in H with bound B if and only if the map*

$$\{f_{w,v}\}_{w\in\Omega, v\in\Omega_w} \to \int_{\Omega}\int_{\Omega} \alpha_{w,v}\pi_{W_w}\Lambda^*_w f_{w,v} d\mu(v)d\mu(w)$$

is a well-defined bounded operator from $\left(\int_{\Omega}\int_{\Omega_w} \oplus V_{w,v}\right)_{l^2}$ to H and its norm is less or equal to $\sqrt{B}$.

Proof. First assume that $\mathcal{R}$ is a Bessel relay fusion sequence for H with bound B. By Lemma 1 the $\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}\pi_{W_w}\Lambda^*_w f_{w,v} d\mu(v)d\mu(w)$ is convergent. Thus $T^*_{\mathcal{R}}(\{f_{w,v}\}_{w\in\Omega, v\in\Omega_w})$ is well defined. A simple calculation as in Lemma 1 shows that $T^*_{\mathcal{R}}$ is bounded and $\|T^*_{\mathcal{R}}\| \leq \sqrt{B}$.

For the opposite implication, suppose that $T^*_{\mathcal{R}}$ is well defined and $\|T^*_{\mathcal{R}}\| \leq \sqrt{B}$. Then

$$\int_{\Omega}\int_{\Omega_w} \alpha^2_{w,v}\|P_{w,v}\Lambda_w\pi_{W_w} f\|^2 d\mu(v)d\mu(w) =$$

$$= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \langle \pi_{W_w}\Lambda_w^* P_{V_{w,v}} \Lambda_w \pi_{W_w} f, f\rangle d\mu(v)d\mu(w) =$$

$$= \langle T_{\mathcal{R}}^* \{\alpha_{w,v} P_{V_{w,v}} \Lambda_w \pi_{W_w} f\}_{w\in\Omega, v\in\Omega_w}, f\rangle \leq$$

$$\leq \left(\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} f\|^2 d\mu(v)d\mu(w)\right)^{\frac{1}{2}} . \|T_{\mathcal{R}}^*\| . \|f\|,$$

so we have

$$\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} f\|^2 d\mu(v)d\mu(w) \leq \|T_{\mathcal{R}}^*\| . \|f\| \leq \sqrt{B} . \|f\|.$$

◀

Theorem 3. *Let $\mathcal{R}$ be a relay fusion frame with bounds A and B. Then the frame operator for $\mathcal{R}$ is bounded, positive, self-adjoint, invertible operator on H with*

$$AI_H \leq S_{\mathcal{R}} \leq BI_H.$$

Proof. $S_{\mathcal{R}}$ is bounded as a composition of two bounded operators:

$$\|S_{\mathcal{R}}\| = \|T_{\mathcal{R}}^* T_{\mathcal{R}}\| = \|T_{\mathcal{R}}^*\|^2 \leq B.$$

Since

$$S_{\mathcal{R}}^* = \left(T_{\mathcal{R}}^* T_{\mathcal{R}}\right)^* = T_{\mathcal{R}}^* T_{\mathcal{R}} = S_{\mathcal{R}},$$

the operator $S_{\mathcal{R}}$ is self-adjoint. The inequality (2) means that

$$A\|f\|^2 \leq \langle S_{\mathcal{R}} f, f\rangle \leq B\|f\|^2, \forall f \in H.$$

This shows that

$$AI_H \leq S_{\mathcal{R}} \leq BI_H,$$

and hence $S_{\mathcal{R}}$ is positive, invertible operator on H. ◀

Theorem 4. *Let $\mathcal{R}$ be a relay fusion frame for H with frame operator $S_{\mathcal{R}}$. Then we have for all $f \in H$*

$$f = \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 S_{\mathcal{R}}^{-1} \pi_{W_w}\Lambda_w^* P_{V_{w,v}} \Lambda_w \pi_{W_w} f d\mu(v)d\mu(w)$$

$$= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \pi_{W_w}\Lambda_w^* P_{V_{w,v}} \Lambda_w \pi_{W_w} S_{\mathcal{R}}^{-1} f d\mu(v)d\mu(w).$$

Proof. As $S_{\mathcal{R}}$ is invertible, for all $f \in H$ we have

$$f = S_{\mathcal{R}}^{-1} S_{\mathcal{R}} f = \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 S_{\mathcal{R}}^{-1} \pi_{W_w} \Lambda_w^* P_{V_{w,v}} \Lambda_w \pi_{W_w} f d\mu(v) d\mu(w),$$

and

$$f = S_{\mathcal{R}} S_{\mathcal{R}}^{-1} f = \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \pi_{W_w} \Lambda_w^* P_{V_{w,v}} \Lambda_w \pi_{W_w} S_{\mathcal{R}}^{-1} f d\mu(v) d\mu(w).$$

◀

Theorem 5. *Let*

$$\mathcal{R} = \{W_w, V_{w,v}, \alpha_{w,v}\}_{w\in\Omega, v\in\Omega_w}$$

and

$$\mathcal{R}' = \{W_w', V_{w,v}', \Lambda_w', \alpha_{w,v}'\}_{w\in\Omega, v\in\Omega_w}$$

be two Bessel relay sequences for H with bounds B and B', respectively. Let $T_{\mathcal{R}}$ and $T_{\mathcal{R}'}$ be their analysis operators such that $T_{\mathcal{R}'}^* T_{\mathcal{R}} = I_H$. Then both $\mathcal{R}$ and $\mathcal{R}'$ are relay fusion frames.

Proof. We have for all $f \in H$

$$\begin{aligned}
\|f\|^4 &= \langle T_{\mathcal{R}} f, T_{\mathcal{R}'} f\rangle^2 \\
&\leq \|T_{\mathcal{R}} f\|^2 \|T_{\mathcal{R}'} f\|^2 \\
&= \left(\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} f\|^2 d\mu(v) d\mu(w)\right) \\
&\quad \cdot\left(\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}'^2 \|P_{V_{w,v}'} \Lambda_w' \pi_{W_w'} f\|^2 d\mu(v) d\mu(w)\right) \\
&\leq \left(\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} f\|^2 d\mu(v) d\mu(w)\right) B' \|f\|^2.
\end{aligned}$$

Thus

$$\frac{1}{B'}\|f\|^2 \leq \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} f\|^2 d\mu(v) d\mu(w).$$

Similarly we obtain a lower bound for $\mathcal{R}'$. ◀

3. Duality of relay fusion frames

Lemma 2. *[6] Let $A \in B(H)$ and $V \subseteq H$ be a closed subspace. Then*

$$\pi_V A^* = \pi_V A^* \pi_{\overline{AV}}.$$

3.1. Global continuous relay dual of continuous relay fusion frames

Let $\mathcal{R}$ be a continuous relay fusion frame for H. We consider global continuous relay space $\mathcal{K} = \left(\int_\Omega \oplus K_w\right)_{l^2}$ and let $\mathcal{F}_\mathcal{K}$ be a frame for $\mathcal{K}$, where every K_w is a local continuous relay space. We use $S_{\mathcal{F}_\mathcal{K}}$ to denote the frame operator for $\mathcal{K}$. Let $\widetilde{V_{w,v}} = S_{\mathcal{F}_\mathcal{K}}^{-1} V_{w,v}$ and $\widetilde{\Lambda_w} = S_{\mathcal{F}_\mathcal{K}}^{-1} P_{V_{w,v}} \Lambda_w$. We now prove that $\widetilde{\mathcal{R}} = \{W_w, \widetilde{V_{w,v}}, \widetilde{\Lambda_w}, v_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is a continuous relay fusion frame for H and we call $\widetilde{\mathcal{R}}$ the global continuous relay dual of continuous relay fusion frame of $\mathcal{R}$.

Theorem 6. *Let $\mathcal{R}$ be a continuous relay fusion frame for H. Then $\widetilde{\mathcal{R}}$ is a continuous relay fusion frame for H, for all $f \in H$,*

$$f = \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 S_{\widetilde{\mathcal{R}}}^{-1} \pi_{W_w} \widetilde{\Lambda_w}^* \widetilde{\Lambda_w} \pi_{W_w} f \, d\mu(v) d\mu(w) =$$

$$= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \pi_{W_w} \widetilde{\Lambda_w}^* \widetilde{\Lambda_w} \pi_{W_w} S_{\widetilde{\mathcal{R}}}^{-1} f \, d\mu(v) d\mu(w).$$

Proof. For each $f \in H$, we have

$$\int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \| P_{S_{\mathcal{F}_\mathcal{K}}^{-1} V_{w,v}} S_{\mathcal{F}_\mathcal{K}}^{-1} P_{V_{w,v}} \Lambda_w \pi_{W_w} f \|^2 d\mu(v) d\mu(w) =$$

$$= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \| S_{\mathcal{F}_\mathcal{K}}^{-1} P_{V_{w,v}} \Lambda_w \pi_{W_w} f \|^2 d\mu(v) d\mu(w) \leq \| S_{\mathcal{F}_\mathcal{K}}^{-1} \|^2 B \|f\|^2.$$

On the other hand, we have

$$\int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \| P_{S_{\mathcal{F}_\mathcal{K}}^{-1} V_{w,v}} S_{\mathcal{F}_\mathcal{K}}^{-1} P_{V_{w,v}} \Lambda_w \pi_{W_w} f \|^2 d\mu(v) d\mu(w) =$$

$$= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \| S_{\mathcal{F}_\mathcal{K}}^{-1} P_{V_{w,v}} \Lambda_w \pi_{W_w} f \|^2 d\mu(v) d\mu(w) \geq$$

$$\geq \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \frac{1}{\|S_{\mathcal{F}_\mathcal{K}}\|^2} \| P_{V_{w,v}} \Lambda_w \pi_{W_w} f \|^2 d\mu(v) d\mu(w) \geq \frac{A}{\|S_{\mathcal{F}_\mathcal{K}}\|^2} \|f\|^2.$$

Further, since $S_{\widetilde{\mathcal{R}}}$ is invertible, for all $f \in H$ we have

$$\begin{aligned}
f &= S_{\widetilde{\mathcal{R}}}^{-1} S_{\widetilde{\mathcal{R}}} f = S_{\widetilde{\mathcal{R}}} S_{\widetilde{\mathcal{R}}}^{-1} f \\
&= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 S_{\widetilde{\mathcal{R}}}^{-1} \pi_{W_w} \widetilde{\Lambda_w}^* P_{\widetilde{V_{w,v}}} \widetilde{\Lambda_w} \pi_{W_w} f \, d\mu(v) d\mu(w) \\
&= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 S_{\widetilde{\mathcal{R}}}^{-1} \pi_{W_w} \Lambda_w^* P_{V_{w,v}} S_{\mathcal{F}_\mathcal{K}}^{-1} P_{S_{\mathcal{F}_\mathcal{K}}^{-1} V_{w,v}} S_{\mathcal{F}_\mathcal{K}}^{-1} P_{V_{w,v}} \Lambda_w \pi_{W_w} f \, d\mu(v) d\mu(w)
\end{aligned}$$

$$= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 S_{\widetilde{\mathcal{R}}}^{-1}\pi_{W_w}\Lambda_w^* P_{V_{w,v}} S_{\mathcal{F}_{\mathcal{K}}}^{-1} S_{\mathcal{F}_{\mathcal{K}}}^{-1} P_{V_{w,v}}\Lambda_w \pi_{W_w} f d\mu(v)d\mu(w)$$
$$= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 S_{\widetilde{\mathcal{R}}}^{-1}\pi_{W_w}\widetilde{\Lambda_w}^*\widetilde{\Lambda_w}\pi_{W_w} f d\mu(v)d\mu(w)$$
$$= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \pi_{W_w}\widetilde{\Lambda_w}^*\widetilde{\Lambda_w}\pi_{W_w} S_{\widetilde{\mathcal{R}}}^{-1} f d\mu(v)d\mu(w).$$

◀

3.2. Local continuous relay dual of continuous relay fusion frames

Let $\widehat{V_{w,v}} = S_w^{-1}V_{w,v}$ and $\widehat{\Lambda_w} = S_w^{-1}P_{w,v}\Lambda_w$, where S_w denotes the frame operator with respect to K_w for each $w \in \Omega$ and we call every S_w local continuous relay frame operator. We now prove that $\widehat{\mathcal{R}} = \{W_w, \widehat{V_{w,v}}, \widehat{\Lambda_w}, \alpha_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is also a continuous relay fusion frame for H and we call $\widehat{\mathcal{R}}$ the local continuous relay dual of continuous relay fusion frame of $\mathcal{R}$.

Theorem 7. *Let $\mathcal{R}$ be a continuous relay fusion frame for H. Then $\widehat{\mathcal{R}}$ is a continuous relay fusion frame for H and, for all $f \in H$,*

$$f = \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 S_{\widehat{\mathcal{R}}}^{-1}\pi_{W_w}\widehat{\Lambda_w}^*\widehat{\Lambda_w}\pi_{W_w} f d\mu(v)d\mu(w) =$$
$$= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \pi_{W_w}\widehat{\Lambda_w}^*\widehat{\Lambda_w}\pi_{W_w} S_{\widehat{\mathcal{R}}}^{-1} f d\mu(v)d\mu(w).$$

Proof. For all $f \in H$ we have

$$\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{S_w^{-1}V_{w,v}} S_w^{-1} P_{V_{w,v}}\Lambda_w\pi_{W_w} f\|^2 d\mu(v)d\mu(w) =$$
$$= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|S_w^{-1} P_{V_{w,v}}\Lambda_w\pi_{W_w} f\|^2 d\mu(v)d\mu(w) \le \max_{w\in\Omega}\{\|S_w^{-1}\|^2\} B\|f\|^2.$$

On the other hand, for each $f \in H$ we have

$$\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{S_w^{-1}V_{w,v}} S_w^{-1} P_{V_{w,v}}\Lambda_w\pi_{W_w} f\|^2 d\mu(v)d\mu(w) =$$
$$= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|S_w^{-1} P_{V_{w,v}}\Lambda_w\pi_{W_w} f\|^2 d\mu(v)d\mu(w) \ge$$
$$\ge \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \frac{1}{\|S_w\|^2} \|P_{V_{w,v}}\Lambda_w\pi_{W_w} f\|^2 d\mu(v)d\mu(w) \ge$$

$$\geq \min_{w\in\Omega}\left\{\frac{1}{\|S_w\|^2}\right\}A\|f\|^2.$$

Since $S_{\widehat{\mathcal{R}}}$ is invertible, for all $f \in H$ we have

$$\begin{aligned}
f &= S_{\widehat{\mathcal{R}}}^{-1}S_{\widehat{\mathcal{R}}}f = S_{\widehat{\mathcal{R}}}S_{\widehat{\mathcal{R}}}^{-1}f \\
&= \int_{\Omega}\int_{\Omega_w}\alpha_{w,v}^2 S_{\widehat{\mathcal{R}}}^{-1}\pi_{W_w}\widehat{\Lambda_w}^* P_{\widehat{V_{w,v}}}\widehat{\Lambda_w}\pi_{W_w}d\mu(v)d\mu(w) \\
&= \int_{\Omega}\int_{\Omega_w}\alpha_{w,v}^2 S_{\widehat{\mathcal{R}}}^{-1}\pi_{W_w}\Lambda_w^* P_{V_{w,v}}S_w^{-1}P_{S_w^{-1}V_{w,v}}S_w^{-1}P_{V_{w,v}}\Lambda_w\pi_{W_w}fd\mu(v)d\mu(w) \\
&= \int_{\Omega}\int_{\Omega_w}\alpha_{w,v}^2 S_{\widehat{\mathcal{R}}}^{-1}\pi_{W_w}\widehat{\Lambda_w}^*\widehat{\Lambda_w}\pi_{W_w}fd\mu(v)d\mu(w) \\
&= \int_{\Omega}\int_{\Omega_w}\alpha_{w,v}^2 \pi_{W_w}\widehat{\Lambda_w}^*\widehat{\Lambda_w}\pi_{W_w}S_{\widehat{\mathcal{R}}}^{-1}fd\mu(v)d\mu(w).
\end{aligned}$$

◀

3.3. Continuous canonical dual of continuous relay fusion frames

Now let $\mathring{W}_w = S_{\mathcal{R}}^{-1}W_w$ and $\mathring{\Lambda}_w = \Lambda_w\pi_{W_w}S_{\mathcal{R}}^{-1}$, where $S_{\mathcal{R}}$ is the frame operator for $\mathcal{R}$. We prove that $\mathring{\mathcal{R}} = \{\mathring{W}_w, V_{w,v}, \mathring{\Lambda}_w, \alpha_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is also a continuous relay fusion frame for H and we call $\mathring{\mathcal{R}}$ the continuous canonical dual of continuous relay fusion frame of $\mathcal{R}$ for H.

Theorem 8. *Let $\mathcal{R}$ be a continuous relay fusion frame for H. Then $\mathring{\mathcal{R}}$ is a continuous relay fusion frame for H.*

Proof. We have for all $f \in H$

$$\int_{\Omega}\int_{\Omega_w}\alpha_{w,v}^2\|P_{V_{w,v}}\Lambda_w\pi_{W_w}S_{\mathcal{R}}^{-1}\pi_{S_{\mathcal{R}}^{-1}W_w}f\|^2 d\mu(v)d\mu(w) =$$

$$= \int_{\Omega}\int_{\Omega_w}\alpha_{w,v}^2\|P_{V_{w,v}}\Lambda_w\pi_{S_{\mathcal{R}}^{-1}}f\|^2 d\mu(v)d\mu(w) \leq$$

$$\leq \|S_{\mathcal{R}}^{-1}\|^2 B\|f\|^2.$$

On the other hand, we have

$$\|f\|^4 = \left|\langle\int_{\Omega}\int_{\Omega_w}\alpha_{w,v}^2\pi_{W_w}\Lambda_w^* P_{V_{w,v}}\Lambda_w\pi_{W_w}S_{\mathcal{R}}^{-1}fd\mu(v)d\mu(w), f\rangle\right|^2 =$$

$$= \left|\int_{\Omega}\int_{\Omega_w}\alpha_{w,v}^2\langle P_{V_{w,v}}\Lambda_w\pi_{W_w}S_{\mathcal{R}}^{-1}f, P_{V_{w,v}}\Lambda_w\pi_{W_w}f\rangle d\mu(v)d\mu(w)\right|^2 \leq$$

$$\leq \left(\int_\Omega\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}}\Lambda_w \pi_{W_w} S_{\mathcal{R}}^{-1}\pi_{S_{\mathcal{R}}^{-1}W_w} f\|^2 d\mu(v)d\mu(w)\right)\times$$

$$\times\left(\int_\Omega\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}}\Lambda_w \pi_{W_w} f\|^2 d\mu(v)d\mu(w)\right) \leq$$

$$\leq \left(\int_\Omega\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}}\Lambda_w \pi_{W_w} S_{\mathcal{R}}^{-1}\pi_{S_{\mathcal{R}}^{-1}W_w} f\|^2 d\mu(v)d\mu(w)\right) B\|f\|^2.$$

Therefore,

$$\frac{1}{B}\|f\|^2 \leq \int_\Omega\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}}\Lambda_w \pi_{W_w} S_{\mathcal{R}}^{-1}\pi_{S_{\mathcal{R}}^{-1}W_w} f\|^2 d\mu(v)d\mu(w).$$

◀

Theorem 9. *Let $\mathcal{R}$ be a continuous relay fusion frame for H with frame operator $S_{\mathcal{R}}$ and let $\mathring{\mathcal{R}}$ be the continuous canonical dual of continuous relay fusion frame of $\mathcal{R}$ with frame operator $S_{\mathring{\mathcal{R}}}$. Then $S_{\mathcal{R}}S_{\mathring{\mathcal{R}}} = I_H$ and $T_{\mathcal{R}}^*T_{\mathring{\mathcal{R}}} = I_H$ and, for all $f \in H$,*

$$f = \int_\Omega\int_{\Omega_w} \alpha_{w,v}^2 \pi_{W_w}\Lambda_w^* P_{V_{w,v}}\mathring{\Lambda}_w \pi_{\mathring{W}_w} f d\mu(v)d\mu(w) =$$

$$= \int_\Omega\int_{\Omega_w} \alpha_{w,v}^2 \pi_{\mathring{W}_w}\mathring{\Lambda}_w^{\,*} P_{w,v}\Lambda_w \pi_{W_w} f d\mu(v)d\mu(w).$$

Proof. We have for all $f \in H$

$$\begin{aligned}
S_{\mathcal{R}}S_{\mathring{\mathcal{R}}}f &= S_{\mathcal{R}}\int_\Omega\int_{\Omega_w} \alpha_{w,v}^2 \pi_{S_{\mathcal{R}}^{-1}W_w} S_{\mathcal{R}}^{-1}\pi_{W_w}\Lambda_w^* P_{V_{w,v}}\Lambda_w\pi_{W_w}S_{\mathcal{R}}^{-1}\pi_{S_{\mathcal{R}}^{-1}W_w} f d\mu(v)d\mu(w)\\
&= S_{\mathcal{R}}\int_\Omega\int_{\Omega_w} \alpha_{w,v}^2 S_{\mathcal{R}}^{-1}\pi_{W_w}\Lambda_w^* P_{V_{w,v}}\Lambda_w\pi_{W_w}S_{\mathcal{R}}^{-1} f d\mu(v)d\mu(w)\\
&= \int_\Omega\int_{\Omega_w} \alpha_{w,v}^2 \pi_{W_w}\Lambda_w^* P_{V_{w,v}}\Lambda_w\pi_{W_w}S_{\mathcal{R}}^{-1} f d\mu(v)d\mu(w)\\
&= S_{\mathcal{R}}S_{\mathcal{R}}^{-1}f\\
&= f
\end{aligned}$$

and

$$\begin{aligned}
T_{\mathcal{R}}^*T_{\mathring{\mathcal{R}}} &= T_{\mathcal{R}}^*\left(\{\alpha_{w,v}P_{V_{w,v}}\Lambda_w\pi_{W_w}S_{\mathcal{R}}^{-1}\pi_{S_{\mathcal{R}}^{-1}W_w} f\}_{w\in\Omega, v\in\Omega_w}\right)\\
&= T_{\mathcal{R}}^*\left(\{\alpha_{w,v}P_{V_{w,v}}\Lambda_w\pi_{W_w}S_{\mathcal{R}}^{-1} f\}_{w\in\Omega, v\in\Omega_w}\right)
\end{aligned}$$

$$= T_{\mathcal{R}}^* T_{\mathcal{R}} S_{\mathcal{R}}^{-1} f$$
$$= f.$$

The last assertion of the theorem follows from the previous steps of the proof. ◀

Theorem 10. *Let $\mathcal{R}$ be a continuous relay fusion frame with continuous canonical dual of continuous relay fusion frame $\mathring{\mathcal{R}}$. Then, for any $g_{w,v} \in V_{w,v}$ satisfying $f = \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \pi_{W_w} \Lambda_w^* g_{w,v} d\mu(v) d\mu(w)$, we have*

$$\int_\Omega \int_{\Omega_w} \|g_{w,v}\|^2 d\mu(v)d\mu(w) = \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \mathring{\Lambda}_w \pi_{\mathring{W}_w} f\|^2 d\mu(v)d\mu(w) + \int_\Omega \int_{\Omega_w} \|g_{w,v} - \alpha_{w,v}^2 P_{V_{w,v}} \mathring{\Lambda}_w \pi_{\mathring{W}_w} f\|^2 d\mu(v)d\mu(w).$$

Proof. For each $f \in H$, we have

$$\int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \mathring{\Lambda}_w \pi_{\mathring{W}_w} f\|^2 d\mu(v)d\mu(w) = \langle S_{\mathring{\mathcal{R}}} f, f\rangle = \langle f, S_{\mathcal{R}}^{-1} f\rangle =$$
$$= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \langle \pi_{W_w} \Lambda_w^* g_{w,v}, S_{\mathcal{R}}^{-1} f\rangle d\mu(v)d\mu(w) =$$
$$= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \langle g_{w,v}, P_{w,v} \Lambda_w \pi_{W_w} S_{\mathcal{R}}^{-1} f\rangle d\mu(v)d\mu(w) =$$
$$= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \langle g_{w,v}, P_{w,v} \mathring{\Lambda}_w \pi_{\mathring{W}_w} f\rangle d\mu(v)d\mu(w) =$$
$$= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \langle P_{V_{w,v}} \mathring{\Lambda}_w \pi_{\mathring{W}_w} f, g_{w,v}\rangle d\mu(v)d\mu(w).$$

◀

Example 1. *Let $\mathcal{R}$ be a continuous relay fusion frame for H and $S_{\mathcal{R}}$ denote the frame operator of $\mathcal{R}$. We have for each $f \in H$*

$$\|f\|^2 = \langle S_{\mathcal{R}}^{-\frac{1}{2}} S_{\mathcal{R}} S_{\mathcal{R}}^{-\frac{1}{2}} f, f\rangle$$
$$= \langle \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 S_{\mathcal{R}}^{-\frac{1}{2}} \pi_{W_w} \Lambda_w^* P_{V_{w,v}} \Lambda_w \pi_{W_w} S_{\mathcal{R}}^{-\frac{1}{2}} f d\mu(v)d\mu(w), f\rangle$$
$$=$$
$$= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} S_{\mathcal{R}}^{-\frac{1}{2}} f\|^2 d\mu(v)d\mu(w)$$

$$= \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} S_{\mathcal{R}}^{-\frac{1}{2}} \pi_{S_{\mathcal{R}}^{-\frac{1}{2}} W_w} f\|^2 d\mu(v) d\mu(w).$$

So, $\{S_{\mathcal{R}}^{\frac{1}{2}} W_w, V_{w,v}, \Lambda_w \pi_{W_w} S_{\mathcal{R}}^{\frac{1}{2}}, \alpha_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ *is a Parseval continuous relay fusion frame for* H.

Theorem 11. *Let* $\mathcal{R}$ *be a continuous relay fusion frame for* H *with continuous relay fusion frame operator* $S_{\mathcal{R}}$ *and let* $Q \in B(H)$ *be an invertible operator. Then* $\mathcal{R}_Q = \{QW_w, V_{w,v}, \Lambda_w, \alpha_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ *is a continuous relay fusion frame for* H *with continuous relay fusion frame operator* $S_{\mathcal{R}_Q}$ *satisfying*

$$\frac{QS_{\mathcal{R}}Q^*}{\|Q\|^2} \le S_{\mathcal{R}_Q} \le \|Q^{-1}\|^2 QS_{\mathcal{R}}Q^*.$$

Proof. For each $f \in H$ we have

$$\|P_{V_{w,v}} \Lambda_w \pi_{W_w} Q^* f\| = \|P_{V_{w,v}} \Lambda_w \pi_{W_w} Q^* \pi_{QW_w} f\| \le \|Q^*\| \|P_{V_{w,v}} \Lambda_w \pi_{QW_w} f\|.$$

Since $Q^* f \in H$ and $\mathcal{R}$ is a continuous relay fusion frame for H, we have

$$\begin{aligned} A\|Q^* f\|^2 &\le \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} Q^* f\| d\mu(v) d\mu(w) \\ &\le \|Q^*\|^2 \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{QW_w} f\|^2 d\mu(v) d\mu(w). \end{aligned}$$

Thus

$$\frac{A}{\|Q^*\|^2 \|(Q^*)^{-1}\|^2} \|f\|^2 \le \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{w,v} \Lambda_w \pi_{QW_w} f\|^2 d\mu(v) d\mu(w).$$

On the other hand, we have

$$\pi_{QW_w} = \pi_{QW_w} (Q^{-1})^* \pi_{W_w} Q^*.$$

So

$$\|P_{V_{w,v}} \Lambda_w \pi_{QW_w} f\| \le \|(Q^{-1})^*\| \|P_{V_{w,v}} \Lambda_w \pi_{W_w} Q^* f\|.$$

Therefore

$$\begin{aligned} &\int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{QW_w} f\|^2 d\mu(v) d\mu(w) \le \\ &\le \|(Q^{-1})^*\|^2 \int_\Omega \int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}} \Lambda_w \pi_{W_w} Q^* f\|^2 d\mu(v) d\mu(w) \le \\ &\le B\|Q^*\|^2 \|(Q^*)^{-1}\|^2 \|f\|^2. \end{aligned}$$

Now show that
$$\frac{QS_{\mathcal{R}}Q^*}{\|Q\|^2} \leq S_{\mathcal{R}_Q} \leq \|Q^{-1}\|^2 QS_{\mathcal{R}}Q^*.$$

For all $f \in H$, we have

$$\begin{aligned}
\langle \frac{QS_{\mathcal{R}}Q^*}{\|Q\|^2} f, f\rangle &= \frac{1}{\|Q\|^2}\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}}\Lambda_w \pi_{W_w} Q^* f\|^2 d\mu(v)d\mu(w)\\
&= \frac{1}{\|Q\|^2}\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}}\Lambda_w \pi_{W_w} Q^* \pi_{QW_w} f\|^2 d\mu(v)d\mu(w)\\
&\leq \frac{\|Q^*\|^2}{\|Q\|^2}\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{w,v}\Lambda_w \pi_{QW_w} f\|^2 d\mu(v)d\mu(w)\\
&= \langle S_{\mathcal{R}_Q} f, f\rangle.
\end{aligned}$$

Since
$$\pi_{QW_w} = \pi_{QW_w}(Q^{-1})^*\pi_{W_w}Q^*,$$
we have

$$\begin{aligned}
\langle S_{\mathcal{R}_Q} f, f\rangle &= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}}\Lambda_w \pi_{QW_w} f\|^2 d\mu(v)d\mu(w)\\
&= \int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}}\Lambda_w \pi_{QW_w} (Q^{-1})^*\pi_{W_w}Q^* f\|^2 d\mu(v)d\mu(w)\\
&\leq \|(Q^{-1})^*\|^2\int_{\Omega}\int_{\Omega_w} \alpha_{w,v}^2 \|P_{V_{w,v}}\Lambda_w \pi_{W_w} Q^* f\|^2 d\mu(v)d\mu(w)\\
&= \|(Q^{-1})^*\|^2 \langle S_{\mathcal{R}_Q} Q^* f, Q^* f\rangle\\
&= \langle \|(Q^{-1})^*\|^2 QS_{\mathcal{R}_Q}Q^* f, f\rangle
\end{aligned}$$

◀

Theorem 12. *Let $\mathcal{R}$ be a continuous relay fusion frame for H with continuous relay fusion frame bounds A and B. If $Q_w \in B(K_w)$'s are invertible operators for each $w \in \Omega$, then $\mathcal{R} = \{W_w, Q_wV_{w,v}, \Lambda_w, \alpha_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is a continuous relay fusion frame for H.*

Proof. For all $f \in H$, we have

$$\|Q_wP_{V_{w,v}}\Lambda_w\pi_{W_w}f\| = \|P_{Q_wV_{w,v}}Q_wP_{V_{w,v}}\Lambda_w\pi_{W_w}f\| \leq \|Q_w\|\|P_{Q_wV_{w,v}}\Lambda_w\pi_{W_w}f\|.$$

So
$$\frac{1}{\|Q_w\|\|Q_w^{-1}\|}\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f\| \leq \|P_{Q_wV_{w,v}}\Lambda_w\pi_{W_w}f\|.$$

Therefore

$$\min_{w\in\Omega}\left\{\frac{A}{\|Q_w\|^2\|Q_w^{-1}\|^2}\right\}\|f\|^2 \leq \int_\Omega\int_{\Omega_w}\frac{\alpha_{w,v}^2}{\|Q_w\|^2\|Q_w^{-1}\|^2}\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f\|^2d\mu(v)d\mu(w)$$
$$\leq \int_\Omega\int_{\Omega_w}\alpha_{w,v}^2\|P_{Q_wV_{w,v}}\Lambda_w\pi_{W_w}f\|^2d\mu(v)d\mu(w).$$

We have

$$P_{Q_wV_{w,v}} = P_{Q_wV_{w,v}}(Q_w^{-1})^*P_{V_{w,v}}Q_w^*.$$

Then

$$\|P_{Q_wV_{w,v}}\Lambda_w\pi_{W_w}f\| = \|P_{Q_wV_{w,v}}(Q_w^{-1})^*P_{V_{w,v}}Q_w^*\Lambda_w\pi_{W_w}\|$$
$$\leq \|Q_w^*\|\|(Q_w^{-1})^*\|\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f\|.$$

Hence

$$\int_\Omega\int_{\Omega_w}\alpha_{w,v}^2\|P_{Q_wV_{w,v}}\Lambda_w\pi_{W_w}f\|^2d\mu(v)d\mu(w)\|^2d\mu(v)d\mu(w)$$
$$\leq \int_\Omega\int_{\Omega_w}\|Q_w^*\|^2\|(Q_w^{-1})^*\|^2\alpha_{w,v}^2\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f\|^2d\mu(v)d\mu(w)$$
$$\leq \max_{w\in\Omega}\left\{\|Q_w^*\|^2\|(Q_w^{-1})^*\|^2\right\}B\|f\|^2.$$

◀

4. Perturbation of the continuous relay fusion frames

Theorem 13. *Let $\mathcal{R}_1 = \{W_w, V_{w,v}, \Lambda_w, v_{w,v}\}_{w\in\Omega,v\in\Omega}$ be a continuous relay fusion frame for H with continuous relay bounds A and B. Suppose that $\{Z_{w,v}\}_{v\in\Omega_w}$ is a family of closed subspaces in K_w for each $w\in\Omega$ and there exist constants C, D, ϵ such that $\max\{C+\frac{\epsilon}{\sqrt{A}}, D\} < 1$ and for all $f\in H$*

$$\left(\int_\Omega\int_{\Omega_w}\alpha_{w,v}^2\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f - P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2d\mu(v)d\mu(w)\right)^{\frac{1}{2}}$$
$$\leq C\left(\int_\Omega\int_{\Omega_w}\alpha_{w,v}^2\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f\|^2d\mu(v)d\mu(w)\right)^{\frac{1}{2}}$$
$$+ D\left(\int_\Omega\int_{\Omega_w}\alpha_{w,v}^2\|P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2d\mu(v)d\mu(w)\right)^{\frac{1}{2}} + \epsilon\|f\|.$$

Then $\mathcal{R}_2 = \{W_w, Z_{w,v}, \Lambda_w, \alpha_{w,v}\}_{w\in\Omega, v\in\omega}$ *is a continuous relay fusion frame for* H *with continuous relay fusion frame bounds*

$$A\Big(\frac{1-C-\frac{\epsilon}{\sqrt{A}}}{1+D}\Big)^2, \quad B\Big(\frac{1+A+\frac{\epsilon}{\sqrt{B}}}{1-D}\Big)^2.$$

Proof. We have for each $f \in H$

$$\begin{aligned}&\Big(\int_\Omega\int_{\Omega_w} \alpha^2_{w,v}\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w)\Big)^{\frac{1}{2}}\\&\quad - \Big(\int_\Omega\int_{\Omega_w} \alpha^2_{w,v}\|P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w)\Big)^{\frac{1}{2}}\\&\quad \leq \Big(\int_\Omega\int_{\Omega_w} \alpha^2_{w,v}\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f - P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w)\Big)^{\frac{1}{2}}\\&\quad \leq \Big(C+\frac{\epsilon}{\sqrt{A}}\Big)\Big(\int_\Omega\int_{\Omega_w} \alpha^2_{w,v}\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w)\Big)^{\frac{1}{2}}\\&\qquad + D\Big(\int_\Omega\int_{\Omega_w} \alpha^2_{w,v}\|P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w)\Big)^{\frac{1}{2}}.\end{aligned}$$

Then

$$A\Big(\frac{1-C-\frac{\epsilon}{\sqrt{A}}}{1+D}\Big)^2\|f\|^2 \leq \int_\Omega\int_{\Omega_w} \alpha^2_{w,v}\|P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w).$$

Similarly we can prove that

$$B\Big(\frac{1+C+\frac{\epsilon}{\sqrt{B}}}{1-D}\Big)^2\|f\|^2 \geq \int_\Omega\int_{\Omega_w} \alpha^2_{w,v}\|P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w).$$

◀

Theorem 14. *Let* $\mathcal{R}_1 = \{W_w, V_{w,v}, \Lambda_w, \alpha_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ *be a continuous relay fusion frame for* H *with continuous relay fusion frame bounds* A *and* B*. Suppose that* $\{Z_{w,v}\}_{v\in\Omega_w}$ *is a family of closed subspaces in* K_w *for each* $w \in \Omega$ *and there exists a constant* $0 < C < A$ *such that*

$$\int_\Omega\int_{\Omega_w} \alpha^2_{w,v}\|P_{V_{w,v}}\Lambda_w\pi_{W_w}f - P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w) \leq C\|f\|^2, \quad \forall f \in H.$$

Then $\mathcal{R}_2 = \{W_w, Z_{w,v}, \Lambda_w, \alpha_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ *is a continuous relay fusion frame for* H *with continuous relay fusion frame bounds*

$$\sqrt{C}-\sqrt{A}, \qquad \sqrt{C}+\sqrt{B}.$$

Proof. For each $f \in H$, we have

$$\begin{aligned}
&\|\{\alpha_{w,v}P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\}_{w\in\Omega,v\in\Omega_w}\| \\
&\qquad \leq \|\{\alpha_{w,v}P_{V_{w,v}}\Lambda_w\pi_{W_w}f - \alpha_{w,v}P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\}_{w\in\Omega,v\in\Omega_w}\| \\
&\qquad + \|\{\alpha_{w,v}P_{V_{w,v}}\Lambda_w\pi_{W_w}f\}_{w\in\Omega,v\in\Omega_w}\| \\
&\qquad \leq (\sqrt{C}+\sqrt{B})\|f\|.
\end{aligned}$$

Then

$$\int_\Omega\int_{\Omega_w}\alpha_{w,v}^2\|P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w) \leq (\sqrt{C}+\sqrt{B})^2\|f\|^2.$$

Similarly we have for each $f \in H$

$$\int_\Omega\int_{\Omega_w}\alpha_{w,v}^2\|P_{Z_{w,v}}\Lambda_w\pi_{W_w}f\|^2 d\mu(v)d\mu(w) \geq (\sqrt{C}-\sqrt{A})^2\|f\|^2.$$

◀

References

[1] S.T. Ali, J.P. Antoine, J.P. Gazeau, *Continuous frames in Hilbert spaces*, Annals of physics, **222**, 1993, 1–37.

[2] I. Daubechies, A. Grossmann, Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27**, 1986, 1271–1283.

[3] R.J. Duffin, A.C. Schaeffer, *A class of nonharmonic fourier series*, Trans. Amer. Math. Soc., **72**, 1952, 341-366.

[4] M.H. Faroughi, R. Ahmadi, *Some properties of $C-$fusion frames*, Turk J Math., **34**, 2010, 393-415.

[5] J-P. Gabardo, D. Han, *Frames associated with measurable space*, Adv. Comp. Math., **18**, 2003, 127-147.

[6] P. Gavruta, *On the duality of fusion frames*, J. Math. Anal. Appl., **333(2)**, 22007, 871-879.

[7] G. Kaiser, *A Friendly Guide to Wavelets*, Birkhauser, Boston, 1994.

[8] A. Rahimi, A. Najati, Y.N. Dehghan, *Continuous frames in Hilbert spaces*, Methods of Functional Analysis and Topology, **12(2)**, 2006, 170-182.

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