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$m-$Convex $(m - cv)$ Functions

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Abstract. The theory of $m-$convex $(m - cv)$ functions is a new direction in the theory of real geometry. However, for $m = 1$ this class coincides with the class of convex functions, and for $m = n$ it coincides with the class of subharmonic functions, which, as is known, have been well studied (A.Aleksandrov, I.Bakelman, A.Pogorelov, N.Ivochkina, I.Privalov, etc.) The definition of $m - cv$ functions for $1 < m < n$ has a very different nature, which uses high-order Hessians. Functions for such m have been considered in a series of works by N.Trudinger, X.Wang and others.
In this article, we establish a connection between $m-$convex functions and strongly $m-$subharmonic (sh_m) functions and, using the well-known properties of sh_m functions, we prove a number of important properties of the class of $m - cv$ functions.

Key Words and Phrases: $m-$convex function, strong $m-$subharmonic function, differential form, Hessian.

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1. Introduction

$m-$convex functions are a generalization of convex functions in $\mathbb{R}^n$. Below we will show that they are directly related to strongly $m-$subharmonic (sh_m) functions in the complex space $\mathbb{C}^n$. The theory of sh_m-functions is an actual direction of research in the pluripotential theory, treated by many mathematicians, such as (Z.Blocki [6], S.Dinew and S.Kolodzej [7, 8, 9], S.Y.Li [10], H.C.Lu [11], A.Sadullaev and his disciples [12, 13, 14] and others).

A twice smooth function $u(z) \in C^2(D),\ D \subset \mathbb{C}^n$ is said to be strongly $m-$subharmonic if the relation

$$\begin{aligned} sh_m(D) &= \left\{u \in C^2 :\ (dd^c u)^s \wedge \beta^{n-s} \geq 0,\ \ s = 1, 2, ..., n - m + 1\right\} = \\ &= \{u \in C^2 :\ dd^c u \wedge \beta^{n-1} \geq 0, (dd^c u)^2 \wedge \beta^{n-2} \geq 0, ... \end{aligned}$$

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$$(dd^c u)^{n-m+1} \wedge \beta^{m-1} \geq 0\} \tag{1}$$

holds at each point of the domain D, where $\beta = dd^c\|z\|^2$ is a standard volume form in $\mathbb{C}^n$.

Operators $(dd^c u)^s \wedge \beta^{n-s}$ are closely related to the Hessians. For a twice smooth function $u \in C^2(D)$, the second-order differential $dd^c u = \frac{i}{2}\sum\limits_{k,t}\frac{\partial^2 u}{\partial z_k \partial \bar{z}_t}dz_k \wedge d\,\bar{z}_t$ is a Hermitian quadratic form. After a suitable unitary coordinate transform, it is reduced to the diagonal form $dd^c u = \frac{i}{2}\left[\lambda_1 dz_1 \wedge d\,\bar{z}_1 + ... + \lambda_n dz_n \wedge d\,\bar{z}_n\right]$, where $\lambda_1\,, ..., \lambda_n$ are the eigenvalues of the Hermitian matrix $\left(\frac{\partial^2 u}{\partial z_k \partial \bar{z}_t}\right)$, which are real: $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$. Note that the unitary transformation does not change the differential form $\beta = dd^c\|z\|^2$. Therefore, it is easy to see that

$$(dd^c u)^s \wedge \beta^{n-s} = s!\,(n-s)!H^s\,(u)\,\beta^n,$$

where $H^s(u) = \sum\limits_{1\leq j_1<...<j_s\leq n}\lambda_{j_1}...\lambda_{j_s}$ is the Hessian of dimension s of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

Hence, the twice smooth function $u(z) \in C^2(D),\ D \subset \mathbb{C}^n$, is strongly $m-$ subharmonic if at each point $o \in D$ it satisfies the following inequalities:

$$H_o^s(u) \geq 0,\ \ s = 1, 2, ..., n-m+1. \tag{2}$$

The following very useful theorem is true.

Theorem 1. *(see [6], [7]). For any twice smooth $sh_m \cap C^2(D)$ functions $w_1, ..., w_s \in sh_m(D) \cap C^2(D),\ 1 \leq s \leq n-m+1$, the relation*

$$dd^c w_1 \wedge ... \wedge dd^c w_s \wedge \beta^{m-1} \geq 0$$

is valid. In particular, for $u \in sh_m(D) \cap C^2(D)$ and for any $w_1, ..., w_{n-m} \in sh_m(D) \cap C^2(D)$ the relation

$$dd^c u \wedge dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1} \geq 0 \tag{3}$$

holds. The last property has a dual character: if a twice smooth function u satisfies (3) for all $w_1, ..., w_{n-m} \in sh_m(D) \cap C^2(D)$, then the function u is certainly sh_m function.

Remark 1. *Since any function $w \in C^2(D)$ uniformly approximates in C^2-norm (on compact sets $K \Subset D$), then in (3) as $w_1, ..., w_{n-m} \in sh_m(D) \cap C^2(D)$ we can take second-order Hermitian polynomials or squares (see also [6, 7])*

$$w_j = \sum_{k,t=1}^{n} c_{kt}^j z_k \bar{z}_t \in sh_m\left(\mathbb{C}^n\right), \quad c_{kt}^j = \bar{c}_{tk}^j. \tag{4}$$

Theorem 1 allows us to define sh_m functions in the class L^1_{loc}.

Definition 1. *The function $u \in L^1_{loc}(D)$ is called sh_m in a domain $D \subset \mathbb{C}^n$, if it is upper semicontinuous and for any twice smooth sh_m functions $w_1, ..., w_{n-m}$ in the form (4), the current $dd^c u \wedge dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1}$*

$$\left[dd^c u \wedge dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1}\right](\omega) =$$
$$= \int u dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \qquad \omega \in F^{0,0}$$

is positive defined, i.e.

$$\int u dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \forall \omega \in F^{0,0}, \, \omega \geq 0.$$

We note the following properties of the sh_m functions:

1) $psh = sh_1 \subset sh_2 \subset ... \subset sh_m \subset ... \subset sh_n = sh$;

2) if $u, v \in sh_m$, then $au + bv \in sh_m$ for any $a, b \geq 0$;

3) if $\gamma(t)$ is convex, increasing function of the parameter $t \in \mathbb{R}$ and $u \in sh_m$, then $\gamma \circ u \in sh_m$;

4) the limit of uniformly convergent or decreasing sequence of sh_m functions is sh_m;

5) a maximum of two sh_m functions is again sh_m;

6) if $u \in sh_m$, then for any complex hyperplane $\Pi \subset \mathbb{C}^n$ the restriction $u|_\Pi$ is a sh_m function. As a consequence, it follows that if $u \in sh_m$, then for any $m-$dimensional plane $\Pi \subset \mathbb{C}^n$, $\dim \Pi = m$, the restriction $u|_\Pi$ is a sh function.

2. $m-$convex functions

Let $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D)$. The matrix $\left(\frac{\partial^2 u}{\partial x_k \partial x_t}\right)$ is symmetric, $\frac{\partial^2 u}{\partial x_k \partial x_t} = \frac{\partial^2 u}{\partial x_t \partial x_k}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form

$$\left(\frac{\partial^2 u}{\partial x_k \partial x_t}\right) \to \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $\left(\frac{\partial^2 u}{\partial x_k \partial x_t}\right)$. Let

$$H^s(u) = H^s(\lambda) = \sum_{1 \leq j_1 < ... < j_s \leq n} \lambda_{j_1} ... \lambda_{j_s}$$

be a Hessian of dimension s of the vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$.

Definition 2. *A twice smooth function* $u \in C^2(D)$ *is called* $m-$*convex in* $D \subset \mathbb{R}^n$, $u \in m-cv(D)$, *if its eigenvalue vector* $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), ..., \lambda_n(x))$ *satisfies the condition*

$$m - cv \cap C^2(D) = \{H^s(\lambda(x)) \geq 0, \ \forall x \in D, \ \ s = 1, ..., n-m+1\}.$$

It is clear that for $m = 1$ the class $1 - cv \cap C^2(D) = \left\{H^1(\lambda) \geq 0\right\} = \{\lambda_1 \geq 0, ..., \lambda_n \geq 0\}$ coincides with the convex functions in $\mathbb{R}^n$. The class of convex functions has been well studied by A.Aleksandrov, I.Bakelman, A.Pogorelov, N.Ivochkina, A.Artikbaev and others [1] – [4]. For $m = n$, the class $n - cv \cap C^2(D) = \left\{H^n(\lambda) = \sum\limits_{j=1}^{n} \lambda_j \geq 0\right\}$ coincides with the subharmonic functions in $\mathbb{R}^n$ (see [5]).

When $m > 1$, the class of $m-$ functions has been studied in a series of works by N.Trudinger, X.Wang and others (see [15] – [20]).

3. Relationship between $m - cv$ and sh_m functions

The study of functional properties of the class of $m - cv$ functions and the construction of a potential theory in it is the main subject of this paper. Our purpose and method of study are somewhat different from the approach of the authors mentioned above, where the main focus was on solving equations in Hessians of type $H^{n-m+1}(\lambda(x)) = f(x, u)$ in the class of $m - cv$ functions. The point is that, in the class of sh_m functions, this Hessian type equation is equivalent to the nonlinear elliptic equation $(dd^c u)^{n-m+1} \wedge \beta^{m-1} = f(x, u)\beta^n$.

We will consider real space $\mathbb{R}^n_x$ in corresponding complex space $\mathbb{C}^n$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y, (z = x + iy)$, as a real $n-$dimensional subspace.

Proposition 1. *A twice smooth* $u(x) \in C^2(D)$, $D \subset \mathbb{R}^n_x$, *is* $m - cv$ *in* D *if and only if the function* $u^c(z) = u^c(x + iy) = u(x)$, *that does not depend on variables* $y \in \mathbb{R}^n_y$, *is* sh_m *in the domain* $D \times \mathbb{R}^n_y$.

Proof. Recall that $u^c(z)$ is sh_m if and only if the eigenvalues $\lambda_j = \lambda_j(z) \in \mathbb{R}-$ of the matrix $\left\| \frac{\partial^2 u^c(z)}{\partial z_k \partial \bar{z}_t} \right\|$ satisfy $H^1(\lambda) \geq 0, ..., H^{n-m+1}(\lambda) \geq 0$. But

$$\frac{\partial^2 u^c(z)}{\partial z_k \partial \bar{z}_t} = \frac{\partial^2 u^c(x + iy)}{\partial z_k \partial \bar{z}_t} = \frac{\partial^2 u(x)}{\partial z_k \partial \bar{z}_t} = \frac{1}{4} \frac{\partial^2 u(x)}{\partial x_k \partial x_t}.$$

Then, the eigenvalues of the matrices $\left\| \frac{\partial^2 u^c(z)}{\partial z_k \partial \bar{z}_t} \right\|$ and $\left\| \frac{\partial^2 u(x)}{\partial x_k \partial x_t} \right\|$ coincides. Therefore, $u \in m-cv(D) \Leftrightarrow u^c \in sh_m(D \times \mathbb{R}^n_y)$ and this implies that $u \in C^2 \cap m-cv(D)$

if and only if in the domain $D^c = D\times\mathbb{R}^n_y \subset \mathbb{C}^n$ the differential forms satisfy

$$(dd^c u^c)^s \wedge \beta^{n-s} \geq 0, \quad s = 1, 2, ..., n-m+1.$$

◀

Below we will need the following lemma.

Lemma 1. *A Hermitian square* $w = \sum\limits_{k,t=1}^{n} c_{kt} z_k \bar{z}_t,\ c_{kt} = \bar{c}_{tk}$ *is a* sh_m-*function,denoted* $w \in sh_m(\mathbb{C}^n)$*, if and only if the real square* $v = \sum\limits_{k,t=1}^{n} d_{kt} x_k x_t$ *is an* $m-cv(\mathbb{R}^n)$ *function, where*

$$d_{kt} = \begin{cases} c_{kt} & if \ \ k \neq t \\ \frac{c_{kt}}{2} & if \ \ k = t. \end{cases}$$

Proof. Since $d_{kt} = d_{tk}$, the function

$$v = \sum_{k,t=1}^{n} d_{kt} x_k x_t = \sum_{k<t} [c_{kt} + c_{tk}]\, x_k x_t + \frac{1}{2}\sum_{k=1}^{n} c_{kk} x_k^2 =$$
$$= \sum_{k<t} 2\mathrm{Re} c_{kt} x_k x_t + \frac{1}{2}\sum_{k=1}^{n} c_{kk} x_k^2 = \sum_{k,t=1}^{n} \mathrm{Re} c_{kt} x_k x_t$$

is real. We show that if $w \in sh_m(\mathbb{C}^n)$, then $v = \sum\limits_{k,t=1}^{n} d_{kt} x_k x_t \in m-cv(\mathbb{R}^n)$ or, which is the same, $v^c(z) = v(x) \in sh_m(\mathbb{C}^n)$. We have $\frac{\partial^2 v^c(z)}{\partial z_k \partial \bar{z}_t} = \frac{1}{4}\frac{\partial^2 v(x)}{\partial x_k \partial x_t}$. Consequently,

$$dd^c v^c = \sum_{k,t} d_{k,t} \frac{\partial^2 [x_k x_t]}{\partial z_k \partial \bar{z}_t} dz_k \wedge d\bar{z}_t = \frac{1}{4}\sum_{k,t} d_{k,t} \frac{\partial^2 [x_k x_t]}{\partial x_k \partial x_t} dz_k \wedge d\bar{z}_t =$$
$$= \frac{1}{4}\sum_{k\neq t} c_{k,t}\, dz_k \wedge d\bar{z}_t + \frac{1}{4}\sum_{k=1}^{n} c_{kk} dz_k \wedge d\bar{z}_k = \frac{1}{4} dd^c w.$$

It follows $v^c(z) = v(x) \in sh_m(D\times\mathbb{R}^n_y)$ for $w = \sum\limits_{k,t=1}^{n} c_{kt} z_k \bar{z}_t \in sh_m(\mathbb{C}^n)$. Therefore, $v(x) \in m-cv(D)$.

Conversely, if $v(x) \in m-cv(D)$, then $v^c(z) = v(x) \in sh_m\left(D\times\mathbb{R}^n_y\right)$. It follows from $dd^c v^c = \frac{1}{4} dd^c w$ that $w \in sh_m(\mathbb{C}^n)$. Lemma 1 is proved. ◀

The following theorem is the main result of our study on $m-cv$ functions.

Theorem 2. *A twice smooth function* $u(x)$, $x \in D \subset \mathbb{R}^n_x$, *is* $m-cv(D)$ *if and only if*

$$dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \geq 0\,, \ \ \forall\, v_1, ..., v_{n-m} \in m-cv(D) \cap C^2(D). \quad (5)$$

Moreover, it suffices here to consider the class of squares

$$v_j = \sum_{k,t=1}^{n} d^j_{kt} x_k x_t \in m-cv(D), \ \ d^j_{kt} \in \mathbb{R}, \ \ d^j_{kt} = d^j_{tk}, \ \ j = 1, 2, ..., n-m. \quad (6)$$

Proof. Necessity. If $u(x) \in m-cv(D)$, then, by Proposition 1, $u^c, v_1^c, ..., v_{n-m}^c \in sh_m(D \times \mathbb{R}^n_y) \cap C^2(D \times \mathbb{R}^n_y)$, and by Theorem 1

$$dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \geq 0.$$

Sufficiency. Let $u(x) \in C^2(D)$ satisfy conditions (5). We need to demonstrate that $u^c(z) = u^c(x+iy) = u(x)$ is a $sh_m(D \times \mathbb{R}^n_y)$ function, which is the same as

$$dd^c u^c \wedge dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1} \geq 0, \ \forall\, w_j = \sum_{k,t=1}^{n} c^j_{kt} z_k \bar{z}_t \in sh_m\left(\mathbb{C}^n\right), \, c^j_{kt} = \bar{c}^j_{tk},$$

$$dd^c w_j = \sum_{k,t=1}^{n} c^j_{kt} dz_k \wedge d\bar{z}_t. \quad (7)$$

According to Lemma 1, the function $v_j = \sum\limits_{k,t=1}^{n} d^j_{kt} x_k x_t$ is $m-$convex, where

$$d^j_{kt} = \begin{cases} c^j_{kt} & if \ \ k \neq t \\ \frac{c^j_{kt}}{2} & if \ \ k = t, \end{cases}$$

$v_j = \sum\limits_{k,t=1}^{n} d^j_{kt} x_k x_t \in m-cv\left(\mathbb{R}^n\right)$ or, which is the same, $v_j^c(z) = v_j(x) \in sh_m\left(\mathbb{C}^n\right)$. According to assumption (5),

$$dd^c u^c \wedge dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1} = \frac{1}{4} dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \geq 0,$$

$\forall w_j = \sum\limits_{k,t=1}^{n} c^j_{kt} z_k \bar{z}_t \in sh_m\left(\mathbb{C}^n\right), \ \ c^j_{kt} = \bar{c}^j_{tk}$. Theorem 2 is proved. ◄

We note that Theorem 2 allows us to give a criterion for $u\,(x) \in m-cv(D)$ to be in the class $L^1_{loc}(D)$.

Definition 3. *The function $u(x) \in L^1_{loc}(D)$ is called $m-$convex function in the domain $D \subset \mathbb{R}^n_x$, $u(x) \in m-cv(D)$, if it is upper semicontinuous and for any twice smooth $m-cv(D)$ functions $v_1, ..., v_{n-m}$, the current $dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1}$ defined as*

$$\left[dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1}\right](\omega) =$$
$$= \int u^c dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0}\left(D \times \mathbb{R}^n_y\right) \tag{8}$$

is positive, i.e.

$$\int u^c dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \forall \omega \in F^{0,0}\left(D \times \mathbb{R}^n_y\right), \omega \geq 0.$$

Definition 3 allows us to obtain the following refinement of Proposition 1.

Proposition $1'$. *Function $u(x) \in L^1_{loc}(D)$, $D \subset \mathbb{R}^n_x$, is $m-cv$ in D if and only if the function $u^c(z) = u^c(x+iy) = u(x)$ is sh_m in domain $D \times \mathbb{R}^n_y$.*

Proof. It is clear that the functions $u(x)$ and $u^c(z) = u^c(x+iy) = u(x)$ both belong to the class L^1_{loc} and are upper semicontinuous at the same time. If $u(x) \in m-cv(D)$, then, according to Definition 3, for any twice smooth $m-cv(D)$ functions $v_1, ..., v_{n-m}$, the current $dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1}$ defined as

$$\left[dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1}\right](\omega) =$$
$$= \int u^c dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0}\left(D \times \mathbb{R}^n_y\right) \tag{9}$$

is positive. In particular, this current is also positive for all functions of the form

$$v_j = \sum_{k,t} d^j_{kt} x_k x_t \in m-cv(D), \ d^j_{kt} = d^j_{tk}, \quad j = 1, 2, ..., n-m.$$

As we have seen above, for any functions $w_j = \sum\limits_{k,t} c^j_{kt} z_k \bar{z}_t \in sh_m(\mathbb{C}^n)$, $c^j_{kt} = \bar{c}^j_{tk}$, we have $dd^c v_j^c = \frac{1}{4} dd^c w_j$ and $v_j^c \in sh_m(\mathbb{C}^n)$, or, which is the same, $v_j(x) \in m-cv(D)$. Here

$$v_j = \sum_{k,t=1}^{n} d^j_{kt} x_k x_t, \quad d^j_{kt} = \begin{cases} c^j_{kt} & if \ \ k \neq t \\ \frac{c^j_{kt}}{2} & if \ \ k = t. \end{cases}$$

Therefore, according to the condition

$$\int u^c \, dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \forall \omega \in F^{0,0}\left(D \times \mathbb{R}^n_y\right), \ \omega \geq 0,$$

we obtain

$$\int u^c dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega =$$

$$= 4^{n-m} \int u^c dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \wedge dd^c \omega \geq 0,\ \forall \omega \in F^{0,0}(D \times \mathbb{R}_y^n),\ \omega \geq 0,$$

i.e. $u^c(z) \in\ sh_m(D \times \mathbb{R}_y^n)$.

Conversely, if $u^c(z) = u(x) \in sh_m(D \times \mathbb{R}_y^n)$, then

$$\int u^c dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \forall \omega \in F^{0,0}\left(D \times \mathbb{R}_y^n\right), \ \omega \geq 0,$$

$$\forall w_j = \sum_{k,t} c_{kt}^j z_k \bar{z}_t \in sh_m\left(\mathbb{C}^n\right), \ c_{kt}^j = \bar{c}_{tk}^j,\ j = 1, 2, ..., n-m.$$

But then for any real function

$$v_j = \sum_{k,t} d_{kt}^j x_k x_t \in m - cv\left(\mathbb{R}^n\right), \ d_{kt}^j = d_{tk}^j, \quad j = 1, 2, ..., n-m,$$

we have $dd^c v_j^c = \frac{1}{4} dd^c w_j$, where $w_j = \sum\limits_{k,t=1}^{n} c_{kt}^j w_k \bar{w}_t$ and

$$c_{kt}^j = \begin{cases} d_{kt}^j & if \ \ k \neq t \\ 2d_{kk}^j & if \ \ k = t. \end{cases}$$

It follows $w_j \in sh_m\left(\mathbb{C}^n\right), \ j = 1, 2, ..., n-m$ and, according to Theorem 1,

$$\int u^c dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \omega \in F^{0,0}\left(D \times \mathbb{R}_y^n\right), \ \omega \geq 0.$$

Therefore,

$$\int u^c dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \wedge dd^c \omega =$$

$$= \frac{1}{4^{n-m}} \int u^c dd^c w_1 \wedge ... \wedge dd^c w_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0,\ \omega \in F^{0,0}(D \times \mathbb{R}_y^n),\ \omega \geq 0,$$

which proves $u(x) \in\ m - cv(D)$. ◀

4. Properties of $m-cv(D)$ functions

We note the following properties of the $m-cv$ functions.

1. $cv = 1-cv \subset 2-cv \subset ... \subset m-cv \subset ... \subset n-cv = sh$.

Proof. We will use Proposition $1'$: an upper semicontinuous function $u(x) \in L^1_{loc}(D), D \subset \mathbb{R}^n$ is in $m-cv(D)$ if and only if the function $u^c(z) = u(x+iy) = u(x)$ is in $sh_m(D \times \mathbb{R}^n)$. Let $u(x) \in (m-1)-cv(D)$. Then $u^c(z) \in sh_{m-1}(D \times \mathbb{R}^n)$. Since the inclusion $sh_{m-1} \subset sh_m$ is whell known, we have $u^c(z) \in sh_m(D \times \mathbb{R}^n)$. Hence $u(x) \in m-cv(D)$. ◀

2. If $u, v \in m-cv$, then $au + bv \in m-cv$ for any $a, b \geq 0$.

Proof. If $u, v \in m-cv$, then the following currents are positive:

$$dd^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \geq 0,$$
$$dd^c v^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \geq 0.$$

From this it follows that

$$dd^c(au^c + bv^c) \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} =$$
$$add^c u^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} +$$
$$bdd^c v^c \wedge dd^c v_1^c \wedge ... \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \geq 0.$$

◀

3. If $\gamma(t)$ is convex, increasing function of the parameter $t \in \mathbb{R}$ and $u \in m-cv$, then $\gamma \circ u \in m-cv$.

4. The limit of uniformly converging or decreasing sequence of $m-cv$ functions is $m-cv$ (obviously).

5. The maximum of finite number of $m-cv$ functions is again $m-cv$.

Proof. It is enough to prove it for a maximum of two functions $u, v \in m-cv$. We again use Proposition $1'$, that the functions $u^c, v^c \in sh_m\left(D \times \mathbb{R}^n_y\right)$. As proved in [13], $\max\{u^c(z), v^c(z)\} \in sh_m\left(D \times \mathbb{R}^n_y\right)$. But then

$$[\max\{u(x), v(x)\}]^c = \max\{u^c(z), v^c(z)\} \in sh_m\left(D \times \mathbb{R}^n_y\right).$$

That is, $\max\{u(x), v(x)\} \in m-cv(D)$. ◀

6. If $u \in m-cv$, then for any hyperplane $\Pi \subset \mathbb{R}^n$ the restriction $u|_\Pi$ is also $m-cv$ function on Π;

Proof. Indeed, we fix a hyperplane $\Pi \subset \mathbb{R}^n$, assuming, without loss of generality, $\Pi = \{(x_1, ..., x_{n-1}, x_n) \in \mathbb{R}^n : x_n = 0\}$. Then $\Pi^{\mathbb{C}} = \Pi \times i\Pi = \{(z_1, ..., z_{n-1}, z_n) \in \mathbb{C}^n : z_n = 0\}$ is a hyperplane in a complex space $\mathbb{C}^n$. In

[13] Sadullaev-Abdullaev proved that if $w(z) \in sh_m(D \times \mathbb{R}^n_y)$, then $w(z)|_{\Pi^{\mathbb{C}}} \in sh_m((D \times \mathbb{R}^n_y) \cap \Pi^{\mathbb{C}})$. Hence, $u^c(z)|_{\Pi^{\mathbb{C}}} \in sh_m((D \times \mathbb{R}^n_y) \cap \Pi^{\mathbb{C}})$, which means $u(x)|_{\Pi} \in m - cv(D \cap \Pi)$. ◀

Corollary 1. *If $u \in m - cv$, then for any plane $\Pi \subset \mathbb{R}^n$, $\dim_{\mathbb{R}} \Pi = m$, the restriction $u|_{\Pi}$ is a subharmonic function, $u|_{\Pi} \in m - cv = sh$.*

The proof is easily obtained by applying property 6 consecutively to the planes $\Pi_{n-1} \supset \Pi_{n-2} \supset ... \supset \Pi_m = \Pi$. ◀

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