Azerbaijan Journal of Mathematics
V. 13, No 2, 2023, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2023.2.248

Generalized (G'/G) - Expansion Method and Its Applications to the Loaded Burgers Equation

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Abstract. This paper is dedicated to finding the traveling wave solutions of the loaded Burgers equation. We show how to find the solutions via the (G'/G) - expansion method which is one of the most effective ways of finding solutions. When the parameters are taken as special values, the solitary waves are also derived from the traveling waves. The traveling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. This method is easy to implement using well-known software packages, which allows you to solve complex nonlinear evolution equations of mathematical physics.

Key Words and Phrases: expansion method, evolution equations, continuous function, loaded equation, Burgers equation.

2010 Mathematics Subject Classifications: 34A34, 65L05

1. Introduction

The Burgers equation appears in various areas of applied mathematics, such as modeling of fluid dynamics, turbulence, boundary layer behavior, shock wave formation and traffic flow.

The Modified Equal Width Wave Equation and Burger-Huxleys equation can be regarded as models to describe the interaction between reaction mechanisms, convection effects and diffusion transports [1]. Many physical problems can be described by Burger-KdV and mBKdV equations.

Typical examples are provided by the behavior of long waves in shallow water and waves in plasmas. McIntosh [1] demonstrated how to describe the average behavior of traveling wave solution of mBKdV in the case of small dissipation.

The nonlinear differential equations with loaded terms, which involve the coefficients and the functionals of the solution, have been well studied in [2, 3,

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4, 5, 6]. The concepts and detailed classification of loaded differential equations and their numerous applications to problems of biology have been initially given in [6] as the general definition of a loaded equation.

Alternatively, the (G'/G) - expansion method [7, 8, 9, 10, 11, 12, 13, 14, 15, 16] is also effective in finding traveling wave solutions of nonlinear evolution equations. In [17], the (G'/G) - expansion method was applied to the Burgers, Burgers–Huxley and modified Burgers–KdV equations. Integration of the loaded modified Korteweg-de Vries (mKdV) equation in the class of periodic functions was treated in [18]. (G'/G) - expansion method was used for the integrations of loaded Korteweg-de Vries (KdV) equation and the loaded modified Korteweg-de Vries (mKdV) equation in [19, 20].

In this paper, we treat the loaded Burgers equation using (G'/G)-expansion method.

Let us consider the following loaded Burgers equation:

$$q_t + qq_x - q_{xx} + \gamma(t)q(0,t)q_x = 0 \tag{1}$$

where $q(x,t)$ is an unknown function, $x \in R$, $t \geq 0$, and $\gamma(t)$ is the given real continuous function.

2. Description of the generalized (G'/G) - expansion method

Consider a nonlinear partial differential equation

$$F(q, q_t, q_x, q_{tt}, q_{xx}, q_{xt}, ...) = 0 \tag{2}$$

with two independent variables x and t, where $q = q(x,t)$ is the unknown function, and F is a polynomial of q and its partial derivatives, with the highest order derivatives and nonlinear terms involved. Now, let us state the main steps of the (G'/G) - expansion method [11, 12]:

Step 1. We look for the q in the travelling form:

$$q(x,t) = q(\xi), \xi = kx - \Omega(t), \tag{3}$$

where k is a parameter and $\Omega(t)$ is a continuous function dependent on t. We reduce equation (2) to the following nonlinear ordinary differential equation:

$$P(q, q', q'', q''', ...) = 0, \tag{4}$$

where P is a polynomial of $q(\xi)$ and its all derivatives, $q' = dq(\xi)/d\xi$, $q'' = d^2q(\xi)/d\xi^2$.

Step 2. We assume that the solution of equation (4) has the form

$$q(\xi)=\sum_{j=0}^{m}a_j\left(\frac{G'}{G}\right)^j, \tag{5}$$

with $G=G(\xi)$ satisfying the following second order ordinary differential equation:

$$G''+\lambda G'+\mu G=0, \tag{6}$$

where $G'=dG(\xi)/d\xi$, $G''=d^2G(\xi)/d\xi^2$ and λ, μ, a_j $(j=1,2,\ldots,m)$ are the constants that can be determined later, provided $a_m\neq 0$.

Step 3. We determine the integer m by balancing the nonlinear terms of the highest order and the partial product of the highest order of (4).

Step 4. We substitute (5) and (6) into (4) and collect all terms with the same order of $(G'(\xi)/G(\xi))$, so the left-hand side of (4) becomes a polynomial of $(G'(\xi)/G(\xi))$. Then, we equate each coefficient of this polynomial to zero to obtain a set of over-determined partial differential equations for a_j $(j=1,2,\ldots,m)$ and ξ.

Step 5. Substituting the values a_j $(j=1,2,\ldots,m)$ and ξ as well as the solution of equation (6) into (5), we have the exact solution of equation (2).

3. Exact solution of the loaded Burgers equation

We will show how to find the exact solution of the loaded Burgers equation using the (G'/G) - expansion method. For this, we perform the steps above for equation (1). The travelling wave variable below

$$q(x,t)=q(\xi),\xi=kx-\Omega(t), \tag{7}$$

allows us converting equation (1) into an ordinary differential equation for $q=q(\xi)$

$$-\Omega_t'(t)q'+kqq'-k^2q''+k\gamma(t)q(0,t)q'=0. \tag{8}$$

Integrating it with respect to ξ yields

$$C-\Omega_t'(t)q+\frac{k}{2}q^2-k^2q'+k\gamma(t)q(0,t)q=0, \tag{9}$$

where C is an integration constant that can be determined later.

We express the solution of equation (9) in the form of a polynomial of (G'/G) as follows:

$$q(\xi)=\sum_{j=0}^{m}a_j\left(\frac{G'}{G}\right)^j, \tag{10}$$

where $G = G(\xi)$ satisfies the second order ordinary differential equation in the form

$$G'' + \lambda G' + \mu G = 0. \tag{11}$$

Using (10) and (11), we can easily find q^2 and q':

$$q^2(\xi) = a_m^2 \left(\frac{G'}{G}\right)^{2m} + \ldots, \tag{12}$$

$$q'(\xi) = m a_m \left(\frac{G'}{G}\right)^{m+1} + \ldots . \tag{13}$$

Considering the homogeneous balance between q' and q^2 in equation (9), based on (12) and (13) we require that $m = 1$. Taking into account the above considerations, q becomes

$$q(\xi) = a_1 \left(\frac{G'}{G}\right) + a_0. \tag{14}$$

Then we obtain the exact expression for q^2:

$$q^2(\xi) = a_1^2 \left(\frac{G'}{G}\right)^2 + 2a_1 a_0 \left(\frac{G'}{G}\right) + a_0^2. \tag{15}$$

Using (14) and (11), q' can be written as

$$q'(\xi) = -a_1 \left(\frac{G'}{G}\right)^2 - a_1 \lambda \left(\frac{G'}{G}\right) - a_1 \mu. \tag{16}$$

By substituting (14)-(16) into equation (9) and collecting all terms with the same power of (G'/G), we convert the left-hand side of equation (9) into another polynomial of (G'/G):

$$(\frac{1}{2}ka_1^2 + k^2 a_1)\left(\frac{G'}{G}\right)^2 + (-\Omega_t'(t)a_1 + ka_1a_0 + \lambda k^2 a_1 + \gamma(t)q(0,t)ka_1)\left(\frac{G'}{G}\right) +$$

$$+(C - \Omega_t'(t)a_0 + \frac{1}{2}ka_0^2 + k^2 a_1 \mu + \gamma(t)q(0,t)ka_0)\left(\frac{G'}{G}\right)^0 = 0. \tag{17}$$

Equating each coefficient of expression (17) to zero yields a set of simultaneous equations for a_0, a_1, $\Omega(t)$ and C as follows:

$\left(\frac{G'}{G}\right)^2 : \frac{1}{2}ka_1^2 + k^2 a_1 = 0,$

$\left(\frac{G'}{G}\right) : -\Omega_t'(t)a_1 + ka_1a_0 + \lambda k^2 a_1 + \gamma(t)q(0,t)ka_1 = 0,$

$\left(\frac{G'}{G}\right)^0 : C - \Omega_t'(t)a_0 + \frac{1}{2}ka_0^2 + k^2 a_1\mu + \gamma(t)q(0,t)ka_0 = 0.$
By solving these equations, we obtain

$$a_0 = -k\lambda \pm k\sqrt{\lambda^2 - 4\mu + \frac{2}{k^3}}, a_1 = -2k,$$

$$\Omega(t) = \pm k^2\sqrt{\lambda^2 - 4\mu + \frac{2}{k^3}}t + k\int_0^t \gamma(\tau)q(0,\tau)d\tau + \Omega^0, \tag{18}$$

where λ, μ, k and Ω^0 are arbitrary constants.

Using (18), the expression (14) can be rewritten as

$$q(\xi) = -2k\left(\frac{G'}{G}\right) - k\lambda \pm k\sqrt{\lambda^2 - 4\mu + \frac{2}{k^3}}, \tag{19}$$

where $\xi = kx \mp k^2\sqrt{\lambda^2 - 4\mu + \frac{2}{k^3}}t - k\int_0^t \gamma(\tau)q(0,\tau)d\tau - \Omega^0$. The function (19) is a solution of equation (9), provided that the integration constant C in equation (9) is taken as that in (18). Substituting the general solutions of equation (11) into (19), we obtain three types of travelling wave solutions of the loaded Burgers equation (1) as follows:

When $(\lambda^2 - 4\mu) > 0$,

$$q(\xi) = \pm k\sqrt{\lambda^2 - 4\mu + \frac{2!}{k^3}} - \sqrt{\lambda^2 - 4\mu}\left(\frac{c_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + c_2\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + c_2\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right), \tag{20}$$

where $\xi = kx \mp k^2\sqrt{\lambda^2 - 4\mu + \frac{2!}{k^3}}t - k\int_0^t \gamma(\tau)q(0,\tau)d\tau - \Omega^0$, c_1, c_2 and Ω^0 are arbitrary constants. It is obvious that the function $q(0,t)$ can be easily found from expression (20).

For example, let $\gamma(t)$ be given as follows:

$$\gamma(t) = \frac{\left(-\frac{1}{k}\sum_{j=1}^n j\alpha_j t^{j-1} \mp k\sqrt{\lambda^2 - 4\mu + \frac{2}{k^3}}\right)}{\pm k^2\sqrt{\lambda^2 - 4\mu + \frac{2}{k^3}} + \sqrt{\lambda^2 - 4\mu}\left(\frac{\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}(\sum_{j=1}^n \alpha_j t^j)}{\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}(\sum_{j=1}^n \alpha_j t^j)}\right)},$$

where α_j $(j = 1, 2, \ldots, n)$ are constants. If $c_1 \neq 0$, $c_2 = 0$ and $(\lambda^2 - 4\mu) > 0$, then $q(x,t)$ becomes

$$q(x,t) = \pm k^2\sqrt{\lambda^2 - 4\mu + \frac{2}{k^3}} - \sqrt{\lambda^2 - 4\mu}\left(\frac{\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(x - \sum_{j=1}^n \alpha_j t^j)}{\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(x - \sum_{j=1}^n \alpha_j t^j)}\right). \tag{21}$$

The function (21) is the solution of the following loaded Burgers equation:

$$q_t + qq_x - q_{xx} + \left(-\frac{1}{k} \sum_{j=1}^{n} j\alpha_j t^{j-1} \mp k\sqrt{\lambda^2 - 4\mu + \frac{2}{k^3}} \right) q_x = 0. \tag{22}$$

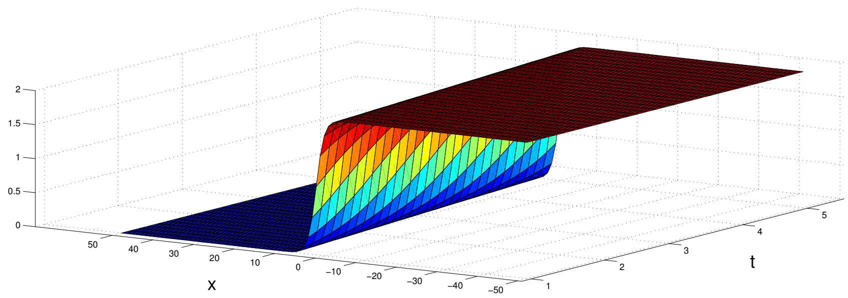

Figure 1: The solution (21) of the loaded Burgers equation (22) for $\lambda = 3$, $\mu = 2$, $k = 1$, $C = 0$, $c_1 = 1$, $c_2 = 0$, $\alpha = 1$.

When $(\lambda^2 - 4\mu) < 0$,

$$q(\xi) = \pm k\sqrt{4\mu - \lambda^2 + \frac{2}{k^3}} - \sqrt{4\mu - \lambda^2} \left(\frac{c_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{c_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi} \right), \tag{23}$$

where $\xi = kx \mp k^2\sqrt{4\mu - \lambda^2 + \frac{2}{k^3}}t - k\int_0^t \gamma(\tau)q(0,\tau)d\tau - \Omega^0$, c_1, c_2 and Ω^0 are arbitrary constants. It is clear that it is not difficult for us to find $q(0,t)$ based on expression (23). Let $\gamma(t)$ be given as below:

$$\gamma(t) = \frac{\left(-\frac{1}{k}\sum_{j=1}^{n} j\alpha_j t^{j-1} \mp k\sqrt{4\mu - \lambda^2 + \frac{2}{k^3}} \right)}{\pm k\sqrt{4\mu - \lambda^2 + \frac{2}{k^3}} + \sqrt{4\mu - \lambda^2}\left(tg\frac{1}{2}\sqrt{4\mu - \lambda^2}\sum_{j=1}^{n} \alpha_j t^j \right)},$$

where α_j $(j = 1, 2, \ \dots, n)$ are constants. In particular, if $c_1 \neq 0$ and $c_2 = 0$, then $q(x,t)$ becomes

$$q(x,t) = \pm k\sqrt{4\mu - \lambda^2 + \frac{2}{k^3}} - \sqrt{4\mu - \lambda^2}\, tg \left(\frac{1}{2}\sqrt{4\mu - \lambda^2}(x - \sum_{j=1}^{n} \alpha_j t^j) \right). \tag{24}$$

The function (24) is the solution of the following loaded Burgers equation:

$$q_t + qq_x - q_{xx} + \left(-\frac{1}{k} \sum_{j=1}^{n} j\alpha_j t^{j-1} \mp k\sqrt{4\mu - \lambda^2 + \frac{2}{k^3}} \right) q_x = 0. \tag{25}$$

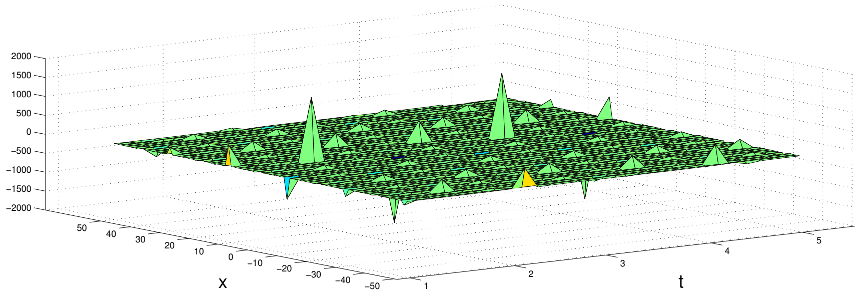

Figure 2: The solution (24) of the loaded Burgers equation (25) for $\lambda = 2$, $\mu = 3$, $k = 1$, $C = 0$, $c_1 = 1$, $c_2 = 0$, $\alpha = 1$

When $(\lambda^2 - 4\mu) = 0$,

$$q(\xi) = \pm\sqrt{\frac{2}{k}} - \frac{2c_2}{c_1 + \xi c_2}, \tag{26}$$

where $\xi = x \mp \sqrt{\frac{2}{k}}t - \int_0^t \gamma(\tau)q(0,\tau)d\tau - \Omega^0$, c_1,c_2 and Ω^0 are arbitrary constants. The function $q(0,t)$ is found based on expression (26). If $c_1 = 0$, $c_2 \neq 0$, $(\lambda^2 - 4\mu) = 0$ and $\gamma(t)$ is given by

$$\gamma(t) = \frac{\left(\mp\sqrt{\frac{2}{k}} - \sum_{j=1}^n j\alpha_j t^{j-1}\right)\sum_{j=1}^n \alpha_j t^j}{\pm\sqrt{\frac{2}{k}}\sum_{j=1}^n \alpha_j t^j - 2},$$

then $q(x,t)$ becomes

$$q(x,t) = \pm\sqrt{\frac{2}{k}} - \frac{2}{x + \sum_{j=1}^n \alpha_j t^j}. \tag{27}$$

We know that the function (27) satisfies the following loaded Burgers equation:

$$q_t + qq_x - q_{xx} + \left(\mp\sqrt{\frac{2}{k}} - \sum_{j=1}^n j\alpha_j t^{j-1}\right) q_x = 0. \tag{28}$$

The exact solutions of the loaded nonlinear differential equations describe different types of nonlinear waves. In particular, the established hyperbolic solutions represent a specific type of solitary wave.

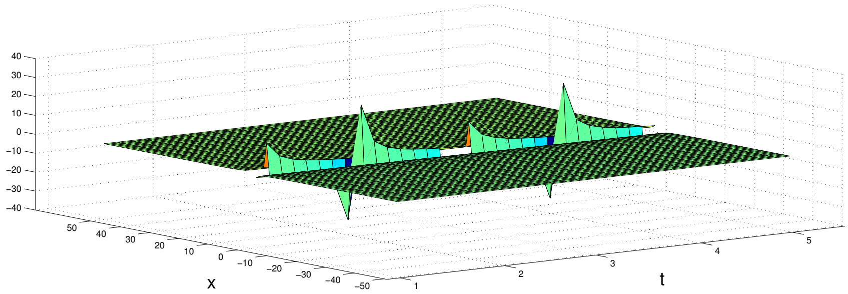

Figure 3: The solution (27) of the loaded Burgers equation (28) for $\lambda = 3$, $\mu = 2$, $k = 1$, $C = 0$, $c_1 = 1$, $c_2 = 0$, $\alpha = 1$

4. Conclusion

The results of this study show that the (G'/G) - expansion method is effective in obtaining the exact solutions of the loaded Burgers equation. Parameters c_1, c_2, λ, μ, κ and arbitrary function $\gamma(t)$ in solutions (20), (23), (26) provide sufficient freedom for the construction of solutions.

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Received 29 November 2022
Accepted 07 May 2023