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Common Attractive Point Approximations for Family of Generic Generalized Bregman Nonspreading Mappings in Banach Spaces

B. Ali, L.Y. Haruna*, M.H. Harbau

Abstract. In this paper, a class of generic generalized Bregman nonspreading mappings which is said to include the classes of generalized Bregman nonspreading, generic generalized nonspreading, generalized hybrid mappings etc. as special cases is investigated. Then, a theorem for existence of attractive point of the said mapping is established in the setting of reflexive Banach spaces. Also, we prove a demiclosedness property and construct a Halpern type iterative algorithm that converges strongly to the common attractive point of finite family of generic generalized Bregman nonspreading mappings in the space. We further apply our main result to approximate common fixed point of the said mappings. Our results improve and generalize many corresponding ones announced in the literature.

Key Words and Phrases: Bregman attractive point, generalized Bregman nonspreading mapping, generic generalized Bregman nonspreading mapping, Halpern type algorithm.

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1. Introduction

Let C be a nonempty subset of a real Hilbert space H and $T : C \to H$ be a map. Denote the set of fixed points of T by $F(T)$ and the set of all attractive points of T by $A(T)$, i.e., $F(T) = \{u \in C : Tu = u\}$ and $A(T) = \{u \in H : \|Tv - u\| \le \|v - u\|, \forall v \in C\}$. A nonlinear mapping $T : C \to H$ is called

(1) nonexpansive if $\|Tx - Ty\| \le \|x - y\| \ \forall \ x, y \in C$;

(2) nonspreading [20] if $2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2 \ \forall \ x, y \in C$;

*Corresponding author.

(3) hybrid [20, 29] if $3\|Tx-Ty\|^2 \leq \|x-y\|^2+\|Tx-y\|^2+\|Ty-x\|^2 \;\forall\, x, y \in C$;

(4) (α, β)-generalized hybrid [17] if for some $\alpha, \beta \in \mathbb{R}$, the inequality

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

holds for all $x, y \in \mathbb{R}$. Observe that a $(1,0)$-generalized hybrid mapping is nonexpansive, a $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping is hybrid and a $(2,1)$-generalized hybrid mapping is nonspreading. The class of generalized hybrid mappings was extended to that of generalized nonspreading mappings in the setting of Banach spaces more general than Hilbert. A mapping T of nonempty subset C of a smooth Banach space E into itself is called a generalized nonspreading mapping [18] if there exists $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\phi(Tx, Ty) &+ (1-\alpha)\phi(x, Ty)+\gamma\{\phi(Ty, Tx)-\phi(Ty, x)\} \\ &\leq \beta\phi(Tx, y)+(1-\beta)\phi(x, y)+\delta\{\phi(y, Tx)-\phi(y, x)\} \end{aligned}$$

for all $x, y \in C$, where $\phi : E \times E \to \mathbb{R}$ is the Lyapunov functional defined by $\phi(x, y) = ||x||^2+2\langle x, Jy\rangle+||y||^2$ and J is the duality mapping from E into E^* defined by $Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$. By calling such mapping T an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping, we see that a (1,1,1,0)-generalized nonspreading mapping is nonspreading [20]. If E is a real Hilbert space, then $\phi(x, y) = \|x-y\|^2$ and subsequently $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping reduces to $(\alpha+\gamma, \beta+\delta)$-generalized hybrid mapping. A class of mappings which is said to include as special case that of generalized nonspreading ones was introduced by Takahashi et al. [30]. A mapping $T : C \to E$ is called generic $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-generalized nonspreading mapping if there exist $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{R}$ such that for all $x, y \in C$, (i) $\alpha+\beta+\gamma+\delta \geq 0$ (ii) $\alpha+\beta > 0$ and

$$\begin{aligned} (iii)\ \alpha\phi(Tx, Ty) &+ \beta\phi(x, Ty)+\gamma\phi(Tx, y)+\delta\phi(x, y) \\ &\leq \epsilon\{\phi(Ty, Tx)-\phi(Ty, x)\}+\zeta\{\phi(y, Tx)-\phi(y, x)\}. \quad (1) \end{aligned}$$

A generic $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-generalized nonspreading mapping reduces to generalized nonspreading if $\alpha+\beta = -\gamma-\delta = 1$.

The existence and approximations of attractive points of the above mentioned generalized nonlinear mappings have been studied by various authors [see, for example, [4, 31, 32, 33] and the references therein]. Several iterative schemes have been proposed for such approximation, one of which is the so called Halpern's scheme introduced by Halpern [14]. Takahashi et al. [31] used the concept of attractive points of nonlinear mappings and obtained a new strong convergence theorem for generalized hybrid mappings using Halpern's type scheme in Hilbert

spaces. The same authors [30] proved weak convergence theorem of Mann's type algorithm for generic generalized nonspreading mappings in Banach spaces.
Let E be a real Banach space and $f : E \to (-\infty, +\infty]$ a strictly convex and Gâteaux differentiable function. The function D_f : dom$f\times$ int(dom$(f)) \to [0, +\infty)$, defined by

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle, \tag{2}$$

is called the Bregman distance with respect to f (see [13]).

Remark 1. *If E is a smooth Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have $\nabla f(x) = 2Jx$ for all $x \in E$, where $J : E \to E^*$ is the normalized duality mapping. Hence $D_f(x, y) = \phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall\ x, y \in E$. And if E is a Hilbert space, then $D_f(x, y) = \|x - y\|^2,\ \forall x, y \in E$.*

Observe that from (2), we have for any $x \in dom f$ and $y, z \in int(dom(f))$.

$$D_f(x, z) = D_f(x, y) + D_f(y, z) + \langle \nabla f(y) - \nabla f(z), x - y\rangle. \tag{3}$$

which is called the three point identity. In 2017, Ali et al. [5] introduced the notion of Bregman attractive point in reflexive Banach spaces. Let the set of Bregman attractive points be denoted by $A_f(T)$, i.e.

$$A_f(T) = \{u \in E : D_f(u, Tx) \leq D_f(u, x),\ \forall x \in C\}, \tag{4}$$

where C is a nonempty subset of $int(dom(f))$. They also established the existence of attractive point of generalized Bregman nonspreading mappings in the space. For more recent results related to Bregman attractive point, see [1, 2, 3].

Motivated and inspired by the works of Takahashi et al. [30], Ali et al. [5] and Takahashi et al. [31], we first prove the existence theorem for attractive point of generic generalized Bregman nonspreading mapping in reflexive Banach spaces. Also, we propose a constructive Halpern-type algorithm that converges strongly to a common attractive point of finite family of generic generalized Bregman nonspreading mappings in reflexive Banach spaces. Our results improve, extend and generalize those of Takahashi et al. [31], Takahashi et al. [30] and Ali et al. [5] in the sense of spaces, mappings and the nature of convergence.

2. Preliminaries

Let E be a real reflexive Banach space with norm $\|\cdot\|$ and E^* the dual space of E. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex

function. The Fenchel conjugate of f is the convex function $f^*: E^* \to (-\infty, +\infty]$ defined by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$. Observe that the Young-Fenchel inequality holds: $\langle x^*, x\rangle \leq f(x) + f^*(x^*)$, $\forall x \in E$, $x^* \in E^*$. It is well known that if $f : E \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, convex and weak* lower semi-continuous function; see, for example, [28]. A sublevel of f is the set of the form $\mathrm{lev}_{\leq}^{f} r := \{x \in E : f(x) \leq r\}$ for $r \in \mathbb{R}$. A function f on E is coercive [16] if every sublevel of f is bounded, equivalently $\lim\limits_{\|x\| \to +\infty} f(x) = +\infty$. Let $B_r := \{x \in E : \|x\| \leq r\}$ for all $r > 0$ and $S_E := \{x \in E : \|x\| = 1\}$. A function f on E is said to be strongly coercive [36] if $\lim\limits_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty$.
For any $x \in \mathrm{int}(\mathrm{dom}(f))$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^{\circ}(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{5}$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t \to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, the gradient of f at x is the function $\nabla f(x) : E \to (-\infty, +\infty]$ defined by $\langle \nabla f(x), y\rangle = f^{\circ}(x, y)$, for any $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at every $x \in \mathrm{int}(\mathrm{dom}(f))$. The function f is said to be Fréchet differentiable at x if the limit in (5) is attained uniformly in y, $\|y\| = 1$. Finally, f is said to be Fréchet differentiable on a subset C of E if the limit (5) is attained uniformly for $x \in E$ and $\|y\| = 1$. It is well known that if a continuous convex function f is Gâteaux differentiable (resp. Fréchet differentiable) in $\mathrm{int}(\mathrm{dom}(f))$, then ∇f is norm-to-weak* continuous (resp. continuous) in $\mathrm{int}(\mathrm{dom}(f))$ (see also [7]).

The following two results are proved in [36]:

Lemma 1 ([36]). *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is bounded on bounded sets. Then the following assertions are equivalent:*

(1) f is strongly coercive and uniformly convex on bounded subsets of E;

(2) $\mathrm{dom} f^* = E^*$, f^* is bounded on bounded subsets and uniformly smooth on bounded sets;

(3) $\mathrm{dom} f^* = E^*$, f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded sets.

Lemma 2 ([36])**.** *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(1) f is bounded on bounded sets and uniformly smooth on bounded sets;

(2) f^* is Fréchet differentiable and f^* is uniformly norm-to-norm continuous on bounded sets.

(3) dom$f^* = E^*$, f^* is strongly coercive and uniformly convex on bounded sets.

Let $x \in \mathrm{int}(\mathrm{dom}(f))$. The subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y),\ \forall y \in E\}.$$

Definition 1 ([9])**.** *The function f is said to be:*
(i) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of dom$f$;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 2. *Let E be a reflexive Banach space. Then we have:*

(i) f is essentially smooth if and only if f^* is essentially strictly convex [see [9], Theorem 5.4];

(ii) $(\partial f)^{-1} = \partial f^*$;

(iii) f is Legendre if and only if f^* is Legendre [see [9],Corrolary 5.5]

(iv) If f is Legendre, then ∇f is a bijection satifying $\nabla f = (\nabla f^*)^{-1}$, ran$\nabla f$ = dom∇f^* = int(dom(f^*)) and ran ∇f^* = dom∇f = int(dom(f)) [see [9], Theorem 5.10].

Various examples of Legendre functions were given in [9, 8]. One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$, when E is a smooth and strictly convex Banach space. In this case, the gradient ∇f of f coincides with the generalized duality mapping of E, i.e, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ is the identity mapping in Hilbert spaces.

Lemma 3 ([9], Theorem 7.3 (vi), (vii))**.** *Suppose $u \in$ dom$f$ and $v \in$ int(dom(f)). Then*

(i) If f is strictly convex, then $D_f(u, v) = 0 \Leftrightarrow u = v$;

(ii) If f is Gâteaux differentiable in int(dom(f)) and essentially strictly convex, then $D_f(u,v) = D_{f^*}(\nabla f(v), \nabla f(u))$.

Lemma 4 ([6], Theorem 1.8)**.** *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable, then f is uniformly continuous on E.*

Lemma 5 ([24])**.** *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

A function $f : E \to \mathbb{R}$ is called a Bregman function [19] if the following conditions are satisfied:

(i) f is continuous, strictly convex and Gâteaux differentiable.

(ii) the set $\{z \in E : D_f(x,z) \leq r\}$ is bounded for all $x \in E$ and $r \geq 0$.

Lemma 6 ([19])**.** *Let E be a reflexive Banach space, let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and let V_f be the function defined by*

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \ \forall \ x \in E, x^* \in E^*.$$

Then the following assertions hold:
(1) $D_f(x, \nabla f^*(x^*)) = V(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.
(2) $V(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.

It follows (see, for example, [34]) from the definition that V is convex in the second argument and

$$V(x, \nabla f(y)) = D_f(x, y).$$

A Bregman projection [10] of $x \in int(dom(f))$ onto the nonempty, closed and convex set $C \subset domf$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

The following is a well-known fact about Bregman projections.

Lemma 7 ([12])**.** *Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*

(a) $z = P_C^f x$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \forall y \in C$;

(b) $D_f(y, P_C^f x) + D_f(P_C^f x, x) \leq D_f(y, x)$ $\forall x \in E, y \in C$.

The following result was proved in [22].

Lemma 8. *Let E be a Banach space, let $r > 0$ be a constant, let ρ_r be the gauge of uniform convexity of g and let $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Then,*

(i) for any $x, y \in B_r$ and $\alpha \in (0,1)$,

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_r(\|x-y\|)$$

(ii) for any $x, y \in B_r$, $\rho_r(\|x-y\|) \leq D_g(x,y)$

(iii) If in addition g is bounded on bounded subsets and uniformly convex on bounded subsets of E, then for any $x \in E$,$y^*, z^* \in B_r^*$ and $\alpha \in (0,1)$,

$$V_g(x, \alpha y^* + (1-\alpha)z^*) \leq \alpha V_g(x, y^*) + (1-\alpha)V_g(x, z^*) - \alpha(1-\alpha)\rho_r^*(\|y^* - z^*\|).$$

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The modulus of total convexity of f at $x \in int(dom(f))$ is the function $v_f(x, .) : int(dom(f)) \times [0, +\infty] \to [0, +\infty]$ defined by

$$v_f(x,t) = \inf\{D_f(y,x) : y \in dom f, \|y-x\| = t\}.$$

The function is totally convex at x if $v_f(x,t) > 0$ whenever $t > 0$. The function f is called totally convex if it is totally convex at every point $x \in int(dom(f))$ and is said to be totally convex on bounded sets if $v_f(B,t) > 0$, for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $V_f : int(dom(f)) \times [0, +\infty] \to [0, +\infty]$ defined by

$$V_f(B,t) = \inf\{v_f(x,t) : x \in B \cap dom f\}.$$

Lemma 9 ([27])**.** *If $x \in int(dom(f))$, then the following statements are equivalent:*

(i) The function f is totally convex at x;

(ii) for any sequence $\{y_n\} \subset dom f$,$\lim_{n\to+\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x\| = 0$.

Recall that the function f is sequentially consistent [11] if for any sequences $\{x_n\}$ and $\{y_n\}$ in E, such that the first one is bounded, the following relation holds:

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Lemma 10 ([11]). *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 11 ([26]). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $int(domf^*)$. Let $x \in int(dom(f))$. If $\{D_f(x, x_n)\}_{n\in\mathbb{N}}$ is bounded, then so is the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Lemma 12 ([25]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and let C be a nonempty, closed and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n, x_0) \le D_f(P_C^f(x_0)x_0, x_0)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $P_C^f(x_0)$.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of the dual space of l^∞ (i.e $\mu \in (l^\infty)^*$). The value of μ at $f = (x_1, x_2, x_3, ...) \in l^\infty$ is denoted by $\mu(f)$ and sometimes by $\mu_n(x_n)$. A linear functional μ_n on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, ...)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. It is known that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, ...) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, ...) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. The following result is well-known, see [15].

Lemma 13. *Let E be a reflexive Banach space, let $\{x_n\}$ be a bounded sequence in E and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \bar{co}\{x_n : n \in \mathbb{N}\}$ such that $\mu_n\langle x_n, y^*\rangle = \langle z_0, y^*\rangle \ \forall \ y^* \in E^*$.*

The following results will play a vital role in establishing our main results.

Lemma 14 ([35]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha)a_n + \alpha_n\delta_n, \ n \ge n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a real sequence satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ *and* $\limsup_{n\to\infty} \delta_n \le 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 15 ([21]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} \le a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there* *exists a non-decreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$*:*

$$a_{m_k} \leq a_{m_k+1},$$

$$a_k \leq a_{m_k+1}.$$

In fact, $m_k = \{j \leq k : a_j < a_{j+1}\}$.

3. Main Results

In this section, we establish a new strong convergence theorem for common attractive point of finite family of generic generalized Bregman nonspreading mappings in a real reflexive Banach space E. But first we prove the following.

Lemma 16. *Let* $f : E \to (-\infty, +\infty]$ *be a Legendre function which is Fréchet differentiable. Let* T *be a mapping of a nonempty subset* C *of* $int(dom(f))$ *into itself. Then the set of attractive points* $A_f(T)$ *is closed and convex.*

Proof. We first show that the set $A_f(T)$ is closed. Let $\{x_n\}$ be a sequence in $A_f(T)$ such that $x_n \to u$ as $n \to \infty$. Then let us show that $u \in A_f(T)$. Now, for any $x \in C$,

$$\begin{aligned}
D_f(u, Tx) &= D_f(\lim_{n\to\infty} x_n, Tx) = \lim_{n\to\infty} D_f(x_n, Tx) \\
&\leq \lim_{n\to\infty} D_f(x_n, x) = D_f(\lim_{n\to\infty} x_n, x) \\
&= D_f(u, x).
\end{aligned}$$

Thus, $u \in A_f(T)$ and so $A_f(T)$ is closed. For convexity, we let $u, v \in A_f(T)$ and $\alpha \in (0, 1)$. Then let us we show that $z := \alpha u + (1 - \alpha)v \in A_f(T)$. Now for any $x \in C$,

$$\begin{aligned}
D_f(z, Tx) &= f(z) - f(Tx) - \langle \nabla f(Tx), z - Tx\rangle \\
&= f(z) - f(Tx) - \langle \nabla f(Tx), \alpha u + (1 - \alpha)v - Tx\rangle \\
&= f(z) - f(Tx) - \alpha\langle \nabla f(Tx), u - Tx\rangle \\
&- (1 - \alpha)\langle \nabla f(Tx), v - Tx\rangle \\
&= f(z) - \alpha f(u) - (1 - \alpha)f(v) \\
&+ \alpha[f(u) - f(Tx) - \langle \nabla f(Tx), u - Tx\rangle] \\
&+ (1 - \alpha)[f(v) - f(Tx) - \langle \nabla f(Tx), v - Tx\rangle \\
&= f(z) - \alpha f(u) - (1 - \alpha)f(v) + \alpha D_f(u, Tx) \\
&+ (1 - \alpha)D_f(v, Tx)
\end{aligned}$$

$$
\begin{aligned}
&\leq f(z) - \alpha f(u) - (1-\alpha)f(v) + \alpha D_f(u,x) + (1-\alpha)D_f(v,x) \\
&= f(z) - \alpha f(x) - (1-\alpha)f(x) - \langle \nabla f(x), \alpha u + (1-\alpha)v - x\rangle \\
&= f(z) - f(x) - \langle \nabla f(x), z - x\rangle \\
&= D_f(z,x).
\end{aligned}
$$

Thus, $D_f(z,Tx) \leq D_f(z,x)$. Therefore, $z \in A_f(T)$. Hence $A_f(T)$ is closed and convex. This completes the proof. ◄

Definition 2. *Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function and C a nonempty subset of int(dom(f)). A mapping $T : C \to C$ is called a generic generalized Bregman nonspreading mapping if there exist real numbers $\alpha, \beta, \gamma, \delta, \epsilon$, ζ such that for all $x, y \in C$, (i) $\alpha + \beta + \gamma + \delta \geq 0$; (ii) $\alpha + \beta > 0$;*

$$
\begin{aligned}
(iii)\ &\alpha D_f(Tx,Ty) + \beta D_f(x,Ty) + \gamma D_f(Tx,y) + \delta D_f(x,y) \\
&\leq \epsilon\big(D_f(Ty,Tx) - D_f(Ty,x)\big) + \zeta\big(D_f(y,Tx) - D_f(y,x)\big). \qquad (6)
\end{aligned}
$$

Remark 3. *Observe that the generic generalized Bregman nonspreading mapping reduces to generalized Bregman nonspreading if $\alpha + \beta = -\gamma - \delta = 1$ and it is Bregman nonspreading [23] if $\alpha = 1, \beta = \delta = \zeta = 0$ and $\gamma = \epsilon = -1$. Also, in view of Remark 1, if E is smooth and $f(x) = \|x\|^2$, then the generic generalized Bregman nonspreading mapping reduces to generic generalized nonspreading mapping in the sense of [30]. We now establish the existence of attractive point of generic generalized Bregman nonspreading mapping.*

Theorem 1. *Let $f : E \to (-\infty, +\infty]$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subsets of E. Let C be a nonempty subset of int(dom(f)) and $T : C \to C$ be a generic generalized Bregman nonspreading mapping. Then $A_f(T) \neq \emptyset$ if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

Proof. Suppose $A_f(T) \neq \emptyset$. Then by taking $z \in A_f(T)$ we see that $D_f(z,Tx) \leq D_f(z,x)$ for all $x \in C$. Thus,

$$
D_f(z,T^n x) \leq D_f(z,T^{n-1}x) \leq ... \leq D_f(z,Tx) \leq D_f(z,x),
$$

for all $x, y \in C$. Therefore, $\{D_f(z,T^n x)\}$ is bounded. Since f is strongly coercive and totally convex which is bounded on a bounded subset of E, it follows from Lemma 1 that ∇f^* is uniformly norm to norm continuous on bounded subsets of dom$f^* = E^*$ and consequently ∇f^* is bounded. Hence, by Lemma 11, $\{T^n x\}$ is bounded.

Conversely, suppose $\{T^n x\}$ is bounded. Replacing x with $T^n x$ in (iii) of Definition 2, we see that for any $y \in C$ and $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \alpha D_f(T^{n+1}x, Ty) &+ \beta D_f(T^n x, Ty) + \gamma D_f(T^{n+1}x, y) + \delta D_f(T^n x, y) \\ &\leq \epsilon\big(D_f(Ty, T^{n+1}x) - D_f(Ty, T^n x)\big) \\ &+ \zeta\big(D_f(y, T^{n+1}x) - D_f(y, T^n x)\big). \end{aligned} \tag{7}$$

Since $\{T^n x\}$ is bounded, then by applying Banach limit μ_n on both sides of the inequality (7) we get

$$\begin{aligned} &\alpha\mu_n D_f(T^n x, Ty) + \beta\mu_n D_f(T^n x, Ty) + \gamma\mu_n D_f(T^n x, y) + \delta\mu_n D_f(T^n x, y) \\ &\leq \epsilon\mu_n\big(D_f(Ty, T^n x) - D_f(Ty, T^n x)\big) + \zeta\mu_n\big(D_f(y, T^n x) - D_f(y, T^n x)\big). \end{aligned}$$

This implies

$$(\alpha+\beta)\mu_n D_f(T^n x, Ty) + (\gamma+\delta)\mu_n D_f(T^n x, y) \leq 0.$$

Using the three point identity (3), we obtain

$$\begin{aligned} (\alpha+\beta)\mu_n\big(D_f(T^n x, y) + D_f(y, Ty) + \langle \nabla f(y) - \nabla f(Ty), T^n x - y\rangle\big) \\ +(\gamma+\delta)\mu_n D_f(T^n x, y) \leq 0. \end{aligned}$$

This implies

$$\begin{aligned} &(\alpha+\beta+\gamma+\delta)\mu_n D_f(T^n x, y) \\ &+ (\alpha+\beta)\big(D_f(y, Ty) + \mu_n\langle \nabla f(y) - \nabla f(Ty), T^n x - y\rangle\big) \leq 0. \end{aligned} \tag{8}$$

By applying condition (i) of Definition 2 in the inequality (8), we get

$$(\alpha+\beta)\big(D_f(y, Ty) + \mu_n\langle \nabla f(y) - \nabla f(Ty), T^n x - y\rangle\big) \leq 0.$$

Thus, there exists $z_0 \in E$ such that by Lemma 13 we get

$$(\alpha+\beta)\big(D_f(y, Ty) + \langle \nabla f(y) - \nabla f(Ty), z_0 - y\rangle\big) \leq 0.$$

Also, applying condition (ii) of Definition 2 together with the use of (3), we get

$$D_f(y, Ty) + D_f(z_0, Ty) - D_f(y, Ty) - D_f(z_0, y) \leq 0.$$

Therefore, $D_f(z_0, Ty) \leq D_f(z_0, y)$. Hence, $A_f(T) \neq \emptyset$. This completes the proof. ◄

The following Lemma gives the demiclosedness property of generic generalized Bregman nonspreading mapping which will play a vital role in proving our main result.

Lemma 17. *(Demiclosedness Property). Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty subset of $int(dom(f))$ and $T : C \to C$ be a generic generalized Bregman nonspreading mapping. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in A_f(T)$.*

Proof. Let $\{x_n\} \subset C$ be a sequence such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Since f is uniformly Fréchet differentiable on bounded subsets of E, then by Lemmas 4 and 5, both f and ∇f are uniformly continuous on a bounded subsets of E, respectively. This implies $f(x_n) - f(Tx_n) \to 0$ and $\nabla f(x_n) - \nabla f(Tx_n) \to 0$ *as* $n \to \infty$. Thus, for any $y \in C$, we have

$$\begin{aligned} D_f(Ty, Tx_n) \quad - \quad & D_f(Ty, x_n) = f(Ty) - f(Tx_n) - \langle \nabla f(Tx_n), Ty - Tx_n \rangle \\ - \quad & f(Ty) + f(x_n) + \langle \nabla f(x_n), Ty - x_n \rangle \\ = \quad & f(x_n) - f(Tx_n) - \langle \nabla f(Tx_n), Ty - Tx_n \rangle \\ + \quad & \langle \nabla f(x_n), Ty - x_n \rangle \\ = \quad & f(x_n) - f(Tx_n) - \langle \nabla f(Tx_n) - \nabla f(x_n), Ty - Tx_n \rangle \\ + \quad & \langle \nabla f(x_n), Tx_n - x_n \rangle \to 0 \text{ as } n \to \infty. \end{aligned} \tag{9}$$

Since $T : C \to C$ is a generic generalized Bregman nonspreading mapping, there exist $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{R}$ such that by replacing x with x_n in (iii) of Definition 2, we obtain

$$\begin{aligned} \alpha D_f(Tx_n, Ty) \quad + \quad & \beta D_f(x_n, Ty) + \gamma D_f(Tx_n, y) + \delta D_f(x_n, y) \\ \leq \quad & \epsilon\big(D_f(Ty, Tx_n) - D_f(Ty, x_n)\big) \\ + \quad & \zeta\big(D_f(y, Tx_n) - D_f(y, x_n)\big), \forall\ y \in C. \end{aligned} \tag{10}$$

Using equation (3) in inequality (10), we get

$$\begin{aligned} & \alpha\big(D_f(Tx_n, y) + D_f(y, Ty) + \langle \nabla f(y) - \nabla f(Ty), Tx_n - y \rangle\big) \\ + \quad & \beta\big(D_f(x_n, y) + D_f(y, Ty) + \langle \nabla f(y) - \nabla f(Ty), x_n - y \rangle\big) + \gamma D_f(Tx_n, y) \\ + \quad & \delta D_f(x_n, y) \\ \leq \quad & \epsilon\big(D_f(Ty, Tx_n) - D_f(Ty, x_n)\big) + \zeta\big(D_f(y, Tx_n) - D_f(y, x_n)\big). \end{aligned}$$

This implies

$$\begin{aligned} & \alpha\big(D_f(Tx_n, y) - D_f(x_n, y) + D_f(x_n, y) + D_f(y, Ty)) \\ + \quad & \alpha\langle \nabla f(y) - \nabla f(Ty), Tx_n - y \rangle\big) \\ + \quad & \beta\big(D_f(x_n, y) + D_f(y, Ty) + \langle \nabla f(y) - \nabla f(Ty), x_n - y \rangle\big) \\ + \quad & \gamma\big(D_f(Tx_n, y) - D_f(x_n, y) + D_f(x_n, y)\big) + \delta D_f(x_n, y) \end{aligned}$$

$$\leq \epsilon\big(D_f(Ty,Tx_n)-D_f(Ty,x_n)\big)+\zeta\big(D_f(y,Tx_n)-D_f(y,x_n)\big).$$

Thus

$$\begin{aligned}
& (\alpha+\beta+\gamma+\delta)D_f(x_n,y) \\
+\ & \alpha\big(D_f(Tx_n,y)-D_f(x_n,y)+D_f(y,Ty)+\langle\nabla f(y)-\nabla f(Ty),Tx_n-y\rangle\big) \\
+\ & \beta\big(D_f(y,Ty)+\langle\nabla f(y)-\nabla f(Ty),x_n-y\rangle\big)+\gamma\big(D_f(Tx_n,y)-D_f(x_n,y)\big) \\
\leq\ & \epsilon\big(D_f(Ty,Tx_n)-D_f(Ty,x_n)\big)+\zeta\big(D_f(y,Tx_n)-D_f(y,x_n)\big). \qquad (11)
\end{aligned}$$

Since $\alpha+\beta+\gamma+\delta\geq 0$, from (11) we obtain

$$\begin{aligned}
& \alpha\big(D_f(Tx_n,y)-D_f(x_n,y)+D_f(y,Ty)+\langle\nabla f(y)-\nabla f(Ty),Tx_n-y\rangle\big) \\
+\ & \beta\big(D_f(y,Ty)+\langle\nabla f(y)-\nabla f(Ty),x_n-y\rangle\big)+\gamma\big(D_f(Tx_n,y)-D_f(x_n,y)\big) \\
\leq\ & \epsilon\big(D_f(Ty,Tx_n)-D_f(Ty,x_n)\big)+\zeta\big(D_f(y,Tx_n)-D_f(y,x_n)\big).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ & \alpha\big(D_f(Tx_n,y)-D_f(x_n,y)+D_f(y,Ty)+\langle\nabla f(y)-\nabla f(Ty),Tx_n-x_n\rangle\big) \\
+\ & \alpha\langle\nabla f(y)-\nabla f(Ty),x_n-y\rangle+\beta\big(D_f(y,Ty)+\langle\nabla f(y)-\nabla f(Ty),x_n-y\rangle\big) \\
+\ & \gamma\big(D_f(Tx_n,y)-D_f(x_n,y)\big) \\
\leq\ & \epsilon\big(D_f(Ty,Tx_n)-D_f(Ty,x_n)\big)+\zeta\big(D_f(y,Tx_n)-D_f(y,x_n)\big).
\end{aligned}$$

Taking limit as $n\to\infty$, we get

$$\begin{aligned}
\alpha\big(D_f(y,Ty) \quad + & \quad \langle\nabla f(y)-\nabla f(Ty),z-y\rangle\big) \\
+ & \quad \beta\big(D_f(y,Ty)+\langle\nabla f(y)-\nabla f(Ty),z-y\rangle\big)\leq 0.
\end{aligned}$$

$$\Rightarrow\quad (\alpha+\beta)\big(D_f(y,Ty)+\langle\nabla f(y)-\nabla f(Ty),z-y\rangle\big)\leq 0.$$

Again using (3), we have

$$(\alpha+\beta)\big(D_f(y,Ty)+D_f(z,Ty)-D_f(y,Ty)+D_f(z,y)\big)\leq 0.$$

Since $\alpha+\beta>0$, we obtain $D_f(z,Ty)\leq D_f(z,y)$ for all $y\in C$. Hence, $z\in A_f(T)$. This completes the proof. ◄

The following Proposition will be used in proving our main result.

Proposition 1. *Let $f:E\to(-\infty,+\infty]$ be a Legendre and uniformly Fréchet differentiable function on bounded subsets of E. Let C be a nonempty subset of $int(dom(f))$ and $T:C\to C$ be a generic generalized Bregman nonspreading mapping such that $A_f(T)\neq\emptyset$. Suppose that $u\in C$ and $\{x_n\}$ is bounded in C with $x_n-Tx_n\to 0$ as $n\to\infty$. Then $\limsup_{n\to\infty}\langle\nabla f(u)-\nabla f(z),x_n-z\rangle\leq 0$; where $z=P_{A_f(T)}(u)$ is the Bregman projection of C onto $A_f(T)$.*

Proof. We have seen from Lemma 16 that $A_f(T)$ is closed and convex. Let $u \in C$ and $z = \mathrm{P}_{A_f(T)}(u)$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in C$ as $k \to \infty$. Since $\lim\limits_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = \lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, by Lemma 17, $v \in A_f(T)$. Now, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\begin{aligned}
\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(z), x_n - z\rangle &= \lim_{k\to\infty} \langle \nabla f(u) - \nabla f(z), x_{n_k} - z\rangle \\
&= \langle \nabla f(u) - \nabla f(z), v - z\rangle.
\end{aligned}$$

Then, by Lemma 7, we get
$\limsup\limits_{n\to\infty} \langle \nabla f(u) - \nabla f(z), x_n - z\rangle = \langle \nabla f(u) - \nabla f(z), v - z\rangle \le 0$, for all $v \in A_f(T)$, where $z = \mathrm{P}^f_{A_f(T)}(u)$ is the Bregman projection of C onto $A_f(T)$. Hence $\limsup\limits_{n\to\infty} \langle \nabla f(u) - \nabla f(z), x_n - z\rangle \le 0$. This completes the proof. ◄

We now prove our main result.

Theorem 2. *Let $f : E \to (-\infty, +\infty]$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of $int(domf)$ and $T_i : C \to C$ for i=1,2,. . .,N be a finite family of generic generalized Bregman nonspreading mappings such that $\mathcal{A} = \cap_{i=1}^N A_f(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$:*

$$\left\{\begin{aligned}
y_n &= \nabla f^*[\beta_n \nabla f x_n + (1-\beta_n)\nabla f T x_n];\\
x_{n+1} &= P^f_C[\nabla f^*(\alpha_n \nabla f u + (1-\alpha_n)\nabla f y_n)],
\end{aligned}\right. \tag{12}$$

where $T = T_N \circ T_{N-1} \circ ... \circ T_1$,$\{\beta_n\} \subset [a, b] \subset (0, 1)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying (C_1) : $\lim_{n\to\infty} \alpha_n = 0$ and (C_2) : $\sum_{n=1}^{\infty} \alpha_n = +\infty$. Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{A}}(u)$.

Proof. From Lemma 16, we see that $A_f(T_i)$ is closed and convex for each i=1,2,...,N and so $\mathcal{A} = \cap_{i=1}^N A_f(T_i)$ is closed and convex. Let $z = \mathrm{P}_{\mathcal{A}}(u) \in \mathcal{A}$. Then from (12), Lemma 6 and equation (4) we have

$$\begin{aligned}
D_f(z, y_n) &= D_f(z, \nabla f^*[\beta_n \nabla f x_n + (1-\beta_n)\nabla f T x_n])\\
&\le \beta_n D_f(z, x_n) + (1-\beta_n) D_f(z, T x_n)\\
&= \beta_n D_f(z, x_n) + (1-\beta_n) D_f(z, T_N \circ T_{N-1} \circ ... \circ T_1(x_n))\\
&\le \beta_n D_f(z, x_n) + (1-\beta_n) D_f(z, T_{N-1} \circ ... \circ T_1(x_n))\\
&\ .
\end{aligned}$$

$$
\begin{aligned}
&.\\
&.\\
\leq\ & \beta_n D_f(z,x_n) + (1-\beta_n)D_f(z,x_n)\\
=\ & D_f(z,x_n).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
D_f(z,x_{n+1}) &= D_f(z, \mathrm{P}_c^f[\nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f y_n)])\\
&\leq D_f(z, \nabla f^*[\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f y_n])\\
&= \alpha_n D_f(z,u) + (1-\alpha_n)D_f(z,y_n) \qquad (13)\\
&\leq \alpha_n D_f(z,u) + (1-\alpha_n)D_f(z,x_n).
\end{aligned}
$$

Thus, by induction we obtain

$$D_f(z,x_{n+1}) \leq max\{D_f(z,u), D_f(z,x_n)\}\ \forall\ n \geq 1.$$

This implies that the sequence $\{D_f(z,x_n)\}$ is bounded. Since $z \in \mathcal{A}$, by boundedness of $\{D_f(z,x_n)\}$ and definition of $A_f(T)$, there exists $k > 0$ such that

$$D_f(z,Tx_n) \leq D_f(z,x_n) \leq k,\ \forall\ n \in \mathbb{N}.$$

Hence by Lemma 12, both $\{x_n\}$ and $\{Tx_n\}$ are bounded. Since f is bounded on bounded subsets of E, by [[11]. Proposition 1.1.11], ∇f is also bounded on bounded subsets of E^*. Hence the sequences $\{\nabla f(x_n)\}$ and $\{\nabla f(Tx_n)\}$ are bounded in E^*. We know from Lemma 2(3) that $dom f^* = E^*$ and f^* is strongly coercive and uniformly convex on bounded subsets of E^*. Let $s = \sup\{\|\nabla f(x_n)\|, \|\nabla f(Tx_n)\|\}$ and $\rho_s^* : E^* \to \mathbb{R}$, the gauge function of uniform convexity of the conjugate function f^*. Then, by Lemmas 6 and 8, we have

$$
\begin{aligned}
D_f(z,y_n) &= D_f(z, \nabla f^*[\beta_n \nabla f x_n + (1-\beta_n)\nabla f T x_n])\\
&= V_f(z, \beta_n \nabla f x_n + (1-\beta_n)\nabla f(Tx_n))\\
&= f(z) - \langle z, \beta_n \nabla f x_n + (1-\beta_n)\nabla f(Tx_n)\rangle\\
&+ f^*(\beta_n \nabla f x_n + (1-\beta_n)\nabla f(Tx_n))\\
&\leq (1-\beta_n)f(z) + \beta_n f(z) - \beta_n\langle z, \nabla f x_n\rangle - (1-\beta_n)\langle z, \nabla f(Tx_n)\rangle\\
&+ \beta_n f^*(\nabla f(x_n)) + (1-\beta_n)f^*(\nabla f(Tx_n))\\
&- \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n) - \nabla f(Tx_n)\|\big)\\
&= (1-\beta_n)[f(z) - \langle z, \nabla f(Tx_n)\rangle + f^*(\nabla f(Tx_n))]\\
&+ \beta_n[f(z) - \langle z, \nabla f x_n\rangle + f^*(\nabla f(x_n))]\\
&- \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n) - \nabla f(Tx_n)\|\big)
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\beta_n)V_f(z,\nabla f(Tx_n)) + \beta_n V_f(z,\nabla f(x_n))\\
&- \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big)\\
&= (1-\beta_n)D_f(z,Tx_n) + \beta_n D_f(z,x_n)\\
&- \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big)\\
&\leq (1-\beta_n)D_f(z,x_n) + \beta_n D_f(z,x_n)\\
&- \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big)\\
&= D_f(z,x_n) - \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big). \qquad (14)
\end{aligned}
$$

It follows from (13) and (14) that

$$
\begin{aligned}
D_f(z,x_{n+1}) &\leq \alpha_n D_f(z,u) + (1-\alpha_n)D_f(z,y_n)\\
&\leq \alpha_n D_f(z,u) + (1-\alpha_n)D_f(z,x_n)\\
&- (1-\alpha_n)\beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big)\\
&= \alpha_n[D_f(z,u) - D_f(z,x_n) + \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big)]\\
&+ D_f(z,x_n) - \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big).
\end{aligned}
$$

Putting $k_1 = \sup\{|D_f(z,u) - D_f(z,x_n)| + \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big)\}$, we see that

$$D_f(z,x_{n+1}) \leq D_f(z,x_n) - \beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big) + \alpha_n k_1.$$

Which implies

$$\beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big) \leq D_f(z,x_n) - D_f(z,x_{n+1}) + \alpha_n k_1 \qquad (15)$$

We now divide the remaining proof into the following cases.
Case I. Let there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the sequence $\{D_f(z,x_n)\}$ is non-increasing and since it is bounded, it is convergent. Thus, we see that

$$D_f(z,x_n) - D_f(z,x_{n+1}) \to 0 \text{ } as \text{ } n \to \infty. \qquad (16)$$

Now, using C_1 and equation (16) in (15) we have

$$\beta_n(1-\beta_n)p_s^*\big(\|\nabla f(x_n)-\nabla f(Tx_n)\|\big) \to 0 \text{ } as \text{ } n \to \infty.$$

Since $\beta_n \in [a,b] \subset (0,1)$, by property of p_s^* we have

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(Tx_n)\| = 0.$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, by Lemma 2(2), f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E. Thus, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|\nabla f^*(\nabla f(x_n)) - \nabla f^*(\nabla f(Tx_n))\| = 0.$$

On the other hand,

$$\begin{aligned} D_f(x_n, y_n) &= D_f(x_n, \nabla f^*[\beta_n \nabla f x_n + (1-\beta_n)\nabla f T x_n]) \\ &\leq \beta_n D_f(x_n, x_n) + (1-\beta_n) D_f(x_n, Tx_n) \\ &= (1-\beta_n) D_f(x_n, Tx_n) \to 0 \ as \ n \to \infty. \end{aligned}$$

It follows from above expression and Lemma 9 that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{17}$$

Now, let $z_n = \nabla f^*[\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)]$. Using Lemmas 6 and 7, we see that

$$\begin{aligned} D_f(z, x_{n+1}) &= D_f(z, \mathrm{P}_C^f(z_n)) \leq D_f(z, z_n) = V_f(z, \nabla f(z_n)) \\ &\leq V_f(z, \nabla f(z_n) - \alpha_n(\nabla f(u) - \nabla f(z)) \\ &+ \alpha_n \langle \nabla f(u) - \nabla f(z), z_n - z\rangle \\ &= V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n) - \alpha_n(\nabla f(u) - \nabla f(z)) \\ &+ \alpha_n \langle \nabla f(u) - \nabla f(z), z_n - z\rangle \\ &= V_f(z, (1-\alpha_n)\nabla f(y_n) + \alpha_n \nabla f(z)) \\ &+ \alpha_n \langle \nabla f(u) - \nabla f(z), z_n - z\rangle \\ &\leq (1-\alpha_n) V_f(z, \nabla f(y_n)) + \alpha_n V_f(z, \nabla f(z)) \\ &+ \alpha_n \langle \nabla f(u) - \nabla f(z), z_n - z\rangle \\ &= (1-\alpha_n) D_f(z, y_n) + \alpha_n D_f(z, z) + \alpha_n \langle \nabla f(u) - \nabla f(z), z_n - z\rangle \\ &\leq (1-\alpha_n) D_f(z, x_n) + \alpha_n \langle \nabla f(u) - \nabla f(z), z_n - z\rangle. \end{aligned} \tag{18}$$

Observe that

$$\|\nabla f(y_n) - \nabla f(z_n)\| = \alpha_n \|\nabla f(y_n) - \nabla f(u)\| \to 0 \ as \ n \to \infty.$$

By the nature of f and ∇f, we obtain

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{19}$$

It follows from (17) and (19) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{20}$$

By Proposition 1 and equation (20), we can conclude that

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(z), z_n - z\rangle = \limsup_{n\to\infty} [\langle \nabla f(u) - \nabla f(z), z_n - x_n\rangle$$

$$
\begin{aligned}
&+ \quad \langle \nabla f(u) - \nabla f(z), x_n - z\rangle] \qquad (21)\\
&= \quad \limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(z), x_n - z\rangle \leq 0.
\end{aligned}
$$

Hence by Lemma 14 and inequalities (18) and (21), we get $x_n \to z = \mathrm{P}_{\mathcal{A}}(u)$ as $n \to \infty$.

Case II: Put $\{a_n\} = \{D_f(z, x_n)\}$ and let there exist a subsequence $\{n_i\}$ of $\{n\}$ such that for all $i \in \mathbb{N}$, $a_{n_i} \leq a_{n_i+1}$. For some sufficiently large N and for all $n \geq N$, define a map $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \{k \leq n : a_k \leq a_{k+1}\}.$$

Then, it follows from Lemma 15 that the sequence $\tau(n)$ is non-decreasing with $\tau(n) \to \infty$ as $n \to \infty$ and $a_{\tau(n)} \leq a_{\tau(n)+1}$, $a_n \leq a_{\tau(n)+1}$. Using the fact that $\alpha_{\tau(n)} \to 0$ as $\tau(n) \to \infty$ and by equation (15), we obtain

$$P_s^*\big(\|\nabla f(x_{\tau(n)}) - \nabla f(Tx_{\tau(n)})\|\big) \to 0.$$

Following similar argument as in Case I, we see that

$$\lim_{n\to\infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0.$$

Also,

$$\limsup_{\tau(n)\to\infty} \langle \nabla f(u) - \nabla f(z), x_{\tau(n)} - z\rangle \leq 0.$$

It follows from equation (16) that

$$a_{\tau(n)+1} \leq a_{\tau(n)} + \alpha_{\tau(n)}[\langle \nabla f(u) - \nabla f(z), x_{\tau(n)} - z\rangle - a_{\tau(n)}].$$

From the fact that $a_{\tau(n)} \leq a_{\tau(n)+1}$ and $a_{\tau(n)} > 0$, the above inequalities give

$$a_{\tau(n)} \leq \langle \nabla f(u) - \nabla f(z), x_{\tau(n)} - z\rangle \to 0 \ as \ \tau(n) \to \infty.$$

Thus, $\lim_{\tau(n)\to\infty} a_{\tau(n)} = \lim_{\tau(n)\to\infty} a_{\tau(n)+1} = 0$. Since $0 \leq a_n \leq a_{\tau(n)+1}$, it implies that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} D_f(z, x_n) = 0$. Therefore, by Lemma 9, we arrived at $x_n \to z$ as $n \to \infty$. Hence in view of the above two cases, we see that the sequence $\{x_n\}$ converges strongly to $z = \mathrm{P}_{\mathcal{A}}(u) \in \mathcal{A}$. This completes the proof. ◄

Corollary 1. *Let $f : E \to (-\infty, +\infty]$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of $int(domf)$ and $T_i : C \to C$ for i=1,2,. . .,N be a finite family of Bregman nonexpansive mappings such that $\mathcal{A} = \cap_{i=1}^N A_f(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by* (12)*. Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{A}}(u)$.*

Proof. Let the mapping in Definition 2 be an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-generic generalized Bregman nonspreading mapping. Observe that $(1, 0, 0, -1, 0, 0)$-generic generalized Bregman nonspreading mapping is a Bregman nonexpansive. Thus, by Theorem 2 we see that the sequence $\{x_n\}$ converges strongly to $z = \mathrm{P}_{\mathcal{A}}(u)$. This completes the proof. ◄

Corollary 2. *Let $f : E \to (-\infty, +\infty]$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of $int(domf)$ and $T_i : C \to C$ for i=1,2,. . .,N be a finite family of Bregman nonspreading mappings; i.e.,*

$$D_f(Tx, Ty) + D_f(Ty, Tx) \leq D_f(Tx, y) + D_f(Ty, x) \ \forall \ x, y \in C.$$

Suppose that $\mathcal{A} = \cap_{i=1}^{N} A_f(T_i) \neq \emptyset$ and let $\{x_n\}$ be a sequence defined by (12)*. Then $\{x_n\}$ converges strongly to $z = P^f_{\mathcal{A}}(u)$.*

Proof. Observe that $(1, 0, -1, 0, -1, 0)$-generic generalized Bregman nonspreading mapping is Bregman nonspreading. Thus, by Theorem 2 we see that the sequence $\{x_n\}$ converges strongly to $z = \mathrm{P}_{\mathcal{A}}(u)$. This completes the proof. ◄

Corollary 3. *Let $f : E \to (-\infty, +\infty]$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of $int(domf)$ and $T_i : C \to C$ for i=1,2,. . .,N be a finite family of generalized Bregman nonspreading mappings; i.e. $\exists$ α, β, γ and $\delta \in \mathbb{R}$ such that for all $x, y \in C$*

$$\begin{aligned} \alpha D_f(Tx, Ty) &+ (1-\alpha) D_f(x, Ty) + \gamma\{D_f(Ty, Tx) - D_f(Ty, x)\} \\ &\leq \beta D_f(Tx, y) + (1-\beta) D_f(x, y) + \delta\{D_f(y, Tx) - D_f(y, x)\}. \end{aligned}$$

Suppose that $\mathcal{A} = \cap_{i=1}^{N} A_f(T_i) \neq \emptyset$ and let $\{x_n\}$ be a sequence defined by (12)*. Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{A}}(u)$.*

Proof. Since an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-generic generalized Bregman nonspreading mapping is generalized Bregman nonspreading for $\alpha + \beta = -\gamma - \delta = 1$ with $\alpha + \beta > 0$ and $\alpha + \beta + \gamma + \delta \geq 0$, it follows from Theorem 2 that the sequence $\{x_n\}$ converges strongly to $z = \mathrm{P}_{\mathcal{A}}(u)$. This completes the proof. ◄

Corollary 4. *Let C be a nonempty, closed and convex subset of a real Hilbert space and $T_i : C \to C$ for i=1,2,. . .,N be a finite family of normally generalized hybrid mappings such that $\mathcal{A} = \cap_{i=1}^{N} A_f(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by* (12)*. Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{A}}(u)$. where $P_{\mathcal{A}}(u)$ is the metric projection of C onto $A_f(T)$.*

Proof. By Remark 1, we see that the generic generalized Bregmen nonspreading mapping reduces to normally generalized hybrid mapping in Hilbert space, i.e. there exist $\alpha_1, \beta_1, \gamma_1$ and $\delta_1 \in \mathbb{R}$ such that

$$\alpha_1\|Tx - Ty\|^2 + \beta_1\|x - Ty\|^2 + \gamma_1\|Tx - y\|^2 + \delta_1\|x - y\|^2 \leq 0 \;\forall\; x, y \in C, \quad (22)$$

where $\alpha_1 = \alpha - \epsilon, \beta_1 = \beta + \epsilon, \gamma_1 = \gamma - \zeta$ and $\delta_1 = \delta + \zeta$ satisfying $\alpha_1 + \beta_1 = \alpha + \beta > 0$ and $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 = \alpha + \beta + \gamma + \delta \geq 0$. Thus, by Theorem 2 we see that the sequence $\{x_n\}$ converges strongly to $z = \mathrm{P}_{\mathcal{A}}(u)$. This completes the proof. ◄

Corollary 5. *Let C be a nonempty, closed and convex subset of a real Hilbert space and $T_i : C \to C$ for i=1,2,. . .,N be a finite family of generalized hybrid mappings such that $\mathcal{A} = \cap_{i=1}^{N} A_f(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by* (12)*. Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{A}}(u)$. where $P_{\mathcal{A}}(u)$ is the metric projection of C onto $A_f(T)$.*

Proof. Observe that from equation (22) it follows that if $\alpha_1 + \beta_1 = -\gamma_1 - \delta_1 = 1$, then an $(\alpha_1, \beta_1, \gamma_1, \delta_1)$-normally generalized hybrid mapping becomes a generalized hybrid mapping satisfying $\alpha_1 + \beta_1 = \alpha_1 + (1 - \alpha_1) > 0$ and $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 = \alpha_1 + (1 - \alpha_1) + \gamma_1 + (-\gamma_1 - 1) \geq 0$. It follows from 2 that the sequence $\{x_n\}$ converges strongly to $z = \mathrm{P}_{\mathcal{A}}(u)$. This completes the proof. ◄

Here, we approximate a common fixed points of generic generalized Bregman nonspreading mappings in Banach space by applying Theorem 2.

Corollary 6. *Let $f : E \to (-\infty, +\infty]$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int$(domf)$ and $T_i : C \to C$ for i=1,2,. . .,N be a finite family of generic generalized Bregman nonspreading mappings such that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by* (12)*. Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}(u)$.*

Proof. Since T_i for $i = 1, 2, ..., N$ are generic generalized Bregman nonspreading mappings with $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, then by letting $v \in \mathcal{F}$ and replacing x with v in (iii) of Definition 2, we have for any $y \in C$

$$\alpha D_f(v, Ty) + \beta D_f(v, Ty) + \gamma D_f(v, y) + \delta D_f(v, y) \leq 0.$$

This implies

$$(\alpha + \beta) D_f(v, Ty) \leq -(\gamma + \delta) D_f(v, y).$$

Using (i) and (ii) of Definition 2, we get

$$D_f(v, Ty) \leq -\frac{(\gamma + \delta)}{(\alpha + \beta)} D_f(v, y) \leq D_f(v, y).$$

$$\implies D_f(v,Ty) \leq D_f(v,y).$$

Thus, $v \in A_f(T)$ and consequently $F(T) \subset A_f(T)$. Since $A_f(T) \neq \emptyset$, it follows from Theorem 2 that $x_n \to z \; as \; n \to 0, z \in A_f(T)$.
Since C is closed and $\{x_n\}$ converges strongly to z, it implies that $z \in C$. Using the fact that $z \in A_f(T) \cap C$, we see that

$$D_f(z,Tz) \leq D_f(z,z) = 0.$$

This implies that $z \in F(T)$. In addition,

$$D_f(u,z) = inf\{D_f(u,v) : v \in A_f(T)\} \leq inf\{D_f(u,v) : v \in F(T)\}.$$

Hence $z = \mathrm{P}_{\mathcal{F}}(u)$. This completes the proof. ◄

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Bashir Ali
Department of Mathematical Sciences, Bayero University, Kano-Nigeria
E-mail: bashiralik@yahoo.com

Lawal Yusuf Haruna
Department of Mathematical Sciences, Kaduna State University, Kaduna-Nigeria
E-mail: yulah121@gmail.com

Murtala Haruna Harbau
Department of Science and Technology Education, Bayero University, Kano-Nigeria
E-mail: murtalaharbau@yahoo.com

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