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The Heat Equation with Piecewise Constant Delay Perturbation

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Abstract. In this paper, we study the heat equation with non-smooth perturbation caused by a delay applied to the time variable through a piecewise constant function, but keeping the time positive, which does not merit changing the phase space as usually occurs when the delay makes the time negative. To do so, we first prove the existence of solutions using Fourier's transform. Next, we prove the uniqueness of solutions by applying the maximum principle method. After that, we study the stability of these solutions. Finally, we propose some problems that can be solved with this or similar techniques.

Key Words and Phrases: heat equation, non-smoothness perturbation, piecewise constant delay, existence and uniqueness, stability, fundamental solution, finite Fourier sine transform, maximum principle.

2010 Mathematics Subject Classifications: 35K05, 35K08, 35A25, 34A25

1. Introduction

We know that when one wants to model real-life problems, these problems are often modeled simply by differential equations, equations involving an unknown function and its derivatives; but sometimes the differential equation is not enough to have a more accurate model since real problems are always subjected to disturbances such as non-local conditions, impulses and delays or time deviation, etc. The differential equations with delay that have been studied the most are those where there is a delay in time, for which it is necessary to know the historical past of the dynamics of the problem; that is, the initial condition is a function in a negative interval with end points minus the delay and zero. So, the initial condition is not a point anymore, which forces us to change the phase space for

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these type of equations. However, there may be cases where the time deviations slow down the dynamics of the problem but the time remains positive. In these cases, there is no historical past, the initial condition is simply a point in the state space. For better clarity in this exposition and the set of the problem to be solved, we will present three simple examples:

$$\begin{cases} \dfrac{d}{dt}x(t) = x(t), \\ x(0) = x_0. \end{cases} \tag{1}$$

In this case the solution of example (1) is given by $x(t) = \exp tx_0$, which is a nice and closed formula.

Now, we consider the same example, but we put delay on time to see what happens:

$$\begin{cases} \dfrac{d}{dt}x(t) = x(t-r), \\ x(\theta) = \eta(\theta),\ \theta \in [-r, 0]. \end{cases} \tag{2}$$

Example (2) seems simple, but the fact that this perturbation on time and the initial condition $\eta : [-r, 0] \longrightarrow \mathbb{R}$ is a continuous function giving the equation an infinite dimensional character, implies there is no closed formula to represent the solutions of this system although we know that the solutions exist and are unique, and using the step-by-step method of integration, we can assert that there is a solution and prove uniqueness trivially. For more information about delay differential equations, one can see the book by Driver, R. [2]

The next example deals with the kind of equations we are interested in, equation under the influence of delay or deviation that keeps time argument positive:

$$\begin{cases} \dfrac{d}{dt}x(t) = x([\![t]\!]), \\ x(0) = x_0, \end{cases} \tag{3}$$

where $[\![t]\!]$ is the largest integer less than t. In this case, fortunately, we can find a nearly closed formula for the solutions of the initial value problem (3), which is given by

$$x(t) = (t - p + 1)2^p x_0, \quad p \le t < p + 1, \quad \text{with} \quad p = 0, 1, 2, \ldots.$$

That is our main motivation to study the following heat equation with a non-smooth perturbation or piecewise constant argument caused by this type of deviation on time:

$$u_t(x, t) = u_{xx}(x, t) + au(x, [\![t]\!]), \tag{4}$$

subject to the conditions

$$\begin{cases} u(0,t) = u(\pi,t) = 0, \\ u(x,0) = f(x), \end{cases} \tag{5}$$

where $[\![t]\!]$ is the largest integer less than t. This term is that causes the delay in (4). We assume that $f(x)$ is twice differentiable in $[0,\pi]$ and that its second derivative is bounded on $[0,\pi]$ by $M > 0$.

We believe that our result can be applied to other equations like the strongly damped wave equation, the BBM equation and others evolution equations. We also hope to apply this kind of perturbations to analyze some mathematical models such as models dealing with different concentrations, population growth, prey-predator population models, control systems and the distribution of prime numbers. A very important application of these models is employed in the solutions of the two body problem of electrodynamics, in which the interaction of two charged particles is considered. Other applications of delay differential equation have been used in the study of nuclear reactors, neutron shielding, transistor circuits, neurology geometrical problems, etc. The study of these equations dates back to S. M. Saha and Wiener [3] and K. L. Cooke and others [4, 5]. But, for partial differential equations with delay we refer the reader to the book by Jianhong Wu [6].

2. Heat equation

The heat equation, also known as the diffusion equation, describes in typical physical applications the evolution in time of the density u of some quantity such as heat, chemical concentration, etc. Let V be any smooth subdomain, in which there is no source or sink, the rate of change of the total quantity within V equals the negative of the net flux $\mathbf{F}$ through ∂V:

$$\frac{d}{dt}\int_V u\,dx = -\int_{\partial V} \mathbf{F}\cdot\nu\,dS.$$

The divergence theorem tells us

$$\frac{d}{dt}\int_V u\,dx = -\int_V \operatorname{div}\mathbf{F}\,dx.$$

Since V is arbitrary, we should have

$$u_t = -\operatorname{div}\mathbf{F}.$$

For many applications, **F** is proportional to the gradient of u, but with points in the opposite direction (flux is from regions of higher concentrations):

$$\mathbf{F} = -a\nabla u \quad (a > 0).$$

Therefore, we obtain the equation

$$u_t = -a\mathrm{div}(\nabla u) = a\Delta u,$$

which is called the heat equation when $a = 1$.

If there is a source in Ω, we obtain the following nonhomogeneous heat equation:

$$u_t - \Delta u = f(x,t),\ x \in \Omega,\ t \in (0,\infty).$$

Definition 1. *The function*

$$\phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{t}} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

is called the fundamental solution of the heat equation $u_t = \Delta u$.

For more details on the heat equation see [1].

3. Maximum principle for the heat equation

In the sequel, for the sake of completeness and the benefit of the reader, we shall give the proof of the so called weak maximum principle.

Let $\Omega \in \mathbb{R}^n$ be an open, bounded set, and fix a time $T > 0$.

1. We define the parabolic cylinder

$$\Omega_T = \Omega \times (0,T] = \{(x,t) : x \in \Omega,\ t \in (0,T]\}$$

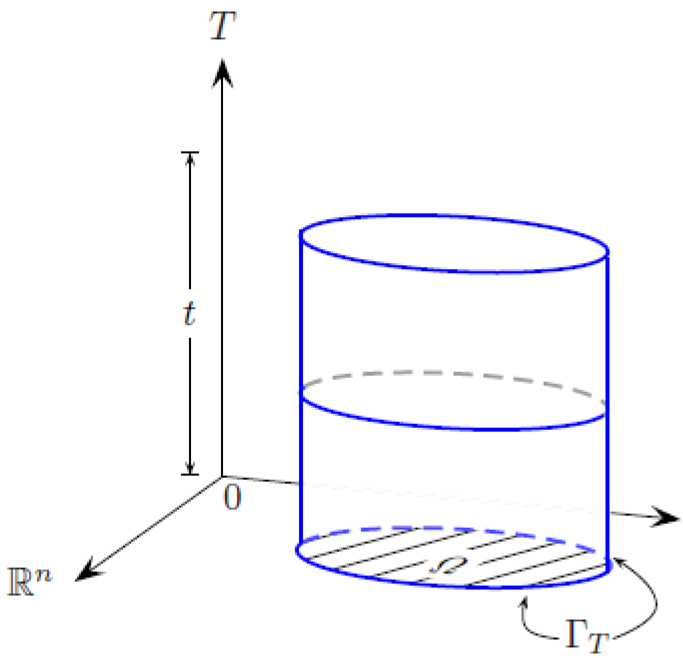

Figure 1: The region Ω_T

2. The parabolic boundary of Ω_T is

$$\Gamma_T = \partial\Omega_T = \overline{\Omega}_T \setminus \Omega_T.$$

Theorem 1 (Weak maximum principle)**.** *Let* $u \in C^2(\Omega) \cap C(\overline{\Omega}_T)$ *and satisfy* $u_t - \Delta u \leq 0$ *in* Ω_T *(in this case we say* u *is a subsolution of the heat equation). Then*

$$\max_{\overline{\Omega}_T} u = \max_{\partial\Omega_T} u.$$

Proof. Consider $v = u - \epsilon t$ for any $\epsilon > 0$. Then

$$v_t - \Delta v = u_t - \Delta u - \epsilon \leq -\epsilon < 0$$

$(x, t) \in \Omega_T$.

Let $v(x^\circ, t^\circ) = \max_{\overline{\Omega}_T} v$. Suppose $(x^\circ, t^\circ) \in \Omega_T$. Then $x^\circ \in \Omega$ and $t^\circ \in (0, T]$; consequently, at the maximum point (x°, t°) we have

$$\Delta v(x^\circ, t^\circ) \geq 0 \qquad \text{and} \qquad v_t(x^\circ, t^\circ) \geq 0;$$

this would imply $v_t - \Delta v \geq 0$ at (x°, t°), which is a contradiction, since $v_t - \Delta v < 0$.

Hence $(x^\circ, t^\circ) \in \partial\Omega_T$ and so

$$\begin{aligned}\max_{\overline{\Omega}_T} u \leq \max_{\overline{\Omega}_T}(v + \epsilon t) &\leq \max_{\overline{\Omega}_T} v + \epsilon T \\ &= \max_{\partial\Omega_T} v + \epsilon T \\ &\leq \max_{\partial\Omega_T} u + \epsilon T.\end{aligned}$$

Setting $\epsilon \to 0^+$, we deduce

$$\max_{\overline{\Omega}_T} u \leq \max_{\partial\overline{\Omega}_T} u. \tag{6}$$

Since $\partial\Omega_T = \overline{\Omega}_T \setminus \Omega_T = \overline{\Omega}_T \cap (\Omega_T)^c \subset \overline{\Omega}_T$, then

$$\max_{\partial\Omega_T} u \leq \max_{\overline{\Omega}_T} u. \tag{7}$$

By (6) and (7), we have

$$\max_{\overline{\Omega}_T} u = \max_{\partial\Omega_T} u,$$

hence the proof is complete. ◀

Corollary 1. *Let $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ and satisfy $u_t - \Delta u \geq 0$. Then*

$$\min_{\overline{\Omega}_T} u = \min_{\partial\Omega_T} u.$$

Corollary 2. *Let $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ and satisfy $u_t - \Delta u = 0$. Then*

$$\min_{\partial\Omega_T} u \leq u \leq \max_{\partial\Omega_T} u.$$

4. Existence of solutions

Now, we are ready to solve problem (4) subject to (5). To this end, we shall apply a finite Fourier sine transform to both sides of (4) which transforms (4) into an ordinary delay differential equation.

We have chosen such an integral transform because of the boundary condition (5).

This should be clear due to a property of the Fourier sine transform given in Theorem 2.

Definition 2. *By a solution of* (4) *we mean a function* $u(x,t)$ *that satisfies the following:*

1. $u(x,t)$ *is continuous on* $[0,\infty)$*;*
2. *The partial derivative* $u_t(x,t)$ *exists at each point in the interval* $[0,\infty)$ *with the possible exception of the* $t = p$ *for* $p = 1,2,3,\ldots$ *where one-side derivative exists.*
3. *The partial derivative* $u_{xx}(x,t)$ *exists at each point* t *in the interval* $[0,\infty)$*,*
4. $u(x,t)$ *satisfies* (4) *and* (5)*.*

The finite Fourier sine transform of a function $g(x)$ is defined by

$$\mathcal{F}_s(g(n)) = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\,dx, \tag{8}$$

where $n = 1,2,3,\ldots$ and $g(x)$ is a piecewise continuous function. The inverse of this transform is given by

$$g(x) = \sum_{n=1}^{\infty} \mathcal{F}_s(n)\sin(nx). \tag{9}$$

The following theorem describes a property of the finite Fourier transform that will be used to solve (4).

Theorem 2. *Let* $f'(x)$ *be continuous and* $f''(x)$ *be piecewise continuous on* $[0,\pi]$*. Let* $\mathcal{F}_s(n)$ *be the finite Fourier sine transform. Then*

$$\mathcal{F}_s(f''(n)) = \frac{2n}{\pi}\left[(-1)^n f'(\pi) - f'(0)\right] - n^2\mathcal{F}_s(n).$$

Proof. Integrating by parts twice, we have

$$\begin{aligned}
\mathcal{F}_s(f''(n)) =& \frac{2}{\pi}\int_0^{\pi} f''(x)\sin(nx)\,dx \\
=& \frac{2}{\pi}\left[f'(x)\sin(nx)\right]_0^{\pi} - \frac{2n}{\pi}\int_0^{\pi} f'(x)\cos(nx)\,dx \\
=& -\frac{2n}{\pi}\left[f(x)\cos(nx)\right]_0^{\pi} - \frac{2n^2}{\pi}\int_0^{\pi} f'(x)\sin(nx)\,dx \\
=& -\frac{2n}{\pi}[(-1)^n f(\pi) - f(0)] - n^2\mathcal{F}_s(n).
\end{aligned}$$

The theorem is proved. ◀

Next, applying the finite Fourier sine transform to both sides of (4), we get

$$\mathcal{F}_s(u_t(n,t)) = \mathcal{F}_s(u_{xx}(n,t)) + a\mathcal{F}_s(u(n,[\![t]\!])), \tag{10}$$

and the left-hand side of (10) is

$$\mathcal{F}_s(u_t(n,t)) = \frac{2}{\pi}\int_0^\pi u_t(x,t)\sin(nx)\,dx. \tag{11}$$

By Leibniz integral rule, (11) can be rewritten as

$$\mathcal{F}_s(u_t(n,t)) = \frac{2}{\pi}\frac{d}{dt}\left(\int_0^\pi u(x,t)\sin(nx)\,dx\right).$$

Hence,

$$\mathcal{F}_s(u_t(n,t)) = \frac{d}{dt}\mathcal{F}_s u(n,t). \tag{12}$$

Let $\mathcal{F}_s(u(n,t)) = U(n,t)$. Then (12) becomes

$$\mathcal{F}_s(u_t(n,t)) = \frac{d}{dt}U(n,t). \tag{13}$$

Consider now the first term on the right-hand side of equation (10). Applying Theorem 2 to this expression, we have

$$\mathcal{F}_s(u_{xx}(n,t)) = \frac{2n}{\pi}[u(0,t) - (-1)^n u(\pi,t)] - n^2U(n,t).$$

Applying the boundary conditions (5) to the last equation, we get

$$\mathcal{F}_s(u_{xx}(n,t)) = -n^2U(n,t), \tag{14}$$

and the last term on the right-hand side of (10) can be written as $U(n,[\![t]\!])$. Finally, by (13), (14) and the fact that $\mathcal{F}_s(u(n,[\![t]\!])) = U(n,[\![t]\!])$, we obtain the ordinary delay differential equation

$$\frac{d}{dt}U(n,t) = -n^2U(n,t) + aU(n,[\![t]\!]). \tag{15}$$

Equation (15) can be solved applying the step by step method. This method consists in solving (15) on successive intervals, where t is defined within each of these intervals. So (15) becomes an ordinary differential equation. This is due to the fact that on any interval of the form $p \le t \le p+1$, $p \ge 0$, $p = 1,2,3,\ldots$, and $[\![t]\!] = p$, we obtain the following ordinary differential equation:

$$\frac{d}{dt}U(n,t) = -n^2U(n,t) + aU(n,p). \tag{16}$$

The solution of (16) is continuous on all the intervals in which equation (16) has been solved. Its derivatives are one-sided derivatives at the end points of each interval. For a more detailed description of this, the reader is advised to consult [2].

We proceed now to solve (16) for $0 \leq t \leq 1$. In this case, the differential equation becomes

$$\frac{d}{dt}U(n,t) = -n^2 U(n,t) + aU(n,0), \qquad 0 \leq t < 1 \tag{17}$$

where

$$U(n,0) = \frac{2}{\pi}\int_0^{\pi} U(y,0)\sin(ny)\,dy,$$

and using the third condition in (5), we can express the last equation as

$$U(n,0) = \frac{2}{\pi}\int_0^{\pi} f(y)\sin(ny)\,dy,$$

and rearranging terms, we get

$$\frac{d}{dt}U(n,t) + n^2(n,t) = aU(n,0),\ 0 \leq t < 1. \tag{18}$$

Let us multiply (18) by the integrating factor $e^{n^2 t}$ to obtain

$$e^{n^2 t}\frac{d}{dt}U(n,t) + n^2 e^{n^2 t}U(n,t) = ae^{n^2 t}U(n,0),$$

which can be expressed as

$$\frac{d}{dt}(e^{n^2 t}U(n,t)) = ae^{n^2 t}U(n.0).$$

Integrating this last equation from 0 to t, where $0 \leq t < 1$, we get

$$U(n,t) = U(n,0)\left(e^{-n^2 t} + a\left(\frac{1-e^{-n^2 t}}{n^2}\right)\right), \tag{19}$$

$0 \leq t < 1$. If we let t approach 1 from the left, (19) becomes

$$U(n,1) = U(n,0)\left(e^{-n^2} + a\left(\frac{1-e^{-n^2}}{n^2}\right)\right). \tag{20}$$

Let us consider the problem of solving (15) in the next interval, i.e. in $1 \leq t < 2$. Equation (15) becomes

$$\frac{dU(n,t)}{dt} = -n^2 U(n,t) + aU(n,1), \quad 1 \leq t < 2. \tag{21}$$

Multiplying (21) by the integrating factor $e^{n^2 t}$, we get

$$e^{n^2 t} U(n,t) + n^2 U(n,t) = a e^{n^2 t} U(n,1),$$

which is equivalent to

$$\frac{d(e^{n^2 t} U(n,t))}{dt} = a e^{n^2 t} U(n,1).$$

Integrating this equation from 1 to t, where $1 \leq t < 2$, we obtain the following expression:

$$U(n,t) = U(n,1)\left(e^{n^2(1-t)} + a\left(\frac{1 - e^{n^2(1-t)}}{n^2}\right)\right), \quad 1 \leq t < 2, \tag{22}$$

and taking (20) into account, we can express (22) as

$$U(n,t) = U(n,0)\left(e^{-n^2} + a\left(\frac{1-e^{-n^2}}{n^2}\right)\right)\left(e^{n^2(1-t)} + a\left(\frac{1-e^{n^2(1-t)}}{n^2}\right)\right) \quad 1 \leq t < 2. \tag{23}$$

Letting t approach 2 from the left, we get

$$U(n,2) = U(n,0)\left(e^{-n^2} + a\left(\frac{1-e^{-n^2}}{n^2}\right)\right)^2. \tag{24}$$

Repeating this process for p intervals, we find

$$U(n,t) = U(n,p)\left(e^{n^2(p-t)} + a\left(\frac{1-e^{n^2(p-t)}}{n^2}\right)\right), \quad p \leq t < p+1, \tag{25}$$

where

$$U(n,p) = U(n,0)\left(e^{-n^2} + a\left(\frac{1-e^{-n^2}}{n^2}\right)\right)^p. \tag{26}$$

The solution of (4) on the interval $p \le t < p+1$ can be obtained by applying the inverse Fourier sine transform to both sides of equation (25). This yields

$$u(x,t) = \sum_{n=1}^{\infty} U(n,p)\left(e^{n^2(p-t)} + a\left(\frac{1-e^{n^2(p-t)}}{n^2}\right)\right)\sin(nx),$$
$$p \le t < p+1. \quad (27)$$

In particular, the value of the solution at the point $t = p+1$ is given by

$$u(x,+1) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\int_0^{\pi} if(y)\sin(ny)\,dy\right)\left(e^{-n^2} + a\left(\frac{1-e^{-n^2}}{n^2}\right)\right)^p \sin(nx). \quad (28)$$

5. Convergence of solution

We proceed to the study of convergence of the series in (27) in the interval $0 \le t < 1$. In this interval $p = 0$, since $p \le t < p+1$, and thus (27) is reduced to

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\int_0^{\pi} f(y)\sin(ny)\,dy\right)\left(e^{-n^2 t} + a\left(\frac{1-e^{-n^2 t}}{n^2}\right)\right)\sin(nx). \quad (29)$$

Taking the absolute value on the both sides of (29) and applying the triangle inequality, we can obtain

$$|u(x,t)| \le \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi} |f(y)||\sin(ny)|\,dy\left|e^{-n^2 t} + a\left(\frac{1-e^{-n^2 t}}{n^2}\right)\right||\sin(nx)|. \quad (30)$$

$f(y)$ is bounded on $[0,\pi]$ since $f(y)$ is continuous on $[0,\pi]$. Let $M > 0$ be such that $|f(y)| \le M$ for any y on $[0,\pi]$. Using this and the fact that $|\sin(ny)| \le 1$, equation (30) becomes

$$|u(x,t)| \le \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\int_0^{\pi} M\,dy\right)\left|e^{-n^2 t} + a\left(\frac{1-e^{-n^2 t}}{n^2}\right)\right|,$$

which, after integration, can be expressed as

$$|u(x,t)| \le 2M\sum_{n=1}^{n}\left|e^{-n^2 t} + a\left(\frac{1-e^{-n^2 t}}{n^2}\right)\right|,$$

and thus

$$|u(x,t)| \leq 2M \sum_{n=1}^{\infty} \left(e^{-n^2 t} + |a| \left(\frac{1-e^{-n^2 t}}{n^2} \right) \right). \tag{31}$$

It is easy to prove by means of the ratio test that the series (31) converges and therefore the result shown in (27) is convergent for $0 \leq t < 1$.

We now proceed to the proof of convergence of the series (27) for the interval $p \leq t < p+1$, $p = 1, 2, 3, \ldots$. Taking the absolute value on both sides of (27) and using the triangle inequality, we get

$$|u(x,t)| \leq 2M \sum_{n=1}^{\infty} \left| e^{-n^2} + |a| \left(\frac{1-e^{-n^2}}{n^2} \right) \right|^p e^{n^2(p-t)} + |a| \left(\frac{1-e^{n^2(p-t)}}{n^2} \right). \tag{32}$$

Observe that $\frac{1-e^{-n^2}}{n^2} \leq \frac{1}{n^2}$. Then, using this in (32) yields

$$|u(x,t)| \leq 2M \sum_{n=1}^{\infty} \left| \frac{1}{n^2} + \frac{|a|}{n^2} \right|^p \left| e^{n^2(p-t)} + |a| \left(\frac{1-e^{n^2(p-t)}}{n^2} \right) \right|,$$

which can be rewritten as

$$|u(x,t)| \leq 2M(1+|a|)^p \sum_{n=1}^{\infty} \left| e^{n^2(p-t)} + |a| \left(\frac{1-e^{n^2(p-t)}}{n^2} \right) \right| \frac{1}{n^{2p}},$$

$p = 1, 2, \ldots$, and since $p \leq t < p+1$, we have $e^{n^2(p-t)} \leq 1$ and

$$\frac{1-e^{n^2(p-t)}}{n^2} \leq 1.$$

Thus,

$$|u(x,t)| \leq 2M(1+|a|)^p |1+|a|| \sum_{n=1}^{\infty} \frac{1}{n^{2p}},$$

for $p = 1, 2, 3, \ldots$. It is easy to prove by means of the ratio test that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$$

is convergent. Hence the series (27) converges for $p = 1, 2, \ldots$, and, since we have already proven that this series converges for $p = 0$, we get the result.

Next, we shall check that the solution given in (27) satisfies (4) and (5). Due to the fact that $\sin(nx) = 0$ at $x = 0$ and at $x = \pi$, (27) satisfies $u(0,t) = u(\pi,t)$. Also, at $t = 0$ (27) becomes

$$\begin{aligned} u(x,0) =& \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\int_0^{\pi} f(y) \sin(ny)\, dy \right) \sin(nx) \\ =& \sum_{n=1}^{\infty} \mathcal{F}_s(n) \sin(nx) \\ =& f(x). \end{aligned}$$

We have thus shown that the series in (27) satisfies the initial and boundary conditions of the problem. We now proceed to show that this series satisfies the partial differential equation with a piecewise constant argument given in (4). The formal partial derivative with respect to t of $u(x,t)$ as in ((27)) is given by

$$u_t(x,t) = \sum_{n=1}^{\infty} U(n,p)(-n^2 e^{n^2(p-t)}) \sin(nx), \tag{33}$$

$p \le t < p+1$, where $u(n,p)$ is given in (26) and $p = 0, 1, 2, \ldots$. Consider the convergence of the series shown in (33) for the interval $0 \le t < 1$, in this case, the series is

$$u_t(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(y) \sin(ny)\, dy \right) (-u^2 e^{-n^2 t}) + a e^{-n^2 t} \sin(nx).$$

Then

$$|u_t(x,t)| \le \frac{2}{\pi} \sum_{n=0}^{\infty} \int_0^{\pi} |f(y)||\sin(ny)|\, dy| - n^2 e^{-n^2 t} + a e^{-n^2 t}||\sin(nx)|.$$

Recall that $|f(y)| \le M$. Using this result and integrating, we obtain

$$|U_t(x,t)| \le 2M \left(\sum_{n=1}^{\infty} n^2 e^{-n^2 t} + |a| \sum_{n=1}^{\infty} e^{-n^2 t} \right).$$

Using the ratio test, it is easy to prove that the series shown in the latter inequality converge. Thus the series given in (33) converges when $0 \le t < 1$. Consider now the series (33) on any interval $p \le t < p+1$, $p = 1, 2, \ldots,$. Then

$$|u_t(x,t)| \le 2M \sum_{n=1}^{\infty} \left| e^{-n^2} + a \left(\frac{1 - e^{-n^2}}{n^2} \right) \right|^p | - n^2 e^{n^2(p-t)} + a e^{n^2(p-t)}|,$$

hence

$$|u_t(x,t)| \leq 2M \sum_{n=1}^{\infty} \left| \frac{1}{n^2} + \frac{|a|}{n^2} \right|^p (n^2 e^{n^2(p-t)} + |a| e^{n^2(p-t)}).$$

After rearranging terms in the previous inequality, we obtain

$$|u_t(x,t)| \leq 2M(1+|a|)^p \left(\sum_{n=1}^{\infty} \frac{e^{n^2(p-t)}}{n^{2(p-1)}} + a \sum_{n=1}^{\infty} \frac{e^{n^2(p-t)}}{n^{2p}} \right),$$

$p = 1, 2, 3, \ldots$

It can be proven by means of the ratio test that the two series in the latter equation are convergent. Thus the series (33) represents $U_t(x,t)$ in any interval $p \leq t < p+1$, $p = 0, 1, 2, \ldots$. Let us consider the other derivatives. Differentiating expression (27) with respect to x, we get

$$u_x(x,t) = \sum_{n=1}^{\infty} U(n,p) \left(e^{n^2(p-t)} + a \left(\frac{1 - e^{n^2(p-t)}}{n^2} \right) \right) n \cos(nx). \tag{34}$$

Recall that $f(y)$ is a twice differentiable function. This allows us integrate the term $\int_0^\pi f(y) \sin(ny)\, dy$ in $U(n,p)$ by parts two times. The first integration by parts yields

$$\int_0^\pi f(y) \sin(ny)\, dy = -f(y) \left. \frac{\cos(ny)}{n} \right|_0^\pi + \int_0^\pi f'(y) \frac{\cos(ny)}{n}\, dy,$$

and since $f(0) = f(\pi) = 0$, the previous expression becomes

$$\int_0^\pi f(y) \sin(ny)\, dy = \int_0^\pi f'(y) \frac{\cos(ny)}{n}\, dy.$$

Integrating by parts once again, we have

$$\begin{aligned} \int_0^\pi f(y) \sin(ny) &= \int_0^\pi f'(y) \frac{\cos(ny)}{n}\, dy \\ &= f'(y) \left. \frac{\sin(ny)}{n^2} \right|_0^\pi - \int_0^\pi f''(y) \frac{\sin(ny)}{n^2}\, dy, \end{aligned}$$

which yields

$$\int_0^\pi f(y) \sin(ny)\, dy = - \int_0^\pi f''(y) \frac{\sin(ny)}{n^2}\, dy. \tag{35}$$

Let $L > 0$ be such that $|f''(y)| \leq L$ for all $y \in [0, \pi]$. Using (35), we obtain the following inequality:

$$|u_x(x,t)| \leq 2L \sum_{n=1}^{\infty} \left| e^{-n^2} + a\left(\frac{1-e^{-n^2}}{n^2}\right)\right|^p \left|\frac{e^{n^2(p-t)}}{n} + a\left(\frac{1-e^{n^2(p-t)}}{n^3}\right)\right|,$$

if $p = 0$, the previous inequality becomes

$$|u_x(x,t)| \leq 2L \sum_{n=1}^{\infty} \left|\frac{e^{-n^2 t}}{n} + a\left(\frac{1-e^{-n^2 t}}{n^3}\right)\right|, \quad 0 < t < 1,$$

and thus

$$|u_x(x,t)| \leq 2L \left(\sum_{n=1}^{\infty} \frac{e^{-n^2 t}}{n} + |a| \sum_{n=1}^{\infty} \frac{1}{n^3} - |a| \sum_{n=1}^{\infty} \frac{e^{-n^2 t}}{n^3}\right).$$

We can prove by means of the ratio test that the three series in the latter inequality converge. This implies that the series in (34) converges in the interval $(0, 1]$. Consider now the convergence of the series in (34) for $p \leq t < p+1$, $p \geq 1$. Then, by (34), we have

$$|u_x(x,t)| \leq 2L \sum_{n=1}^{\infty} \left| e^{-n^2} + a\left(\frac{1-e^{-n^2}}{n^2}\right)\right|^p \left|\frac{e^{n^2(p-t)}}{n} + a\left(\frac{1-e^{n^2(p-t)}}{n^3}\right)\right|,$$

and since $e^{-n^2} \leq \dfrac{1}{n^2}$ and $e^{n^2(p-t)} \leq 1$ for $p \leq t < p+1$, we find that

$$|u_x(x,t)| \leq 2L|1+|a||^p \left(\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} + \frac{|a|}{n^{2p+3}}\right), \quad p = 1, 2, 3, \ldots$$

The series in the latter inequality is known to converge and thus the series given in (34) converges for $p = 1, 2, 3, \ldots$, and hence it represents the partial derivative of $U(x,t)$ with respect to x. Differentiating $U_x(x,t)$ with respect to x, we obtain the series

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} U(n,p) \left(e^{n^2(p-t)} + a\left(\frac{1-e^{n^2(p-t)}}{n^2}\right)\right) n^2 \sin(nx). \tag{36}$$

Consider the series (36) for $p \leq t < p+1$, $p = 0, 1, 2, \ldots$. Then we have

$$|u_{xx}(x,t)| \leq \sum_{n=1}^{\infty} \left| e^{-n^2} + a\left(\frac{1-e^{-n^2}}{n^2}\right)\right|^p \left| e^{n^2(p-t)} + a\left(\frac{1-e^{n^2(p-t)}}{n^2}\right)\right|.$$

We have shown already in previous pages that this series converges. Thus, expression (36) represents the second partial derivative of $U(x,t)$ with respect to x. Now let us check that (27) satisfies (4). On the one hand, we know that

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} U(n,p)\left(e^{n^2(p-t)} + a\left(\frac{1-e^{n^2(p-t)}}{n^2}\right)\right) n^2 \sin(nx).$$

On the other hand, $u(x,[\![t]\!]) = U(x,p)$, $t \in [p,p+1)$. Recall that

$$u(x,p) = \sum_{n=1}^{\infty} U(n,p)\sin(nx).$$

Hence

$$\begin{aligned} &u_{xx}(x,t) + au(x,p) \\ &\quad = -\sum_{n=1}^{\infty} U(n,p)(n^2 e^{n^2} e^{n^2(p-t)} + a(1-e^{n^2(p-t)}))\sin(nx) \\ &\qquad\qquad + a\sum_{n=1}^{\infty} U(n,p)\sin(nx). \end{aligned}$$

Rearranging terms we find

$$u_{xx}(x,t) + au(x,p) = \sum_{n=1}^{\infty} U(n,p)(a-n^2)(1-e^{n^2(p-t)})\sin(nx)$$

By (33) we have

$$u_t(x,t) = u_{xx} + au(x,p).$$

Thus, $u(x,t)$ given in (27) does satisfy equation (4) and the boundary conditions (5).

6. Uniqueness of the solution

In case of one dimension, we proceed to prove that (4) has a unique solution. In order to attain this goal, we shall use a maximum principle given in Theorem 1. The domain Ω_T mentioned in Theorem 1 is a closed interval, say $[0,\pi]$ of axis x. The boundary Γ_T consists of the endpoints $x=0$ and $x=\pi$. Let u be a solution of

$$u_t = u_{xx}$$

for $0 < x < \pi$, $t > 0$. Fix $T > 0$ and let $\overline{\Omega}_T = [0, \pi] \times [0, \pi]$. Furthermore, let

$$\begin{aligned} M_0 &= \max_{0<x<\pi} u(x,0); \\ M_1 &= \max_{0\le x\le T} u(0,t); \\ M_2 &= \max_{0\le x\le T} u(\pi,t); \\ M &= \max\{M_0, M_1, M_2\}, \end{aligned}$$

and

$$\begin{aligned} m_0 &= \min_{0<x<\pi} u(x,0); \\ m_1 &= \min_{0\le x\le T} u(0,t); \\ m_2 &= \min_{0\le x\le T} u(\pi,t); \\ m &= \min\{m_0, m_1, m_2\}. \end{aligned}$$

Corollary 2 states that

$$m \le u(x,t) \le M \quad \text{in } \Omega_T.$$

Consider the first interval $0 < t < 1$. Here we have

$$u_t(x,t) - u_{xx}(x,t) = au(x,0), \quad 0 < x < \pi,\ 0 < t < 1.$$

Recall that

$$u(x,0) = f(x)$$

and

$$u(0,t) = 0, \quad u(\pi,t) = 0 \quad \text{for } t \ge 0.$$

Then $f(0) = f(\pi) = 0$, and

$$u_t(x,t) - u_{xx} = af(x).$$

Let u and v be two solutions of (4) and set $w = u - v$. Then

$$u_t(x,t) - u_{xx}(x,t) = af(x), \tag{37}$$

and

$$v_t(x,t) - v_{xx}(x,t) = af(x) \tag{38}$$

Subtracting (38) from (37), we get

$$u_t(x,t) - v_t(x,t) - (u_{xx}(x,t) - v_{xx}(x,t)) = 0,$$

which is equivalent to

$$\omega_t(x,t) - \omega_{xx}(x,t) = 0, \quad 0 < x < \pi. \tag{39}$$

Hence, ω is a solution of the homogeneous heat equation with the initial-boundary conditions

$$\begin{aligned} \omega(x,0) =& 0, \quad 0 \leq \pi \\ \omega(0,t) =& \omega(\pi,t) = 0, \quad 0 \leq t \leq 1. \end{aligned} \tag{40}$$

From Corollary 2 we know that

$$m \leq \omega \leq M,$$

where

$$m = \min\{\omega(x,0), \omega(0,t), \omega(\pi,t)\}$$

and

$$M = \max\{\omega(x,0), \omega(0,t), \omega(\pi,t)\}$$

for $0 < x < \pi$, $0 \leq t \leq 1$. But (40) implies that $m = 0$ and $M = 0$, hence

$$\omega(x,t) = 0 \quad \text{for all } t \in [0,1]$$

and all $x \in [0,\pi]$.

So we have $u(x,t) = v(x,t)$ for all $t \in [0,1]$ and all $x \in [0,\pi]$. This equality can easily be extended to the intervals $1 \leq t \leq 2$, $2 \leq t \leq 3$, etc. Observe that at $t = 1$ we have $u(x,1) = v(x,1)$, $0 \leq x \leq \pi$.

Consider the next interval, $1 \leq t \leq 2$. We have

$$u_t(x,t) - u_{xx}(x,t) = au(x,1) \tag{41}$$

and

$$v_t(x,t) - v_{xx}(x,t) = av(x,1). \tag{42}$$

Subtracting (42) from (41), we obtain

$$u_t(x,t) - v_t(x,t) - (u_{xx}(x,t) - v_{xx}(x,t)) = 0.$$

Due to the fact that

$$u(x,1) = v(x,1),$$

and since $\omega(x,t) = u(x,t) - v(x,t)$, we find that

$$\omega_t(x,t) - \omega_{xx}(x,t) = 0 \quad \text{for } 0 < x < \pi,\ 1 \leq t \leq 2,$$

where $\omega(0,t) = \omega(\pi,t) = 0$. Applying Corollary 2, we have

$$m \le \omega(x,t) \le M,$$

where

$$m = \min\{\omega(x,1), \omega(0,t), \omega(\pi,t)\}, \quad 0 < x < \pi,\ 1 \le t \le 2,$$

and

$$M = \max\{\omega(x,1), \omega(0,t), \omega(\pi,t)\}, \quad 0 < x < \pi,\ 1 \le t \le 2,$$

Observe that

$$\omega(x,1) = \omega(0,t) = \omega(\pi,t) = 0.$$

Hence $m = M = 0$ which implies that

$$\omega(x,t) = 0 \quad \text{for } 0 < x < \pi,\ 1 \le t \le 2,$$

and so

$$u(x,t) = v(x,t) \quad \text{for } 0 \le x \le \pi,\ 1 \le t \le 2.$$

Repeating this process, it is easy to show that

$$u(x,t) = v(x,t) \quad \text{for } 0 \le x \le \pi$$

and $p \le t < p+1$, $p = 0, 1, 2, 3, \ldots$. Thus (4) has a unique solution.

7. A special case

Consider

$$u_t = u_{xx} + au(x, [\![t]\!]), \quad p < t < p+1,\ p = 1, 2, \ldots,$$

subject to the conditions

$$u(0,t) = U(\pi,t) = 0,$$
$$u(x,0) = \sin x.$$

Then, since

$$\int_0^\pi \sin x \sin(nx)\, dx = \frac{\pi}{2}\delta_{n,1},$$

where $\delta_{n,1}$ is the Kronecker delta, the solution shown in (27) becomes

$$u(x,t) = (e^{-1} + a(1-e^{-1}))^p (e^{p-t} + a(1-e^{p-t})) \sin x \tag{43}$$

for $p \le t < p+1$. In particular,

$$u(x,p) = (e^{-1} + a(1-e^{-1}))^p \sin x. \tag{44}$$

Let $T_1 = e^{-1} + a(1 - e^{-1})$. We shall study the conditions under which (44) oscillates with respect to p when x is fixed. Oscillations with respect to p occur if once x has been fixed value, say x_c, where $x_c \neq 0$, the following holds:

$$u(x_c, p)u(x_c, p + 1) < 0, \tag{45}$$

but

$$u(x_c, p)u(x_c, p + 1) = (\sin^2 x_c)T_1^{2p+1}. \tag{46}$$

(45) and (46) imply that $T_1 < 0$, so

$$e^{-1} + a(1 - e^{-1}) < 0 \tag{47}$$

and hence

$$a < \frac{-e^{-1}}{1 - e^{-1}}. \tag{48}$$

Thus, if (48) holds, then $u(x_c, p)$ is periodic in p. Assume that $u(x_c, p)$ is periodic with period k, i.e.

$$u(x_c, p) = u(x_c, p + k).$$

This means that

$$T_1^p \sin x_c = T_1^{p+k} \sin x_c,$$

which implies

$$T_1^k = 1. \tag{49}$$

If k is odd, then (48) implies that $T_1 > 0$. But in this case T_1 does not fulfil condition (47) and thus no oscillation with respect to p occurs, which implies that k cannot be odd. If k is even, then $|T_1| = 1$. (49) indicates that only $T_1 = -1$ will produce oscillations. This implies that

$$e^{-1} + a(1 - e^{-1}) = -1.$$

From this we find that

$$a = \frac{1 + e^{-1}}{1 - e^{-1}}. \tag{50}$$

Hence, (50) oscillates periodically with respect to p of (34). The period of oscillation is $k = 2$.

8. Stability of special solution

Consider

$$u(x,p) = T_1^p \sin x. \tag{51}$$

We are going to study the behavior of this function as p increases without any bound and x is such that $0 < x < \pi$. First of all, let us introduce the following definitions.

Definition 3. *Let $u(x,p)$ be a solution of (4) subject to (5). Let x and x_0 belong to $(0,\pi)$ and let p be a non-negative integer. Then $u(x,p)$ is said to be stable if there exists $\epsilon > 0$ such that for any $x \in (0,\pi)$*

$$|u(x,p) - u(x_0,p)| < \epsilon, \quad p \geq 0.$$

A solution that is not stable is said to be unstable.

Definition 4. *Let $u(x,p)$ be a stable solution in the sense of Definition 3. $u(x,p)$ is said to be asymptotically stable if*

$$\lim_{p\to\infty} |u(x,p) - u(x_0,p)| = 0.$$

First case. Let $|T_1| = 1$. This occurs when $a = 1$ and $a = \dfrac{(1+e^{-1})}{1-e^{-1}}$. In this case, $u(x,p) = \sin x$ and $u(x_0,p) = \sin x_0$, which are independent of p and such that

$$|u(x,p) - u(x_0,p)| = |\sin x - \sin x_0| < 1.$$

Thus, the solution $u(x,p)$ is stable.

Second case. $|T_1| > 1$. This occurs when $a > 1$ and $a < \dfrac{(1+e^{-1})}{1-e^{-1}}$. In this case

$$\lim_{p\to\infty} |u(x,p) - u(x_0,p)| = \lim_{p\to\infty} |T_1|^p |\sin x - \sin x_0| = \infty,$$

hence the solution $u(x,p) = T_1^p \sin x$ is unstable.

Third case. Let $|T_1| < 1$. This happens when $\dfrac{-(1+e^{-1})}{1-e^{-1}} < a < 1$. In this case, we have

$$|u(x,p) - u(x_0,p)| = |T_1|^p |\sin x - \sin x_0| \leq |T_1|^p < 1.$$

Thus, $u(x,p)$ is stable in the sense of Definition 3. Furthermore,

$$\lim_{p\to\infty} |u(x,p) - u(x_0,p)| = \lim_{p\to\infty} |T_1|^p |\sin x - \sin x_0| = 0.$$

Hence $u(x,p)$ is asymptotically stable.

9. Conclusion and final remark

In this paper, we study the heat equation with non-smooth perturbation caused by a delay applied to the time variable through a piecewise constant function $[\![\cdot]\!]$. But if we want the delay time to be smaller, we can multiply the time by a positive constant greater than one, that is, consider the function $[\![\lambda t]\!]$, with $\lambda > 1$, in stead. We first prove the existence of solutions using Fourier's transform. Next, we prove the uniqueness of solutions by applying the maximum principle method. After that, we study the stability of these solutions. Finally, we propose some problems that can be solved with this or similar techniques.
Our methodology is simple and can be applied to those diffusive processes with these kind of piecewise constant delay function. For example, the Benjamin - Bona-Mohany equation, the strongly damped wave equations, the beam equations with piecewise constant delay function, etc.

Open Problem 9.1. *The Benjamin-Bona-Mohany equation is a source of non-linear heat equation. So, our next work deals with the existence of solution of the BBM equation with piecewise constant delay function:*

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = z([\![\lambda t]\!], x), & \text{in} \quad (0,\tau)\times\Omega, \\ z(t,x) = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(0,x) = z_0, & , \end{cases}$$

where $a \geq 0$ and $b > 0$ are constants, Ω is a domain in $\mathbb{R}^N$.

Open Problem 9.2. *We believe that this technique can be applied to prove the existence of solutions for the strongly damped wave equation with piecewise constant delay function:*

$$\begin{cases} \dfrac{\partial^2 w(t,x)}{\partial t^2} + \eta(-\Delta)^{1/2}\dfrac{\partial w(t,x)}{\partial t} + \gamma(-\Delta)w = w([\![\lambda t]\!], x), \\ w = 0, \text{on} \qquad (0,\tau)\times\partial\Omega, \\ w(0,x) = f(x), \quad w_t(0,x) = g(x), \qquad x\in\Omega. \end{cases}$$

Open Problem 9.3. *Another example where this technique may be applied is a partial differential equations modeling the structural damped vibrations of a string or a beam with piecewise constant delay function:*

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} - 2\beta\Delta\dfrac{\partial y(t,x)}{\partial t} + \Delta^2 y = y([\![\lambda t]\!], x), & \text{on} \quad (0,+\infty)\times\Omega, \\ y = \Delta y = 0, & \text{on} \quad (0,+\infty)\times\partial\Omega, \\ y(0,x) = f(x), \quad y_t(0,x) = g(x), & x\in\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$.

Moreover, some of these particular problems can be formulated in a more general setting. Indeed, we can consider the following semilinear evolution equation in a general Hilbert space Z with piecewise constant delay function:

$$\begin{cases} z' = -Az + z([\![\lambda t]\!]), \quad z \in Z, \quad t \in (0,\infty), \\ z(0) = z_0, \end{cases} \tag{52}$$

where $A : D(A) \subset Z \to Z$ is an unbounded linear operator in Z with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k},$$

with the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ of A having finite multiplicity γ_j equal to the dimension of the corresponding eigenspaces, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenfunctions of A. The operator $-A$ generates a strongly continuous compact semigroup $\{T_A(t)\}_{t\geq 0}$ given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}.$$

References

[1] L. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics. 19, American Mathematical Society, Providence R.I. 1998.

[2] R.D. Driver, *Ordynary Delay Differential Equation*, Springer Verlag. 1977.

[3] S.M. Shah, J. Wiener, *Distributional and entaire solutions for ordinary differential and functional equations*, Inter. J. Math. Sci., **6(2)**, 1983, 243–270.

[4] G. Ladas, *Oscillations of equations with piecewise constant mixed argument*, Proceedings of International conference on theory and Applications of Differential Equations. March 21–25-1988. Ohio University, Athens, Ohio. USA arcel Dekker Inc.

[5] E.C. Parthemiades, *Stability and Oscillation of neutral delay differential equation with piecewise constant argument*, Differential and Integral Equations, **1(4)**, 1988, 459–472.

[6] J. Wu, *Theory and applications of partial functional differential equations*, **119**, Springer Science & Business Media, 1996.

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