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Norms of Maximal Functions Between Generalized and Classical Lorentz Spaces

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Abstract. The aim of the paper is to find the norm of the generalized maximal operator $M_{\phi,\Lambda^\alpha(b)}$, defined for all measurable functions f on $\mathbb{R}^n$, with $0 < \alpha < \infty$ and functions b, $\phi : (0,\infty) \to (0,\infty)$, by

$$M_{\phi,\Lambda^\alpha(b)} f(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_{\Lambda^\alpha(b)}}{\phi(|Q|)}, \qquad x \in \mathbb{R}^n,$$

from generalized Lorentz spaces $G\Gamma(p,m,v)$ into classical Lorentz spaces $\Lambda^q(w)$.
In order to achieve the goal, we reduce the problem to the solution of the inequality

$$\left(\int_0^\infty \left[T_{u,b} f^*(y)\right]^q w(y)\,dy \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left(\int_0^x [f^*(s)]^p\,ds \right)^{\frac{m}{p}} v(x)\,dx \right)^{\frac{1}{m}}$$

where w and v are weight functions on $(0,\infty)$. Here f^* is the non-increasing rearrangement of a measurable function f defined on $\mathbb{R}^n$ and $T_{u,b}$ is the iterated Hardy-type operator involving suprema, which is defined for a measurable non-negative function g on $(0,\infty)$ by

$$(T_{u,b} g)(t) := \sup_{\tau \in [t,\infty)} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(s) b(s)\,ds, \qquad t \in (0,\infty),$$

where u and b are weight functions on $(0,\infty)$ such that u is continuous on $(0,\infty)$ and the function $B(t) := \int_0^t b(s)\,ds$ satisfies $0 < B(t) < \infty$ for every $t \in (0,\infty)$.

Key Words and Phrases: generalized maximal functions, classical and generalized Lorentz spaces, iterated Hardy inequalities involving suprema, weights.

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1. Introduction

Let $(\mathcal{R}, \mu)$ be a σ-finite non-atomic measure space. Denote by $\mathfrak{M}(\mathcal{R})$ the set of all μ-measurable functions on $\mathcal{R}$ and by $\mathfrak{M}_0(\mathcal{R})$ the class of functions in $\mathfrak{M}(\mathcal{R})$ that are finite μ - a.e. on $\mathcal{R}$. The symbol $\mathfrak{M}^+(\mathcal{R})$ stands for the collection of all $f \in \mathfrak{M}(\mathcal{R})$ which are non-negative on $\mathcal{R}$.

The non-increasing rearrangement f^* of $f \in \mathfrak{M}_0(\mathcal{R})$ is given by

$$f^*(t) = \inf \big\{ \lambda \ge 0 : \mu(\{x \in \mathcal{R} : |f(x)| > \lambda\}) \le t \big\}, \qquad t \in [0, \mu(\mathcal{R})).$$

The maximal non-increasing rearrangement of f is defined as follows:

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(\tau)\, d\tau, \qquad t \in (0, \mu(\mathcal{R})).$$

The majority of the functions which we shall deal with will be defined on $\mathbb{R}^n$ or $(0,\infty)$. In this case, $(\mathcal{R}, \mu)$ is $\mathbb{R}^n$ or $(0,\infty)$ endowed with either the n-dimensional Lebesgue measure or the one-dimensional Lebesgue measure, respectively. We shall write just $\mathfrak{M}^+$ instead of $\mathfrak{M}^+(0,\infty)$.

Let Ω be any measurable subset of $\mathbb{R}^n$, $n \ge 1$. The family of all weight functions (also called just weights) on Ω, that is, locally integrable non-negative functions on Ω will be denoted by $\mathcal{W}(\Omega)$.

For $p \in (0,\infty]$ and $w \in \mathfrak{M}^+(\Omega)$, we define the functional $\|\cdot\|_{p,w,\Omega}$ on $\mathfrak{M}(\Omega)$ by

$$\|f\|_{p,w,\Omega} := \begin{cases} \left(\int_\Omega |f(x)|^p w(x)\, dx \right)^{1/p} & \text{if} \quad p < \infty, \\ \operatorname{ess\,sup}_\Omega |f(x)| w(x) & \text{if} \quad p = \infty. \end{cases}$$

If, in addition, $w \in \mathcal{W}(\Omega)$, then the weighted Lebesgue space $L^p(w,\Omega)$ is given by

$$L^p(w,\Omega) = \{ f \in \mathfrak{M}(\Omega) : \ \|f\|_{p,w,\Omega} < \infty \}$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,w,\Omega}$.

When $w \equiv 1$ on Ω, we write simply $L^p(\Omega)$ and $\|\cdot\|_{p,\Omega}$ instead of $L^p(w,\Omega)$ and $\|\cdot\|_{p,w,\Omega}$, respectively.

Quite many familiar function spaces can be defined using the non-increasing rearrangement of a function. One of the most important classes of such spaces are the so-called classical Lorentz spaces.

Let $p \in (0,\infty)$ and $w \in \mathcal{W}(0, \mu(\mathcal{R}))$. Then the classical Lorentz spaces $\Lambda^p(w)$ and $\Gamma^p(w)$ consist of all functions $f \in \mathfrak{M}(\mathcal{R})$ for which

$$\|f\|_{\Lambda^p(w)} := \left(\int_0^{\mu(\mathcal{R})} [f^*(s)]^p w(s)\, ds \right)^{\frac{1}{p}} < \infty$$

and

$$\|f\|_{\Gamma^p(w)} := \left(\int_0^{\mu(\mathcal{R})} [f^{**}(s)]^p w(s)\, ds \right)^{\frac{1}{p}} < \infty,$$

respectively. For more information about the Lorentz Λ and Γ see, e.g., [3] and the references therein.

The study of particular problems in the regularity theory of PDE's led to the definition of spaces involving inner integral means which, in turn, involve powers of the non-increasing rearrangements of functions.

The generalized Lorentz $\mathrm{G}\Gamma(p, m, v)(\mathcal{R}, \mu)$ space, simply denoted by $\mathrm{G}\Gamma(p, m, v)$, introduced and studied in [11] and [12], is defined as the collection of all $g \in \mathfrak{M}(\mathcal{R})$ such that

$$\|g\|_{\mathrm{G}\Gamma(p,m,v)} = \left(\int_0^{\mu(\mathcal{R})} \left(\int_0^x [g^*(\tau)]^p\, d\tau \right)^{\frac{m}{p}} v(x)\, dx \right)^{\frac{1}{m}} < \infty,$$

where m, $p \in (0, \infty)$, $v \in \mathcal{W}(0, \mu(\mathcal{R}))$.

The spaces $\mathrm{G}\Gamma(p, m, v)$ cover several types of important function spaces and have plenty of applications. For example, when $\mu(\mathcal{R}) = \infty$, $p = 1$, $m > 1$ and $v(t) = t^{-m} w(t)$, $t \in (0, \infty)$, where w is another weight on $(0, \infty)$, then $\mathrm{G}\Gamma(p, m, v)$ reduces to the spaces $\Gamma^m(w)$. Another important example is obtained when $\mu(\mathcal{R}) = 1$, $m = 1$, $p \in (1, \infty)$ and $v(t) = t^{-1}\big(\log(2/t)\big)^{-1/p}$ for $t \in (0, 1)$. In this case the space $\mathrm{G}\Gamma(p, m, v)$ coincides with the small Lebesgue space, which was originally studied by Fiorenza in [9]. In the same paper it was proved that this space is the associate space of the grand Lebesgue space, which is introduced in [28] in connection with integrability properties of Jacobians. Subsequently, Fiorenza and Karadzhov in [10] derived an equivalent form of the norm in the small Lebesgue space written in the form of the norm in the $\mathrm{G}\Gamma(p, m, v)$ space with the above mentioned parameters and weight. Recently, the relationship between the $\mathrm{G}\Gamma(p, m, v)$ space and some well-known function spaces have been studied in [1]. In the present paper we take $(\mathbb{R}^n, dx)$ as underlying measure space and use the notation $\mathrm{G}\Gamma(p, m, v)$ for $\mathrm{G}\Gamma(p, m, v)(\mathbb{R}^n, dx)$.

The study of maximal operators occupies an important place in harmonic analysis. Behaviors of these important non-linear operators are very informative particularly in differentiation theory, providing the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance, [57, 27, 13, 60, 58, 25, 26]).

The main example is the Hardy-Littlewood maximal function which is defined for locally integrable functions f on $\mathbb{R}^n$ by

$$(Mf)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q containing x. By a cube, we mean an open cube with sides parallel to the coordinate axes.

Another important example is the fractional maximal operator, M_γ, $\gamma \in (0,n)$, defined for locally integrable functions f on $\mathbb{R}^n$ by

$$(M_\gamma f)(x) := \sup_{Q \ni x} |Q|^{\gamma/n-1} \int_Q |f(y)|\, dy, \qquad x \in \mathbb{R}^n.$$

One more example is the fractional maximal operator $M_{s,\gamma,\mathbb{A}}$ defined in [7] for all measurable functions f on $\mathbb{R}^n$ by

$$(M_{s,\gamma,\mathbb{A}} f)(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_s}{\|\chi_Q\|_{sn/(n-\gamma),\mathbb{A}}}, \qquad x \in \mathbb{R}^n.$$

Here $s \in (0,\infty)$, $\gamma \in [0,n)$, $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$ and

$$\ell^{\mathbb{A}}(t) := (1 + |\log t|)^{A_0} \chi_{[0,1]}(t) + (1 + |\log t|)^{A_\infty} \chi_{[1,\infty)}(t), \qquad t \in (0,\infty).$$

Recall that the following equivalency holds:

$$(M_{s,\gamma,\mathbb{A}} f)(x) \approx \sup_{Q \ni x} \frac{\|f\chi_Q\|_s}{|Q|^{(n-\gamma)/(sn)} \ell^{\mathbb{A}}(|Q|)}, \qquad x \in \mathbb{R}^n.$$

Hence, if $s = 1$, $\gamma = 0$ and $\mathbb{A} = (0,0)$, then $M_{s,\gamma,\mathbb{A}}$ is equivalent to M. If $s = 1$, $\gamma \in (0,n)$ and $\mathbb{A} = (0,0)$, then $M_{s,\gamma,\mathbb{A}}$ is equivalent to M_γ. Moreover, if $s = 1$, $\gamma \in [0,n)$ and $\mathbb{A} \in \mathbb{R}^2$, then $M_{s,\gamma,\mathbb{A}}$ is the fractional maximal operator which corresponds to potentials with logarithmic smoothness treated in [43, 44]. In particular, if $\gamma = 0$, then $M_{1,\gamma,\mathbb{A}}$ is the maximal operator of purely logarithmic order.

Given $0 < p$, $q < \infty$, let $M_{p,q}$ denote the maximal operator associated to the Lorentz $L^{p,q}$ spaces defined for all measurable function f on $\mathbb{R}^n$ by

$$M_{p,q} f(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_{p,q}}{\|\chi_Q\|_{p,q}},$$

where $\|\cdot\|_{p,q}$ is the usual Lorentz norm

$$\|f\|_{p,q} := \left(\int_0^\infty \left[\tau^{1/p} f^*(\tau)\right]^q \frac{d\tau}{\tau} \right)^{1/q}.$$

This operator was introduced by Stein in [56] in order to obtain certain endpoint results in differentiation theory. The operator $M_{p,q}$ have been also considered by other authors, see, for instance, [41, 35, 2, 46, 36].

Let $0 < \alpha < \infty$, $b \in \mathcal{W}(0,\infty)$ and $\phi : (0,\infty) \to (0,\infty)$. Recall the definition of the generalized maximal function introduced in [38] and denoted for all measurable function f on $\mathbb{R}^n$ by

$$M_{\phi,\Lambda^\alpha(b)} f(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_{\Lambda^\alpha(b)}}{\phi(|Q|)}, \qquad x \in \mathbb{R}^n. \tag{1}$$

Obviously, $M_{\phi,\Lambda^\alpha(b)} = M$, where M is the Hardy-Littlewood maximal operator, when $\alpha = 1$, $b \equiv 1$ and $\phi(t) = t$ $(t > 0)$. Note that $M_{\phi,\Lambda^\alpha(b)} = M_\gamma$, where M_γ is the fractional maximal operator, when $\alpha = 1$, $b \equiv 1$ and $\phi(t) = t^{1-\gamma/n}$ $(t > 0)$ with $0 < \gamma < n$. Moreover, $M_{\phi,\Lambda^\alpha(b)} \approx M_{s,\gamma,\mathbb{A}}$, when $\alpha = s$, $b \equiv 1$ and $\phi(t) = t^{(n-\gamma)/(sn)} \ell^{\mathbb{A}}(t)$ $(t > 0)$ with $0 < \gamma < n$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$. It is worth also to mention that $M_{\phi,\Lambda^\alpha(b)} = M_{p,q}$, when $\alpha = q$, $b(t) = t^{q/p-1}$ and $\phi(t) = t^{1/p}$ $(t > 0)$.

The boundedness of $M_{\phi,\Lambda^\alpha(b)}$ between classical Lorentz spaces Λ was completely characterized in [38]. The norm of $M_{\phi,\Lambda^\alpha(b)}$ between two GΓ's was calculated in [40] for a wide range of parameters under additional conditions on weight functions. In view of recent increasing interest in generalized Lorentz spaces, in our opinion, it will be interesting to obtain a characterization of the boundedness of this maximal function between generalized and classical Lorentz spaces.

The iterated Hardy-type operator involving suprema $T_{u,b}$ is defined for non-negative measurable function g on the interval $(0,\infty)$ by

$$(T_{u,b} g)(t) := \sup_{\tau \in [t,\infty)} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(y) b(y)\, dy, \qquad t \in (0,\infty),$$

where u and b are weight functions on $(0,\infty)$ such that u is continuous on $(0,\infty)$ and the function $B(t) := \int_0^t b(s)\, ds$ satisfies $0 < B(t) < \infty$ for every $t \in (0,\infty)$. Such operators are essential in the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds (cf. [29]). They are an effective tool for characterization of the associate norm of an operator-induced norm, which appears as an optimal domain norm in a Sobolev embedding (cf. [45, 47]). Supremum operators are also very useful in limiting interpolation theory as can be seen from their appearance, for example in [4, 5, 8, 52].

In the present paper, it is shown under appropriate conditions on parameters and weight functions that the inequality

$$\bigg(\int_0^\infty \big[\big(M_{\phi,\Lambda^\alpha(b)} f \big)^*(x) \big]^q w(x)\, dx \bigg)^{\frac{1}{q}} \le C \bigg(\int_0^\infty \bigg(\int_0^x [f^*(\tau)]^p \, d\tau \bigg)^{\frac{m}{p}} v(x)\, dx \bigg)^{\frac{1}{m}}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\left(\int_0^\infty \left[T_{B/\phi^\alpha,b} h^*(t) \right]^{\frac{q}{\alpha}} w(x)\,dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left(\int_0^x [h^*(\tau)]^{\frac{p}{\alpha}}\,d\tau \right)^{\frac{m}{p}} v(x)\,dx \right)^{\frac{1}{m}}$$

holds for all $h \in \mathfrak{M}(\mathbb{R}^n)$ (see Theorem 15).

The above-mentioned observation motivates the investigation of the following restricted inequality for $T_{u,b}$:

$$\left(\int_0^\infty \left[T_{u,b} f^*(x) \right]^q w(x)\,dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left(\int_0^x [f^*(\tau)]^p\,d\tau \right)^{\frac{m}{p}} v(x)\,dx \right)^{\frac{1}{m}}. \quad (2)$$

Here m, p, q are positive real numbers and w, v are weight functions on $(0,\infty)$.

The method used for solution of inequalty (2) is based on the combination of the duality techniques with the formula

$$\sup_{g:\, \int_0^x g \le \int_0^x f} \int_0^\infty g(x) w(x)\,dx = \int_0^\infty f(x) \left(\sup_{t \in [x,\infty)} w(t) \right) dx$$

from [54], which holds for f, $w \in \mathfrak{M}^+(0,\infty)$. On the other hand, it uses estimates of optimal constants in weighted Hardy-type inequalites, as well as in weighted inequalities for a superposition of the supremal or Copson operator with the Hardy operator or the Copson operator. Detailed information on materials that are used in the proofs of the main results is given in the following section.

However, we are not able to solve the inequality under the restrictions $1 < p \le q < m < \infty$ or $1 < q < \min\{p,m\} < \infty$, since for these values of parameters the conditions that characterize the weighted iterated Hardy-type inequalities contain more complicated expressions and the approach used in our paper needs an improvement.

Throughout the paper, we denote by C a positive constant, which is independent of main parameters but it may vary from line to line. However, a constant with subscript such as C_1 does not change in different occurrences. By $a \lesssim b$, we mean that $a \le \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent.

As usual, we put $0 \cdot \infty = 0$, $\infty/\infty = 0$ and $0/0 = 0$. If $p \in [1,+\infty]$, we define p' by $1/p + 1/p' = 1$.

The paper is organized as follows. We start with formulations of background material in Section 2. In Section 3 we present solution of the restricted inequality. Finally, in Section 4, we calculate the norm of generalized maximal operator from $G\Gamma$ spaces into Λ spaces.

2. Background material

In this section we collect background material that will be used in the proofs of the main theorems.

We begin with the following characterization of the norm of the associate space of $\mathrm{G}\Gamma(p,m,v)$ given in [24]. Recall that the associate space $\mathrm{G}\Gamma(p,m,v)'$ of $\mathrm{G}\Gamma(p,m,v)$ is defined as the collection of all functions $g \in \mathfrak{M}(\mathbb{R}^n)$ such that

$$\|g\|_{\mathrm{G}\Gamma(p,m,v)'} = \sup_{\|f\|_{\mathrm{G}\Gamma(p,m,v)} \le 1} \int_0^\infty f^*(t) g^*(t)\, dt < \infty.$$

As it is mentioned in [24], it is reasonable to adopt a general assumption that p, m and v are such that

$$\int_0^t v(s) s^{\frac{m}{p}}\, ds + \int_t^\infty v(s)\, ds < \infty, \qquad t \in (0,\infty), \tag{3}$$

because if this requirement is not satisfied, then $\mathrm{G}\Gamma(p,m,v) = \{0\}$.

Under the assumption (3), we denote

$$v_0(t) := t^{\frac{m}{p}-1} \int_0^t v(s) s^{\frac{m}{p}}\, ds \int_t^\infty v(s)\, ds, \qquad t \in (0,\infty), \tag{4}$$

and

$$v_1(t) := \int_0^t v(s) s^{\frac{m}{p}}\, ds + t^{\frac{m}{p}} \int_t^\infty v(s)\, ds, \qquad t \in (0,\infty). \tag{5}$$

Moreover, we assume that a weight v is non-degenerate (with respect to the power function $t^{m/p}$), that is,

$$\int_0^1 v(s)\, ds = \int_1^\infty v(s) s^{\frac{m}{p}}\, ds = \infty. \tag{6}$$

We denote the set of all weight functions satisfying conditions (3) and (6) by $\mathcal{W}_{m,p}(0,\infty)$.

Theorem 1. *[24, Theorem 1.1] Assume that* $0 < m$, $p < \infty$ *and* $v \in \mathcal{W}_{m,p}(0,\infty)$.

(i) *Let* $0 < m \le 1$ *and* $0 < p \le 1$. *Then*

$$\|g\|_{\mathrm{G}\Gamma(p,m,v)'} \approx \sup_{t \in (0,\infty)} g^{**}(t) \frac{t}{v_1(t)^{\frac{1}{m}}};$$

(ii) *Let* $0 < m \le 1$ *and* $1 < p < \infty$. *Then*

$$\|g\|_{\mathrm{G}\Gamma(p,m,v)'} \approx \sup_{t \in (0,\infty)} \left(\int_t^\infty g^{**}(s)^{p'}\, ds \right)^{\frac{1}{p'}} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}};$$

(iii) *Let* $1 < m < \infty$ *and* $0 < p \le 1$. *Then*

$$\|g\|_{\mathrm{G}\Gamma(p,m,v)'} \approx \left(\int_0^\infty g^{**}(t)^{m'} \frac{t^{m'} v_0(t)}{v_1(t)^{m'+1}}\, dt \right)^{\frac{1}{m'}};$$

(iv) *Let* $1 < m < \infty$ *and* $1 < p < \infty$. *Then*

$$\|g\|_{\mathrm{G}\Gamma(p,m,v)'} \approx \left(\int_0^\infty \left(\int_t^\infty g^{**}(s)^{p'}\, ds \right)^{\frac{m'}{p'}} \frac{t^{\frac{m'}{p'}} v_0(t)}{v_1(t)^{m'+1}}\, dt \right)^{\frac{1}{m'}}.$$

We recall the following well-known duality principle in weighted Lebesgue spaces.

Theorem 2. *Let* $p > 1$, $f \in \mathfrak{M}^+$ *and* $w \in \mathcal{W}(0,\infty)$. *Then*

$$\left(\int_0^\infty f(t)^p w(t)\, dt \right)^{\frac{1}{p}} = \sup_{h \in \mathfrak{M}^+} \frac{\int_0^\infty f(t)h(t)\, dt}{\left(\int_0^\infty h(t)^{p'} w(t)^{1-p'}\, dt \right)^{\frac{1}{p'}}}.$$

We will use the following statement.

Theorem 3. *[54, Theorem 2.1] Suppose* f, $w \in \mathfrak{M}^+$. *Then*

$$\sup_{g:\, \int_0^x g \le \int_0^x f} \int_0^\infty g(x) w(x)\, dx = \int_0^\infty f(x) \left(\sup_{t \in [x,\infty)} w(t) \right) dx.$$

Let us now recall (now classical) well-known characterizations of weights for which Hardy and Copson inequalities hold. The following two theorems, which are, incidentally, exactly one hundred years old, are absolutely indispensable in various parts of mathematics (cf. [42, 33, 32, 34]).

Theorem 4. *Let* $1 < p$, $q < \infty$ *and* v, $w \in \mathcal{W}(0,\infty)$. *Denote the best constant in the inequality*

$$\left(\int_0^\infty \left(\int_0^x f(s)\, ds \right)^q w(x)\, dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty f(x)^p v(x)\, dx \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^+,$$

by

$$C_1 := \sup_{f \in \mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_0^x f(s)\, ds \right)^q w(x)\, dx \right)^{\frac{1}{q}}}{\left(\int_0^\infty f(x)^p v(x)\, dx \right)^{\frac{1}{p}}}.$$

(a) *Let* $p \le q$. *Then*

$$C_1 \approx \sup_{x\in(0,\infty)} \left(\int_x^\infty w(s)\,ds\right)^{\frac{1}{q}} \left(\int_0^x v(s)^{1-p'}\,ds\right)^{\frac{1}{p}}.$$

(b) *Let* $q < p$. *Then*

$$C_1 \approx \left(\int_0^\infty \left(\int_x^\infty w(s)\,ds\right)^{\frac{q}{p-q}} w(x) \left(\int_0^x v(s)^{1-p'}\,ds\right)^{\frac{q(p-1)}{p-q}} dx\right)^{\frac{p-q}{pq}}.$$

Theorem 5. *Let* $1 < p,\, q < \infty$ *and* $v,\, w \in \mathcal{W}(0,\infty)$. *Denote the best constant in the inequality*

$$\left(\int_0^\infty \left(\int_x^\infty f(s)\,ds\right)^q w(x)\,dx\right)^{\frac{1}{q}} \le C \left(\int_0^\infty f(x)^p v(x)\,dx\right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^+,$$

by

$$C_2 := \sup_{f\in\mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_x^\infty f(s)\,ds\right)^q w(x)\,dx\right)^{\frac{1}{q}}}{\left(\int_0^\infty f(x)^p v(x)\,dx\right)^{\frac{1}{p}}}.$$

(a) *Let* $p \le q$. *Then*

$$C_2 \approx \sup_{x\in(0,\infty)} \left(\int_0^x w(s)\,ds\right)^{\frac{1}{q}} \left(\int_x^\infty v(s)^{1-p'}\,ds\right)^{\frac{1}{p}}.$$

(b) *Let* $q < p$. *Then*

$$C_2 \approx \left(\int_0^\infty \left(\int_0^x w(s)\,ds\right)^{\frac{q}{p-q}} w(x) \left(\int_x^\infty v(s)^{1-p'}\,ds\right)^{\frac{q(p-1)}{p-q}} dx\right)^{\frac{p-q}{pq}}.$$

We next quote results concerning characterizations of inequalities involving supremum operators in the following two statements.

Theorem 6. *Let* $1 < p < \infty$. *Given* $t \in [0,\infty)$, *assume that* $u \in \mathcal{W}(t,\infty) \cap C(t,\infty)$ *and* $a, v, w \in \mathcal{W}(t,\infty)$ *such that* $0 < \int_t^x v < \infty$ *and* $0 < \int_t^x w < \infty$ *for* $x \in (t,\infty)$. *Then the inequality*

$$\int_t^\infty \left[\sup_{y\in[x,\infty)} u(y) \int_t^y g(s)\,ds\right] w(x)\,dx \le C \left(\int_t^\infty g(x)^p v(x)\,dx\right)^{\frac{1}{p}} \qquad (7)$$

holds for all $g \in \mathfrak{M}^+[t,\infty)$ *if and only if* $D_1 + D_2 < \infty$, *where*

$$D_1 := \left(\int_t^\infty \left[\sup_{\tau \in [x,\infty)} \left[\sup_{y \in [\tau,\infty)} u(y)^{p'} \right] \left(\int_t^\tau v^{1-p'} \right) \right] \left(\int_t^x w \right)^{p'-1} w(x)\,dx \right)^{\frac{1}{p'}}$$

and

$$D_2 := \left(\int_t^\infty \left(\int_x^\infty \left[\sup_{\tau \in [y,\infty)} u(\tau) \right] w(y)\,dy \right)^{p'-1} \left[\sup_{\tau \in [x,\infty)} u(\tau) \right] \left(\int_t^x v^{1-p'} \right) w(x)\,dx \right)^{\frac{1}{p'}}.$$

Moreover, the least constant C *such that* (7) *holds for all* $g \in \mathfrak{M}^+$ *satisfies* $C \approx D_1 + D_2$.

Proof. The statement was formulated in [22, Theorem 4.4] for $t = 0$. The proof directly follows by using change of variables of the type $x + t = y$ several times when $t > 0$. ◀

Theorem 7. *Let* $1 < p < \infty$. *Given* $t \in [0,\infty)$, *assume that* $u \in \mathcal{W}(t,\infty) \cap C(t,\infty)$ *and* $a, v, w \in \mathcal{W}(t,\infty)$ *such that* $0 < \int_t^x v < \infty$ *and* $0 < \int_t^x w < \infty$ *for* $x \in (t,\infty)$. *Then the inequality*

$$\int_t^\infty \left[\sup_{y \in [x,\infty)} u(y) \int_y^\infty g(s)\,ds \right] w(x)\,dx \le C \left(\int_t^\infty g(x)^p v(x)\,dx \right)^{\frac{1}{p}} \tag{8}$$

holds for all $g \in \mathfrak{M}^+[t,\infty)$ *if and only if* $E_1 + E_2 < \infty$, *where*

$$E_1 := \left(\int_t^\infty \left[\sup_{\tau \in [x,\infty)} u(\tau)^{p'} \left(\int_\tau^\infty v^{1-p'} \right) \right] \left(\int_t^x w \right)^{p'-1} w(x)\,dx \right)^{\frac{1}{p'}}$$

and

$$E_2 := \left(\int_t^\infty \left(\int_t^x \left[\sup_{y \in [s,x]} u(y) \right] w(s)\,ds \right)^{p'-1} \left[\sup_{\tau \in [x,\infty)} u(\tau) \left(\int_\tau^\infty v^{1-p'} \right) \right] w(x)\,dx \right)^{\frac{1}{p'}}.$$

Moreover, the least constant C *such that* (8) *holds for all* $g \in \mathfrak{M}^+$ *satisfies* $C \approx E_1 + E_2$.

Proof. The statement was formulated in [30, Theorem 6] for $t = 0$. The proof directly follows by using change of variables of the type $x + t = y$ several times when $t > 0$. ◀

Investigation of weighted iterated Hardy-type inequalities started with the study of the inequality

$$\bigg(\int_0^\infty \bigg(\int_0^t \bigg(\int_s^\infty h(y)\,dy\bigg)^m u(s)\,ds\bigg)^{\frac{q}{m}} w(t)\,dt\bigg)^{\frac{1}{q}} \le C\bigg(\int_0^\infty h(t)^p v(t)\,dt\bigg)^{\frac{1}{p}}, \qquad h \in \mathfrak{M}^+. \tag{9}$$

Note that the inequality (9) has been considered in the case $m = 1$ in [6] (see also [14]), where the result was presented without proof, and in the case $p = 1$ in [15] and [55], where the special type of weight function v was considered. Recall that the inequality has been completely characterized in [17] and [18] in the case $0 < m < \infty$, $0 < q \le \infty$, $1 \le p < \infty$ by using discretization and anti-discretization methods. Another approach to get the characterization of inequality (9) was presented in [48]. The characterization of the inequality can be reduced to the characterization of the weighted Hardy inequality on the cones of non-increasing functions (see, [19] and [20]). Different approach to solve iterated Hardy-type inequalities has been given in [37].

As it was mentioned in [19], the characterization of "dual" inequality

$$\bigg(\int_0^\infty \bigg(\int_t^\infty \bigg(\int_0^s h(y)\,dy\bigg)^m u(s)\,ds\bigg)^{\frac{q}{m}} w(t)\,dt\bigg)^{\frac{1}{q}} \le C\bigg(\int_0^\infty h(t)^p v(t)\,dt\bigg)^{\frac{1}{p}}, \qquad h \in \mathfrak{M}^+.$$

can be easily obtained from the solutions of inequality (9), which was presented in [16].

Theorem 8. *[16, Theorem 2.9, (a) and (c)] Let* $p, q, m \in (1,\infty)$ *and* $u, w, v \in \mathcal{W}(0,\infty)$. *Assume that the following non-degeneracy conditions are satisfied:*

- *u is strictly positive,* $\int_t^\infty u(s)\,ds < \infty$ *for all* $t \in (0,\infty)$, $\int_0^\infty u(s)\,ds = \infty$,
- $\int_0^t w(s)\,ds < \infty$, $\int_t^\infty w(s)\Big(\int_s^\infty u(y)\,dy\Big)^{\frac{q}{m}} ds < \infty$ *for all* $t \in (0,\infty)$,
- $\int_0^1 w(s)\Big(\int_s^\infty u(y)\,dy\Big)^{\frac{q}{m}} ds = \infty$, $\int_1^\infty w(s)\,ds = \infty$.

Let

$$C = \sup_{h \in \mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s h(y)\,dy \right)^m u(s)\,ds \right)^{\frac{q}{m}} w(t)\,dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty h(t)^p v(t)\,dt \right)^{\frac{1}{p}}}.$$

(a) *If* $p \le \min\{m,\, q\}$, *then* $C \approx F_1 + F_2$, *where*

$$F_1 = \sup_{t \in (0,\infty)} \left(\int_0^t w \right)^{\frac{1}{q}} \left(\int_t^\infty u \right)^{\frac{1}{m}} \left(\int_0^t v^{1-p'} \right)^{\frac{1}{p'}}$$

and

$$F_2 = \sup_{t \in (0,\infty)} \left(\int_t^\infty \left(\int_s^\infty u \right)^{\frac{q}{m}} w(s)\,ds \right)^{\frac{1}{q}} \left(\int_0^t v^{1-p'} \right)^{\frac{1}{p'}}.$$

(b) *If* $m < p \le q$, *then* $C \approx F_2 + F_3$, *where*

$$F_3 = \sup_{t \in (0,\infty)} \left(\int_0^t w \right)^{\frac{1}{q}} \left(\int_t^\infty \left(\int_s^\infty u \right)^{\frac{p}{p-m}} \left(\int_0^s v^{1-p'} \right)^{\frac{p(m-1)}{p-m}} v(s)^{1-p'}\,ds \right)^{\frac{p-m}{pm}}.$$

Another pair of "dual" weighted iterated Hardy-type inequalities are

$$\left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty h(y)\,dy \right)^m u(s)\,ds \right)^{\frac{q}{m}} w(t)\,dt \right)^{\frac{1}{q}} \le C \left(\int_0^\infty h(t)^p v(t)\,dt \right)^{\frac{1}{p}}, \qquad h \in \mathfrak{M}^+ \tag{10}$$

and

$$\left(\int_0^\infty \left(\int_0^t \left(\int_0^s h(y)\,dy \right)^m u(s)\,ds \right)^{\frac{q}{m}} w(t)\,dt \right)^{\frac{1}{q}} \le C \left(\int_0^\infty h(t)^p v(t)\,dt \right)^{\frac{1}{p}}, \qquad h \in \mathfrak{M}^+.$$

Both of them were characterized in [19] by so-called "flipped" conditions. The "classical" conditions ensuring the validity of (10) was recently presented in [31].

Theorem 9. *[31, Theorem 1.1, (a) and (c)] Let* $p,\, q,\, m \in (1,\infty)$ *and* $u,\, w,\, v$ *be weights such that the pair* (u,w) *is admissible with respect to* $(m.q)$, *that is,*

$$0 < \int_0^t \left(\int_s^t u \right)^{\frac{q}{m}} w(s)\,ds < \infty, \qquad t \in (0,\infty).$$

Let

$$C = \sup_{h \in \mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty h(y)\,dy \right)^m u(s)\,ds \right)^{\frac{q}{m}} w(t)\,dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty h(t)^p v(t)\,dt \right)^{\frac{1}{p}}}.$$

(a) *If* $p \le \min\{m, q\}$, *then* $C \approx G_1$, *where*

$$G_1 = \sup_{t \in (0,\infty)} \left(\int_0^t w(s) \left(\int_s^t u \right)^{\frac{q}{m}} ds \right)^{\frac{1}{q}} \left(\int_t^\infty v^{1-p'} \right)^{\frac{1}{p'}}.$$

(b) *If* $m < p \le q$, *then* $C \approx G_1 + G_2$, *where*

$$G_2 = \sup_{t \in (0,\infty)} \left(\int_0^t w \right)^{\frac{1}{q}} \left(\int_t^\infty \left(\int_t^s u \right)^{\frac{m}{p-m}} u(s) \left(\int_s^\infty v^{1-p'} \right)^{\frac{m(p-1)}{p-m}} ds \right)^{\frac{p-m}{pm}}.$$

We will apply the following "gluing" lemma.

Lemma 1. *[21, Lemma 2.7] Let* α *and* β *be positive numbers. Suppose that* $g, h \in \mathfrak{M}^+$ *and* $a \in \mathcal{W}(0,\infty)$ *is non-decreasing. Then*

$$\operatorname*{ess\,sup}_{x \in (0,\infty)} \left(\int_0^\infty \left(\frac{a(x)}{a(x)+a(t)} \right)^\beta g(t)\,dt \right)^{\frac{1}{\beta}} \left(\int_0^\infty \left(\frac{a(t)}{a(x)+a(t)} \right)^\alpha h(t)\,dt \right)^{\frac{1}{\alpha}}$$
$$\approx \operatorname*{ess\,sup}_{x \in (0,\infty)} \left(\int_0^x g \right)^{\frac{1}{\beta}} \left(\int_x^\infty h \right)^{\frac{1}{\alpha}} + \operatorname*{ess\,sup}_{x \in (0,\infty)} \left(\int_x^\infty a^{-\beta} g \right)^{\frac{1}{\beta}} \left(\int_0^x a^\alpha h \right)^{\frac{1}{\alpha}}.$$

We recall the following "an integration by parts" formula.

Theorem 10. *[39, Theorem 2.1] Let* $\alpha > 0$. *Let* g *be a non-negative function on* $(0,\infty)$ *such that* $0 < \int_0^t g < \infty$ *for all* $t \in (0,\infty)$ *and let* f *be a non-negative non-increasing right-continuous function on* $(0,\infty)$. *Then*

$$A_1 := \int_0^\infty \left(\int_0^t g \right)^\alpha g(t)[f(t) - \lim_{t \to +\infty} f(t)]\,dt < \infty$$

holds if and only if

$$A_2 := \int_{(0,\infty)} \left(\int_0^t g \right)^{\alpha+1} d[-f(t)] < \infty$$

holds. Moreover, $A_1 \approx A_2$.

We are going to make use of the following remark in order to shorten some calculations in the proofs.

Remark 1. *Let $w \in \mathcal{W}(0,\infty)$ and F be any non-negative continuous function on $(0,\infty)$.*

Since

$$\sup_{x\le\tau} F(\tau)\chi_{(0,t]}(\tau) = \begin{cases} \sup_{x\le\tau\le t} F(\tau) & \text{for} \quad x \le t, \\ 0 & \text{for} \quad t < x, \end{cases}$$

for any $0 < x,\, t < \infty$, the relation

$$\int_0^\infty \left(\sup_{x\le\tau} F(\tau)\chi_{(0,t]}(\tau)\right) w(x)\,dx \approx \int_0^t \left(\sup_{x\le\tau\le t} F(\tau)\right) w(x)\,dx$$

holds for $0 < t < \infty$.

Similarly, since

$$\sup_{x\le\tau} F(\tau)\chi_{[t,\infty)}(\tau) = \begin{cases} \sup_{t\le\tau} F(\tau) & \text{for} \quad x \le t, \\ \sup_{x\le\tau} F(\tau) & \text{for} \quad t < x, \end{cases}$$

for any $0 < x,\, t < \infty$, the relation

$$\int_0^\infty \left(\sup_{x\le\tau} F(\tau)\chi_{[t,\infty)}(\tau)\right) w(x)\,dx \approx \left(\sup_{t\le\tau} F(\tau)\right) \int_0^t w(x)\,dx + \int_t^\infty \left(\sup_{x\le\tau} F(\tau)\right) w(x)\,dx$$

holds for $0 < t < \infty$.

3. Characterizations of restricted inequalities for $T_{u,b}$

We start this section with some historical remarks concerning restricted inequalities related to the operator $T_{u,b}$.

The notation $\mathfrak{M}^+((0,\infty);\downarrow)$ is used to denote the subset of those functions from $\mathfrak{M}^+(0,\infty)$ which are non-increasing on $(0,\infty)$.

Recall that the inequality

$$\|T_{u,b}f\|_{q,w,(0,\infty)} \le C\|f\|_{p,v,(0,\infty)}, \qquad f \in \mathfrak{M}^+((0,\infty);\downarrow) \tag{11}$$

was characterized in [22, Theorem 3.5] under condition

$$\sup_{t\in(0,\infty)} \frac{u(t)}{B(t)} \int_0^t \frac{b(\tau)}{u(\tau)}\,d\tau < \infty.$$

However, the case where $0 < p \le 1 < q < \infty$ was not considered in [22]. It is also worth to mention that in the case where $1 < p < \infty$, $0 < q < p < \infty$, $q \neq 1$ [22, Theorem 3.5] contains only discrete condition. In [23], the new reduction theorem was obtained when $0 < p \le 1$, and this technique allowed to characterize inequality (11) when $b \equiv 1$, and in the case where $0 < q < p \le 1$, [23] contains only discrete condition. The complete characterizations of inequality (11) for $0 < q \le \infty$, $0 < p \le \infty$ were given in [20] and [38]. Using the results in [48, 49, 50, 51], another characterization of (11) was obtained in [59] and [53].

Note that the inequality

$$\bigg(\int_0^\infty \bigg(\int_0^x T_{u,b} f(t)\,dt\bigg)^q w(x)\,dx\bigg)^{\frac{1}{q}} \le C\bigg(\int_0^\infty f(x)^p v(x)\,dx\bigg)^{\frac{1}{p}}, \qquad f \in \mathfrak{M}^+((0,\infty);\downarrow),$$

was characterized in [39, Theorem 6.1] for $1 < p$, $q < \infty$, where w and v are weight functions on $(0,\infty)$.

Recall that the inequality

$$\bigg(\int_0^\infty \bigg(\int_0^x \big[T_{u,b} f^*(t)\big]^r\,dt\bigg)^{\frac{q}{r}} w(x)\,dx\bigg)^{\frac{1}{q}} \le C\bigg(\int_0^\infty \bigg(\int_0^x [f^*(\tau)]^p\,d\tau\bigg)^{\frac{m}{p}} v(x)\,dx\bigg)^{\frac{1}{m}}, \qquad f \in \mathfrak{M}(\mathbb{R}^n), \tag{12}$$

was investigated in [40, Theorems 3.3 and 3.4] for $1 < m < p \le r < q < \infty$ or $1 < m \le r < \min\{p,q\} < \infty$, where w and v are appropriate weight functions on $(0,\infty)$.

In this section we give the characterization of the inequality

$$\bigg(\int_0^\infty \big[T_{u,b} f^*(x)\big]^q\,w(x)\,dx\bigg)^{\frac{1}{q}} \le C\bigg(\int_0^\infty \bigg(\int_0^x [f^*(\tau)]^p\,d\tau\bigg)^{\frac{m}{p}} v(x)\,dx\bigg)^{\frac{1}{m}}, \qquad f \in \mathfrak{M}(\mathbb{R}^n). \tag{13}$$

As it was mentioned in [40], inequality (13) is a special case of inequality (12) for $r = q$.

Denote the best constant in inequality (13) by K, that is,

$$K := \sup_{f \in \mathfrak{M}(\mathbb{R}^n)} \frac{\Big(\int_0^\infty \big[T_{u,b} f^*(x)\big]^q w(x)\,dx\Big)^{\frac{1}{q}}}{\|f\|_{\mathrm{G}\Gamma(p,m,v)}}.$$

Lemma 2. *Let* $0 < p < \infty$, $0 < m < \infty$, $1 < q < \infty$ *and* $b \in \mathcal{W}(0,\infty)$ *be such that the function* $B(t)$ *satisfies* $0 < B(t) < \infty$ *for every* $t \in (0,\infty)$. *Assume that* $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$ *and* v, $w \in \mathcal{W}(0,\infty)$. *Then we have*

$$K = \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{\varphi \in \mathfrak{M}^+} \frac{1}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \sup_{f \in \mathfrak{M}(\mathbb{R}^n)} \frac{\int_0^\infty f^*(y) b(y) \left(\int_y^\infty \varphi \frac{u}{B} \right) dy}{\|f\|_{\mathrm{G\Gamma}(p,m,v)}}.$$

Proof. Applying Theorem 3, we have

$$\begin{aligned} K &= \sup_{f \in \mathfrak{M}(\mathbb{R}^n)} \frac{\left(\int_0^\infty \sup_{\tau \in [x,\infty)} \left[\frac{u(\tau)}{B(\tau)} \int_0^\tau f^* b \right]^q w(x)\, dx \right)^{\frac{1}{q}}}{\|f\|_{\mathrm{G\Gamma}(p,m,v)}} \\ &= \sup_{f \in \mathfrak{M}(\mathbb{R}^n)} \frac{1}{\|f\|_{\mathrm{G\Gamma}(p,m,v)}} \sup_{h:\,\int_0^x h \le \int_0^x w} \left(\int_0^\infty h(x) \left(\frac{u(x)}{B(x)} \int_0^x f^* b \right)^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

By duality and Fubini theorem, we get

$$\begin{aligned} K &= \sup_{f \in \mathfrak{M}(\mathbb{R}^n)} \frac{1}{\|f\|_{\mathrm{G\Gamma}(p,m,v)}} \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{\varphi \in \mathfrak{M}^+} \frac{\int_0^\infty \varphi(x) \frac{u(x)}{B(x)} \left(\int_0^x f^* b \right) dx}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \\ &= \sup_{f \in \mathfrak{M}(\mathbb{R}^n)} \frac{1}{\|f\|_{\mathrm{G\Gamma}(p,m,v)}} \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{\varphi \in \mathfrak{M}^+} \frac{\int_0^\infty f^*(y) b(y) \left(\int_y^\infty \varphi \frac{u}{B} \right) dy}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}}. \end{aligned}$$

Interchanging the suprema yields

$$K = \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{\varphi \in \mathfrak{M}^+} \frac{1}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \sup_{f \in \mathfrak{M}(\mathbb{R}^n)} \frac{\int_0^\infty f^*(y) b(y) \left(\int_y^\infty \varphi \frac{u}{B} \right) dy}{\|f\|_{\mathrm{G\Gamma}(p,m,v)}}.$$

This completes the proof. ◀

Theorem 11. *Let* $0 < m \le 1, 0 < p \le 1, 1 < q < \infty$ *and* $b \in \mathcal{W}(0,\infty) \cap \mathfrak{M}^+((0,\infty);\downarrow)$ *be such that the function* $B(t)$ *satisfies* $0 < B(t) < \infty$ *for every* $t \in (0,\infty)$. *Suppose that* $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$, $v \in \mathcal{W}_{m,p}(0,\infty)$ *and* $w \in \mathcal{W}(0,\infty)$. *Then*

$$K \approx \sup_{t \in (0,\infty)} \frac{1}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t \left(\sup_{\tau \in [x,t]} u(\tau)^q \right) w(x)\, dx \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{B(t)}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)} \right) \left(\int_0^t w \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{B(t)}{v_1(t)^{\frac{1}{m}}} \left(\int_t^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)} \right)^q \right) w(x)\, dx \right)^{\frac{1}{q}}.$$

Proof. By Lemma 2, Theorem 1, (i), and Fubini theorem, we have

$$K \approx \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{\varphi\in\mathfrak{M}^+} \frac{1}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \sup_{t\in(0,\infty)} \frac{\int_0^t b(y) \left(\int_y^\infty \varphi \frac{u}{B} \right) dy}{v_1(t)^{\frac{1}{m}}}$$

$$\approx \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{\varphi\in\mathfrak{M}^+} \frac{1}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \sup_{t\in(0,\infty)} \frac{\int_0^t \varphi u}{v_1(t)^{\frac{1}{m}}}$$

$$+ \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{\varphi\in\mathfrak{M}^+} \frac{1}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \sup_{t\in(0,\infty)} \frac{B(t) \int_t^\infty \varphi \frac{u}{B}}{v_1(t)^{\frac{1}{m}}}.$$

Interchanging suprema, by duality, we get

$$K \approx \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{1}{v_1(t)^{\frac{1}{m}}} \sup_{\varphi\in\mathfrak{M}^+} \frac{\int_0^\infty \varphi(x) u(x) \chi_{(0,t]}(x)\, dx}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}}$$

$$+ \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{B(t)}{v_1(t)^{\frac{1}{m}}} \sup_{\varphi\in\mathfrak{M}^+} \frac{\int_0^\infty \varphi(x) \frac{u(x)}{B(x)} \chi_{[t,\infty)}(x)\, dx}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}}$$

$$\approx \sup_{t\in(0,\infty)} \frac{1}{v_1(t)^{\frac{1}{m}}} \sup_{h:\, \int_0^x h \le \int_0^x w} \left(\int_0^\infty h(x) u(x)^q \chi_{(0,t]}(x)\, dx \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{B(t)}{v_1(t)^{\frac{1}{m}}} \sup_{h:\, \int_0^x h \le \int_0^x w} \left(\int_0^\infty h(x) \left(\frac{u(x)}{B(x)} \right)^q \chi_{[t,\infty)}(x)\, dx \right)^{\frac{1}{q}}.$$

Applying Theorem 3, by Remark 1, we arrive at

$$K \approx \sup_{t\in(0,\infty)} \frac{1}{v_1(t)^{\frac{1}{m}}} \left(\int_0^\infty \left(\sup_{\tau\in[x,\infty)} u(\tau)^q \chi_{(0,t]}(\tau) \right) w(x)\, dx \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{B(t)}{v_1(t)^{\frac{1}{m}}} \left(\int_0^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)} \right)^q \chi_{[t,\infty)}(\tau) \right) w(x)\, dx \right)^{\frac{1}{q}}$$

$$\approx \sup_{t\in(0,\infty)} \frac{1}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t \left(\sup_{\tau\in[x,t]} u(\tau)^q \right) w(x)\, dx \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{B(t)}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t w\right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)}$$

$$+ \sup_{t\in(0,\infty)} \frac{B(t)}{v_1(t)^{\frac{1}{m}}} \left(\int_t^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)}\right)^q\right) w(x)\,dx\right)^{\frac{1}{q}}.$$

The proof is completed. ◀

Theorem 12. *Let* $0 < m \le 1, 1 < p < \infty, 1 < q < \infty$ *and* $b \in \mathcal{W}(0,\infty) \cap \mathfrak{M}^+((0,\infty);\downarrow)$ *be such that the function* $B(t)$ *satisfies* $0 < B(t) < \infty$ *for every* $t \in (0,\infty)$. *Suppose that* $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$, $v \in \mathcal{W}_{m,p}(0,\infty)$ *and* $w \in \mathcal{W}(0,\infty)$.

i) If $p \le q$, *then*

$$K \approx \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} s^{-\frac{1}{p}} \left(\int_0^s \left(\sup_{\tau\in[x,s]} u(\tau)^q\right) w(x)\,dx\right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} \left(\int_t^s \left(\frac{B(\tau)}{\tau}\right)^{p'} d\tau\right)^{\frac{1}{p'}} \left(\int_0^s w\right)^{\frac{1}{q}} \sup_{y\in[s,\infty)} \frac{u(y)}{B(y)}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} \left(\int_t^s \left(\frac{B(\tau)}{\tau}\right)^{p'} d\tau\right)^{\frac{1}{p'}}$$

$$\times \left(\int_s^\infty \left(\sup_{y\in[x,\infty)} \left(\frac{u(y)}{B(y)}\right)^q\right) w(x)\,dx\right)^{\frac{1}{q}};$$

ii) If $q < p$, *then*

$$K \approx \sup_{t\in(0,\infty)} \frac{1}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t \left(\sup_{\tau\in[x,t]} u(\tau)^q\right) w(x)\,dx\right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t w\right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} u(\tau)\tau^{-\frac{1}{p}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\int_t^\infty \left(\sup_{\tau\in[s,\infty)} u(\tau)^{\frac{pq}{p-q}} \tau^{\frac{q}{q-p}}\right) \left(\int_t^s w\right)^{\frac{q}{p-q}} w(s)\,ds\right)^{\frac{p-q}{pq}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\int_t^\infty \left(\int_t^s \left(\sup_{\tau\in[x,s]} u(\tau)^q\right) w(x)\,dx\right)^{\frac{q}{p-q}}\right.$$

$$\left.\times \left(\sup_{y\in[s,\infty)} u(y)^q y^{\frac{q}{q-p}}\right) w(s)\,ds\right)^{\frac{p-q}{pq}}$$

$$
\begin{aligned}
&+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_0^t w\bigg)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)} \bigg(\int_t^\tau \mathcal{B}(t,s)\,ds\bigg)^{\frac{p-q}{pq}} \\
&+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \Bigg(\int_t^\infty \Bigg(\sup_{\tau\in[x,\infty)} \Bigg(\sup_{y\in[\tau,\infty)} \bigg(\frac{u(y)}{B(y)}\bigg)^{\frac{pq}{p-q}}\Bigg) \bigg(\int_t^\tau \mathcal{B}(t,s)\,ds\bigg)\Bigg) \\
&\qquad\qquad \times \bigg(\int_t^x w\bigg)^{\frac{q}{p-q}} w(x)\,dx\Bigg)^{\frac{p-q}{pq}} \\
&+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \Bigg(\int_t^\infty \Bigg(\int_x^\infty \Bigg(\sup_{\tau\in[y,\infty)} \bigg(\frac{u(\tau)}{B(\tau)}\bigg)^q\Bigg) w(y)\,dy\Bigg)^{\frac{q}{p-q}} \\
&\qquad\qquad \times \Bigg(\sup_{y\in[x,\infty)} \bigg(\frac{u(y)}{B(y)}\bigg)^q\Bigg) \bigg(\int_t^x \mathcal{B}(t,s)\,ds\bigg) w(x)\,dx\Bigg)^{\frac{p-q}{pq}},
\end{aligned}
$$

where

$$
\mathcal{B}(t,s) := \bigg(\int_t^s \bigg(\frac{B(y)}{y}\bigg)^{p'} dy\bigg)^{\frac{p(q-1)}{p-q}} \bigg(\frac{B(s)}{s}\bigg)^{p'}, \qquad 0<t<s<\infty.
$$

Proof. By Lemma 2 and Theorem 1, (ii), we have

$$
\begin{aligned}
K &\approx \sup_{h:\,\int_0^x h\le\int_0^x w}\ \sup_{\varphi\in\mathfrak{M}^+} \frac{1}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \\
&\qquad \times \sup_{t\in(0,\infty)} \Bigg(\int_t^\infty \bigg(\frac{1}{s}\int_0^s b(y)\bigg(\int_y^\infty \varphi\frac{u}{B}\bigg)dy\bigg)^{p'} ds\Bigg)^{\frac{1}{p'}} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \\
&\approx \sup_{h:\,\int_0^x h\le\int_0^x w}\ \sup_{\varphi\in\mathfrak{M}^+} \frac{1}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \\
&\qquad \times \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_t^\infty \bigg(\frac{1}{s}\int_0^s \varphi\,u\bigg)^{p'} ds\bigg)^{\frac{1}{p'}} \\
&+ \sup_{h:\,\int_0^x h\le\int_0^x w}\ \sup_{\varphi\in\mathfrak{M}^+} \frac{1}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \\
&\qquad \times \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_t^\infty \bigg(\frac{B(s)}{s}\int_s^\infty \varphi\frac{u}{B}\bigg)^{p'} ds\bigg)^{\frac{1}{p'}}.
\end{aligned}
$$

Interchanging the suprema gives

$$
K \approx \sup_{h:\,\int_0^x h\le\int_0^x w}\ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{\varphi\in\mathfrak{M}^+} \frac{\Big(\int_0^\infty \Big(\frac{1}{s}\int_0^s \varphi\,u\Big)^{p'} \chi_{[t,\infty)}(s)\,ds\Big)^{\frac{1}{p'}}}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}}
$$

$$+ \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{\varphi\in\mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\frac{B(s)}{s} \int_s^\infty \varphi \frac{u}{B} \right)^{p'} \chi_{[t,\infty)}(s)\, ds \right)^{\frac{1}{p'}}}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}}.$$

First consider the case where $p \le q$. By Theorems 4 and 5, we have

$$K \approx \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in(0,\infty)} \left(\int_s^\infty \tau^{-p'} \chi_{[t,\infty)}(\tau)\, d\tau \right)^{\frac{1}{p'}} \left(\int_0^s h\, u^q \right)^{\frac{1}{q}}$$
$$+ \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}$$
$$\times \sup_{s\in(0,\infty)} \left(\int_0^s \left(\frac{B(\tau)}{\tau} \right)^{p'} \chi_{[t,\infty)}(\tau)\, d\tau \right)^{\frac{1}{p'}} \left(\int_s^\infty h \left(\frac{u}{B} \right)^q \right)^{\frac{1}{q}}.$$

Since

$$\begin{aligned}
&\sup_{s\in(0,\infty)} \left(\int_s^\infty \tau^{-p'} \chi_{[t,\infty)}(\tau)\, d\tau \right)^{\frac{1}{p'}} \left(\int_0^s h\, u^q \right)^{\frac{1}{q}} \\
&\quad = \max \left\{ \sup_{s\in(0,t)} \left(\int_s^\infty \tau^{-p'} \chi_{[t,\infty)}(\tau)\, d\tau \right)^{\frac{1}{p'}} \left(\int_0^s h\, u^q \right)^{\frac{1}{q}}, \right. \\
&\qquad\qquad \left. \sup_{s\in[t,\infty)} \left(\int_s^\infty \tau^{-p'} \chi_{[t,\infty)}(\tau)\, d\tau \right)^{\frac{1}{p'}} \left(\int_0^s h\, u^q \right)^{\frac{1}{q}} \right\} \\
&\quad = \max \left\{ \left(\int_t^\infty \tau^{-p'}\, d\tau \right)^{\frac{1}{p'}} \left(\int_0^t h\, u^q \right)^{\frac{1}{q}}, \right. \\
&\qquad\qquad \left. \sup_{s\in[t,\infty)} \left(\int_s^\infty \tau^{-p'}\, d\tau \right)^{\frac{1}{p'}} \left(\int_0^s h\, u^q \right)^{\frac{1}{q}} \right\} \\
&\quad = \sup_{s\in[t,\infty)} \left(\int_s^\infty \tau^{-p'}\, d\tau \right)^{\frac{1}{p'}} \left(\int_0^s h\, u^q \right)^{\frac{1}{q}} \\
&\quad \approx \sup_{s\in[t,\infty)} s^{-\frac{1}{p}} \left(\int_0^s h\, u^q \right)^{\frac{1}{q}}
\end{aligned}$$

and

$$\sup_{s\in(0,\infty)} \left(\int_0^s \left(\frac{B(\tau)}{\tau} \right)^{p'} \chi_{[t,\infty)}(\tau)\, d\tau \right)^{\frac{1}{p'}} \left(\int_s^\infty h \left(\frac{u}{B} \right)^q \right)^{\frac{1}{q}}$$

$$= \max\Bigg\{ \sup_{s\in(0,t)} \bigg(\int_0^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} \chi_{[t,\infty)}(\tau)\, d\tau \bigg)^{\frac{1}{p'}} \bigg(\int_s^\infty h\Big(\frac{u}{B}\Big)^q \bigg)^{\frac{1}{q}},$$

$$\sup_{s\in[t,\infty)} \bigg(\int_0^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} \chi_{[t,\infty)}(\tau)\, d\tau \bigg)^{\frac{1}{p'}} \bigg(\int_s^\infty h\Big(\frac{u}{B}\Big)^q \bigg)^{\frac{1}{q}} \Bigg\}$$

$$= \max\Bigg\{ 0, \sup_{s\in[t,\infty)} \bigg(\int_t^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} d\tau \bigg)^{\frac{1}{p'}} \bigg(\int_s^\infty h\Big(\frac{u}{B}\Big)^q \bigg)^{\frac{1}{q}} \Bigg\}$$

$$= \sup_{s\in[t,\infty)} \bigg(\int_t^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} d\tau \bigg)^{\frac{1}{p'}} \bigg(\int_s^\infty h\Big(\frac{u}{B}\Big)^q \bigg)^{\frac{1}{q}},$$

by interchanging the suprema we get

$$K \approx \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} s^{-\frac{1}{p}} \sup_{h:\, \int_0^x h \le \int_0^x w} \bigg(\int_0^s h\, u^q \bigg)^{\frac{1}{q}}$$
$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} \bigg(\int_t^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} d\tau \bigg)^{\frac{1}{p'}} \sup_{h:\, \int_0^x h \le \int_0^x w} \bigg(\int_s^\infty h\Big(\frac{u}{B}\Big)^q \bigg)^{\frac{1}{q}}.$$

Finally, by Theorem 3 and Remark 1, we arrive at

$$K \approx \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} s^{-\frac{1}{p}} \bigg(\int_0^\infty \Big(\sup_{\tau\in[x,\infty)} u(\tau)^q \chi_{(0,s]}(\tau) \Big)\, dx \bigg)^{\frac{1}{q}}$$
$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} \bigg(\int_t^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} d\tau \bigg)^{\frac{1}{p'}}$$
$$\times \bigg(\int_0^\infty \bigg(\sup_{y\in[x,\infty)} \bigg(\frac{u(y)}{B(y)}\bigg)^q \chi_{[s,\infty)}(y) \bigg)\, dx \bigg)^{\frac{1}{q}}$$

$$\approx \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} s^{-\frac{1}{p}} \bigg(\int_0^s \Big(\sup_{\tau\in[x,s]} u(\tau)^q \Big)\, w(x)\, dx \bigg)^{\frac{1}{q}}$$
$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} \bigg(\int_t^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} d\tau \bigg)^{\frac{1}{p'}} \bigg(\sup_{y\in[s,\infty)} \frac{u(y)}{B(y)} \bigg) \bigg(\int_0^s w \bigg)^{\frac{1}{q}}$$
$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} \bigg(\int_t^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} d\tau \bigg)^{\frac{1}{p'}}$$
$$\times \bigg(\int_s^\infty \bigg(\sup_{y\in[x,\infty)} \bigg(\frac{u(y)}{B(y)}\bigg)^q \bigg)\, w(x)\, dx \bigg)^{\frac{1}{q}}.$$

Next consider the case where $q < p$. By Theorems 4 and 5, we have

$$K \approx \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_0^\infty \bigg(\int_s^\infty \tau^{-p'} \chi_{[t,\infty)}(\tau)\,d\tau \bigg)^{\frac{(q-1)p}{p-q}} s^{-p'} \chi_{[t,\infty)}(s) \times \bigg(\int_0^s h\,u^q \bigg)^{\frac{p}{p-q}} ds \bigg)^{\frac{p-q}{pq}}$$
$$+ \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_0^\infty \bigg(\int_0^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} \chi_{[t,\infty)}(\tau)\,d\tau \bigg)^{\frac{(q-1)p}{p-q}} \times \bigg(\frac{B(s)}{s}\bigg)^{p'} \chi_{[t,\infty)}(s) \bigg(\int_s^\infty h \bigg(\frac{u}{B}\bigg)^q \bigg)^{\frac{p}{p-q}} ds \bigg)^{\frac{p-q}{pq}}.$$

Since

$$\int_0^\infty \bigg(\int_s^\infty \tau^{-p'} \chi_{[t,\infty)}(\tau)\,d\tau \bigg)^{\frac{(q-1)p}{p-q}} s^{-p'} \chi_{[t,\infty)}(s) \bigg(\int_0^s h\,u^q \bigg)^{\frac{p}{p-q}} ds$$
$$= \int_t^\infty \bigg(\int_s^\infty \tau^{-p'}\,d\tau \bigg)^{\frac{(q-1)p}{p-q}} s^{-p'} \bigg(\int_0^s h\,u^q \bigg)^{\frac{p}{p-q}} ds$$
$$\approx \int_t^\infty s^{\frac{p}{q-p}} \bigg(\int_0^s h\,u^q \bigg)^{\frac{p}{p-q}} ds$$

and

$$\int_0^\infty \bigg(\int_0^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} \chi_{[t,\infty)}(\tau)\,d\tau \bigg)^{\frac{(q-1)p}{p-q}} \bigg(\frac{B(s)}{s}\bigg)^{p'} \chi_{[t,\infty)}(s) \bigg(\int_s^\infty h \bigg(\frac{u}{B}\bigg)^q \bigg)^{\frac{p}{p-q}} ds$$
$$= \int_t^\infty \bigg(\int_t^s \bigg(\frac{B(\tau)}{\tau}\bigg)^{p'} d\tau \bigg)^{\frac{(q-1)p}{p-q}} \bigg(\frac{B(s)}{s}\bigg)^{p'} \bigg(\int_s^\infty h \bigg(\frac{u}{B}\bigg)^q \bigg)^{\frac{p}{p-q}} ds$$
$$= \int_t^\infty \mathcal{B}(t,s) \bigg(\int_s^\infty h \bigg(\frac{u}{B}\bigg)^q \bigg)^{\frac{p}{p-q}} ds,$$

we have

$$K \approx \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_t^\infty s^{\frac{p}{q-p}} \bigg(\int_0^s h\,u^q \bigg)^{\frac{p}{p-q}} ds \bigg)^{\frac{p-q}{pq}}$$
$$+ \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_t^\infty \mathcal{B}(t,s) \bigg(\int_s^\infty h \bigg(\frac{u}{B}\bigg)^q d\tau \bigg)^{\frac{p}{p-q}} ds \bigg)^{\frac{p-q}{pq}}.$$

By duality, applying Fubini's Theorem and interchanging the suprema, we get

$$K \approx \sup_{h:\int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\int_t^\infty \varphi(s) \Big(\int_0^s h\, u^q \Big)\, ds}{\Big(\int_t^\infty \varphi(s)^{\frac{p}{q}} s^{\frac{p}{q}}\, ds \Big)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{h:\int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\int_t^\infty \varphi(s) \Big(\int_s^\infty h \Big(\frac{u}{B}\Big)^q \Big)\, ds}{\Big(\int_t^\infty \varphi(s)^{\frac{p}{q}} \mathcal{B}(t,s)^{\frac{q-p}{q}}\, ds \Big)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$\approx \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\sup_{h:\int_0^x h \le \int_0^x w} \int_0^t h\, u^q \Big(\int_t^\infty \varphi \Big)}{\Big(\int_t^\infty \varphi(s)^{\frac{p}{q}} s^{\frac{p}{q}}\, ds \Big)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\sup_{h:\int_0^x h \le \int_0^x w} \int_t^\infty h(\tau) u(\tau)^q \Big(\int_\tau^\infty \varphi \Big)\, d\tau}{\Big(\int_t^\infty \varphi(s)^{\frac{p}{q}} s^{\frac{p}{q}}\, ds \Big)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\sup_{h:\int_0^x h \le \int_0^x w} \int_t^\infty h(\tau) \Big(\frac{u(\tau)}{B(\tau)}\Big)^q \Big(\int_t^\tau \varphi \Big)\, d\tau}{\Big(\int_t^\infty \varphi(s)^{\frac{p}{q}} \mathcal{B}(t,s)^{\frac{q-p}{q}}\, ds \Big)^{\frac{q}{p}}} \right)^{\frac{1}{q}}.$$

On using Theorem 3, we arrive at

$$K \approx \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\Big(\int_0^\infty \Big(\sup_{\tau\in[x,\infty)} u(\tau)^q \chi_{(0,t]}(\tau) \Big) w(x)\, dx \Big) \Big(\int_t^\infty \varphi \Big)}{\Big(\int_t^\infty \varphi(s)^{\frac{p}{q}} s^{\frac{p}{q}}\, ds \Big)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\int_0^\infty \Big(\sup_{\tau\in[x,\infty)} u(\tau)^q \chi_{[t,\infty)}(\tau) \int_\tau^\infty \varphi \Big) w(x)\, dx}{\Big(\int_t^\infty \varphi(s)^{\frac{p}{q}} s^{\frac{p}{q}}\, ds \Big)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\int_0^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)}\right)^q \chi_{[t,\infty)}(\tau) \int_t^\tau \varphi\right) w(x)\,dx}{\left(\int_t^\infty \varphi(s)^{\frac{p}{q}} \mathcal{B}(t,s)^{\frac{q-p}{r}}\,ds\right)^{\frac{q}{p}}} \right)^{\frac{1}{q}}.$$

By Remark 1 and interchanging the suprema, we get

$$K \approx \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t \left(\sup_{\tau\in[x,t]} u(\tau)^q\right) w(x)\,dx\right)^{\frac{1}{q}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\int_t^\infty \varphi(s)\,ds}{\left(\int_t^\infty \varphi(s)^{\frac{p}{q}} s^{\frac{p}{q}}\,ds\right)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t w\right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} u(\tau) \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\left(\int_\tau^\infty \varphi(s)\,ds\right)}{\left(\int_t^\infty \varphi(s)^{\frac{p}{q}} s^{\frac{p}{q}}\,ds\right)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\int_t^\infty \sup_{\tau\in[x,\infty)} u(\tau)^q \left(\int_\tau^\infty \varphi\right) w(x)\,dx}{\left(\int_t^\infty \varphi(s)^{\frac{p}{q}} s^{\frac{p}{q}}\,ds\right)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t w\right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\int_t^\tau \varphi(s)\,ds}{\left(\int_t^\infty \varphi(s)^{\frac{p}{q}} \mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\right)^{\frac{q}{p}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)} \frac{\int_t^\infty \sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)}\right)^q \left(\int_t^\tau \varphi\right) w(x)\,dx}{\left(\int_t^\infty \varphi(s)^{\frac{p}{q}} \mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\right)^{\frac{q}{p}}} \right)^{\frac{1}{q}}.$$

By duality, we have

$$K \approx \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t \left(\sup_{\tau\in[x,t]} u(\tau)^q\right) w(x)\,dx\right)^{\frac{1}{q}} \left(\int_t^\infty s^{\frac{p}{q-p}}\,ds\right)^{\frac{p-q}{pq}}$$
$$+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \left(\int_0^t w\right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} u(\tau) \left(\int_t^\infty s^{\frac{p}{q-p}} \chi_{[\tau,\infty)}(s)\,ds\right)^{\frac{p-q}{pq}}$$

$$+\sup_{t\in(0,\infty)}\frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}\left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)}\frac{\int_t^\infty \sup_{\tau\in[x,\infty)} u(\tau)^q\Big(\int_\tau^\infty\varphi\Big)\,w(x)\,dx}{\Big(\int_t^\infty \varphi(s)^{\frac{p}{q}}s^{\frac{p}{q}}\,ds\Big)^{\frac{q}{p}}}\right)^{\frac{1}{q}}$$

$$+\sup_{t\in(0,\infty)}\frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}\left(\int_0^t w\right)^{\frac{1}{q}}\sup_{\tau\in[t,\infty)}\frac{u(\tau)}{B(\tau)}\left(\int_t^\infty \mathcal{B}(t,s)\chi_{[t,\tau]}(s)\,ds\right)^{\frac{p-q}{pq}}$$

$$+\sup_{t\in(0,\infty)}\frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}\left(\sup_{\varphi\in\mathfrak{M}^+[t,\infty)}\frac{\int_t^\infty \sup_{\tau\in[x,\infty)}\Big(\frac{u(\tau)}{B(\tau)}\Big)^q\Big(\int_t^\tau\varphi\Big)\,w(x)\,dx}{\Big(\int_t^\infty \varphi(s)^{\frac{p}{q}}\mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\Big)^{\frac{q}{p}}}\right)^{\frac{1}{q}}.$$

Applying Theorem 6 and Theorem 7 yields

$$\begin{aligned}
K\approx{}&\sup_{t\in(0,\infty)}\frac{1}{v_1(t)^{\frac{1}{m}}}\left(\int_0^t\left(\sup_{\tau\in[x,t]}u(\tau)^q\right)w(x)\,dx\right)^{\frac{1}{q}}\\
&+\sup_{t\in(0,\infty)}\frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}\left(\int_0^t w\right)^{\frac{1}{q}}\sup_{\tau\in[t,\infty)}u(\tau)\tau^{-\frac{1}{p}}\\
&+\sup_{t\in(0,\infty)}\frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}\left(\int_t^\infty\left(\sup_{\tau\in[s,\infty)}u(\tau)^{\frac{pq}{p-q}}\tau^{\frac{q}{q-p}}\right)\left(\int_t^s w\right)^{\frac{q}{p-q}}w(s)\,ds\right)^{\frac{p-q}{pq}}\\
&+\sup_{t\in(0,\infty)}\frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}\left(\int_t^\infty\left(\int_t^s\left(\sup_{y\in[x,s]}u(y)^q\right)w(x)\,dx\right)^{\frac{q}{p-q}}\right.\\
&\qquad\qquad\left.\times\left(\sup_{\tau\in[s,\infty)}u(\tau)^q\tau^{\frac{q}{q-p}}\right)w(s)\,ds\right)^{\frac{p-q}{pq}}\\
&+\sup_{t\in(0,\infty)}\frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}\left(\int_0^t w\right)^{\frac{1}{q}}\sup_{\tau\in[t,\infty)}\frac{u(\tau)}{B(\tau)}\left(\int_t^\tau\mathcal{B}(t,s)\,ds\right)^{\frac{p-q}{pq}}\\
&+\sup_{t\in(0,\infty)}\frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}\left(\int_t^\infty\left(\sup_{\tau\in[x,\infty)}\sup_{y\in[\tau,\infty)}\left(\frac{u(y)}{B(y)}\right)^{\frac{pq}{p-q}}\left(\int_t^\tau\mathcal{B}(t,s)\,ds\right)\right)\right.\\
&\qquad\qquad\left.\times\left(\int_t^x w\right)^{\frac{q}{p-q}}w(x)\,dx\right)^{\frac{p-q}{pq}}\\
&+\sup_{t\in(0,\infty)}\frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}}\left(\int_t^\infty\left(\int_x^\infty\left(\sup_{\tau\in[y,\infty)}\left(\frac{u(\tau)}{B(\tau)}\right)^q\right)w(y)\,dy\right)^{\frac{q}{p-q}}\right.
\end{aligned}$$

$$\times \left(\sup_{z \in [x,\infty)} \left(\frac{u(z)}{B(z)} \right)^q \right) \left(\int_t^x \mathcal{B}(t,s)\,ds \right) w(x)\,dx \Bigg)^{\frac{p-q}{pq}}.$$

The proof is completed. ◀

Theorem 13. *Let $1 < m < \infty, 0 < p \le 1, 1 < q < \infty$ and $b \in \mathcal{W}(0,\infty) \cap \mathfrak{M}^+((0,\infty);\downarrow)$ be such that the function $B(t)$ satisfies $0 < B(t) < \infty$ for every $t \in (0,\infty)$. Suppose that $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$, $v \in \mathcal{W}_{m.p}(0,\infty)$ and $w \in \mathcal{W}(0,\infty)$.*

i) If $m \le q$, then

$$\begin{aligned} K \approx & \sup_{t \in (0,\infty)} \left(\int_t^\infty \frac{v_0}{v_1^{m'+1}} \right)^{\frac{1}{m'}} \left(\int_0^t \left(\sup_{\tau \in [x,t]} u(\tau)^q \right) w(x)\,dx \right)^{\frac{1}{q}} \\ & + \sup_{t \in (0,\infty)} \left(\int_0^t \frac{B^{m'} v_0}{v_1^{m'+1}} \right)^{\frac{1}{m'}} \left(\int_0^t w \right)^{\frac{1}{q}} \left(\sup_{\tau \in [t,\infty)} \frac{u(\tau)}{B(\tau)} \right) \\ & + \sup_{t \in (0,\infty)} \left(\int_0^t \frac{B^{m'} v_0}{v_1^{m'+1}} \right)^{\frac{1}{m'}} \left(\int_t^\infty \left(\sup_{\tau \in [x,\infty)} \left(\frac{u(\tau)}{B(\tau)} \right)^q \right) w(x)\,dx \right)^{\frac{1}{q}}; \end{aligned}$$

ii) If $q < m$, then

$$\begin{aligned} K \approx & \left(\int_0^\infty \left(\sup_{\tau \in [t,\infty)} u(\tau)^{\frac{mq}{m-q}} \left(\int_\tau^\infty \mathfrak{B}_1 \right) \right) \left(\int_0^t w \right)^{\frac{q}{m-q}} w(t)\,dt \right)^{\frac{m-q}{mq}} \\ & + \left(\int_0^\infty \left(\int_0^t \left(\sup_{y \in [x,t]} u(y)^q \right) w(x)\,dx \right)^{\frac{q}{m-q}} \right. \\ & \qquad \times \left. \left(\sup_{\tau \in [t,\infty)} u(\tau)^q \left(\int_\tau^\infty \mathfrak{B}_1 \right) \right) w(t)\,dt \right)^{\frac{m-q}{mq}} \\ & + \left(\int_0^\infty \left(\sup_{\tau \in [t,\infty)} \sup_{s \in [\tau,\infty)} \left(\frac{u(s)}{B(s)} \right)^{\frac{mq}{m-q}} \left(\int_0^\tau \mathfrak{B}_2 \right) \right) \left(\int_0^t w \right)^{\frac{q}{m-q}} w(t)\,dt \right)^{\frac{m-q}{mq}} \\ & + \left(\int_0^\infty \left(\int_t^\infty \left(\sup_{\tau \in [x,\infty)} \left(\frac{u(\tau)}{B(\tau)} \right)^q \right) w(x)\,dx \right)^{\frac{q}{m-q}} \right. \\ & \qquad \times \left. \left(\sup_{s \in [t,\infty)} \left(\frac{u(s)}{B(s)} \right)^q \right) \left(\int_0^t \mathfrak{B}_2 \right) w(t)\,dt \right)^{\frac{m-q}{mq}}, \end{aligned}$$

where the functions $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are defined for all $s \in (0,\infty)$ by

$$\mathfrak{B}_1(s) := \bigg(\int_s^\infty \frac{v_0}{v_1^{m'+1}}\bigg)^{\frac{m(q-1)}{m-q}} \frac{v_0(s)}{v_1(s)^{m'+1}}$$

and

$$\mathfrak{B}_2(s) := \bigg(\int_0^s \frac{B^{m'} v_0}{v_1^{m'+1}}\bigg)^{\frac{m(q-1)}{m-q}} \frac{B(s)^{m'} v_0(s)}{v_1(s)^{m'+1}},$$

respectively.

Proof. By Lemma 2, Theorem 1, (iii), and Fubini's theorem, we have

$$\begin{aligned} K &\approx \sup_{h:\int_0^x h \le \int_0^x w} \sup_{\varphi\in\mathfrak{M}^+} \frac{\bigg(\int_0^\infty \bigg(\frac{1}{t}\int_0^t b(y)\bigg(\int_y^\infty \varphi\frac{u}{B}\bigg)\,dy\bigg)^{m'} \frac{t^{m'} v_0(t)}{v_1(t)^{m'+1}}\,dt\bigg)^{\frac{1}{m'}}}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \\ &\approx \sup_{h:\int_0^x h \le \int_0^x w} \sup_{\varphi\in\mathfrak{M}^+} \frac{\bigg(\int_0^\infty \bigg(\int_0^t \varphi\, u\bigg)^{m'} \frac{v_0(t)}{v_1(t)^{m'+1}}\,dt\bigg)^{\frac{1}{m'}}}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}} \\ &\quad + \sup_{h:\int_0^x h \le \int_0^x w} \sup_{\varphi\in\mathfrak{M}^+} \frac{\bigg(\int_0^\infty \bigg(\int_t^\infty \varphi\frac{u}{B}\bigg)^{m'} \frac{B(t)^{m'} v_0(t)}{v_1(t)^{m'+1}}\,dt\bigg)^{\frac{1}{m'}}}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}}. \end{aligned}$$

Let $m \le q$. By Theorems 4 and 5, we have

$$\begin{aligned} K &\approx \sup_{h:\int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \bigg(\int_t^\infty \frac{v_0}{v_1^{m'+1}}\bigg)^{\frac{1}{m'}} \bigg(\int_0^t h\, u^q\bigg)^{\frac{1}{q}} \\ &\quad + \sup_{h:\int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \bigg(\int_0^t \frac{B^{m'} v_0}{v_1^{m'+1}}\bigg)^{\frac{1}{m'}} \bigg(\int_t^\infty h\bigg(\frac{u}{B}\bigg)^q\bigg)^{\frac{1}{q}}. \end{aligned}$$

Interchanging the suprema, using Theorem 3 and Remark 1, we arrive at

$$\begin{aligned} K &\approx \sup_{t\in(0,\infty)} \bigg(\int_t^\infty \frac{v_0}{v_1^{m'+1}}\bigg)^{\frac{1}{m'}} \sup_{h:\int_0^x h \le \int_0^x w} \bigg(\int_0^\infty h(\tau)u(\tau)^q \chi_{(0,t]}(\tau)\,d\tau\bigg)^{\frac{1}{q}} \\ &\quad + \sup_{t\in(0,\infty)} \bigg(\int_0^t \frac{B^{m'} v_0}{v_1^{m'+1}}\bigg)^{\frac{1}{m'}} \sup_{h:\int_0^x h \le \int_0^x w} \bigg(\int_0^\infty h(\tau)\bigg(\frac{u(\tau)}{B(\tau)}\bigg)^q \chi_{[t,\infty)}(\tau)\,d\tau\bigg)^{\frac{1}{q}} \\ &\approx \sup_{t\in(0,\infty)} \bigg(\int_t^\infty \frac{v_0}{v_1^{m'+1}}\bigg)^{\frac{1}{m'}} \bigg(\int_0^\infty \bigg(\sup_{\tau\in[x,\infty)} u(\tau)^q \chi_{(0,t]}(\tau)\bigg) w(x)\,dx\bigg)^{\frac{1}{q}} \end{aligned}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_0^t \frac{B^{m'} v_0}{v_1^{m'+1}} \right)^{\frac{1}{m'}} \left(\int_0^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)} \right)^q \chi_{[t,\infty)}(\tau) \right) w(x)\,dx \right)^{\frac{1}{q}}$$

$$\approx \sup_{t\in(0,\infty)} \left(\int_t^\infty \frac{v_0}{v_1^{m'+1}} \right)^{\frac{1}{m'}} \left(\int_0^t \left(\sup_{\tau\in[x,t]} u(\tau)^q \right) w(x)\,dx \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_0^t \frac{B^{m'} v_0}{v_1^{m'+1}} \right)^{\frac{1}{m'}} \left(\int_0^t w \right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_0^t \frac{B^{m'} v_0}{v_1^{m'+1}} \right)^{\frac{1}{m'}} \left(\int_t^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)} \right)^q \right) w(x)\,dx \right)^{\frac{1}{q}}.$$

Now let $q < m$. By Theorems 4 and 5, we have

$$K \approx \sup_{h:\, \int_0^x h \le \int_0^x w} \left(\int_0^\infty \mathfrak{B}_1(s) \left(\int_0^s h\, u^q \right)^{\frac{m}{m-q}} ds \right)^{\frac{m-q}{mq}}$$

$$+ \sup_{h:\, \int_0^x h \le \int_0^x w} \left(\int_0^\infty \mathfrak{B}_2(s) \left(\int_s^\infty h \left(\frac{u}{B} \right)^q \right)^{\frac{m}{m-q}} ds \right)^{\frac{m-q}{mq}}.$$

By duality, we get

$$K \approx \sup_{h:\, \int_0^x h \le \int_0^x w} \left(\sup_{\varphi\in\mathfrak{M}^+} \frac{\int_0^\infty \varphi(s) \left(\int_0^s h\, u^q \right) ds}{\left(\int_0^\infty \varphi(s)^{\frac{m}{q}} \mathfrak{B}_1(s)^{\frac{q-m}{q}}\, ds \right)^{\frac{q}{m}}} \right)^{\frac{1}{q}}$$

$$+ \sup_{h:\, \int_0^x h \le \int_0^x w} \left(\sup_{\varphi\in\mathfrak{M}^+} \frac{\int_0^\infty \varphi(s) \left(\int_s^\infty h \left(\frac{u}{B} \right)^q \right) ds}{\left(\int_0^\infty \varphi(s)^{\frac{m}{q}} \mathfrak{B}_2(s)^{\frac{q-m}{q}}\, ds \right)^{\frac{q}{m}}} \right)^{\frac{1}{q}}.$$

By Fubini's theorem, interchanging the suprema, on using Theorem 3, we arrive at

$$K \approx \left(\sup_{\varphi\in\mathfrak{M}^+} \frac{\sup_{h:\, \int_0^x h \le \int_0^x w} \int_0^\infty h(\tau) u(\tau)^q \left(\int_\tau^\infty \varphi \right) d\tau}{\left(\int_0^\infty \varphi(s)^{\frac{m}{q}} \mathfrak{B}_1(s)^{\frac{q-m}{q}}\, ds \right)^{\frac{q}{m}}} \right)^{\frac{1}{q}}$$

$$+\left(\sup_{\varphi\in\mathfrak{M}^+}\frac{\sup_{h:\,\int_0^x h\le\int_0^x w}\int_0^\infty h(\tau)\left(\frac{u(\tau)}{B(\tau)}\right)^q\left(\int_0^\tau\varphi\right)d\tau}{\left(\int_0^\infty\varphi(s)^{\frac{m}{q}}\mathfrak{B}_2(s)^{\frac{q-m}{q}}\,ds\right)^{\frac{q}{m}}}\right)^{\frac{1}{q}}$$

$$\approx\left(\sup_{\varphi\in\mathfrak{M}^+}\frac{\int_0^\infty\left(\sup_{\tau\in[x,\infty)}u(\tau)^q\left(\int_\tau^\infty\varphi\right)\right)w(x)\,dx}{\left(\int_0^\infty\varphi(s)^{\frac{m}{q}}\mathfrak{B}_1(s)^{\frac{q-m}{q}}\,ds\right)^{\frac{q}{m}}}\right)^{\frac{1}{q}}$$

$$+\left(\sup_{\varphi\in\mathfrak{M}^+}\frac{\int_0^\infty\left(\sup_{\tau\in[x,\infty)}\left(\frac{u(\tau)}{B(\tau)}\right)^q\left(\int_0^\tau\varphi\right)\right)w(x)\,dx}{\left(\int_0^\infty\varphi(s)^{\frac{m}{q}}\mathfrak{B}_2(s)^{\frac{q-m}{q}}\,ds\right)^{\frac{q}{m}}}\right)^{\frac{1}{q}}.$$

Applying Theorem 6 and Theorem 7 yields the result. Hence the proof is completed. ◀

Theorem 14. *Let* $1<m<\infty, 1<p<\infty, 1<q<\infty$ *and* $b\in\mathcal{W}(0,\infty)\cap\mathfrak{M}^+((0,\infty);\downarrow)$ *be such that the function* $B(t)$ *satisfies* $0<B(t)<\infty$ *for every* $t\in(0,\infty)$*. Assume that* $u\in\mathcal{W}(0,\infty)\cap C(0,\infty)$*,* $v\in\mathcal{W}_{m,p}(0,\infty)$ *and* $w\in\mathcal{W}(0,\infty)$*. Suppose that*

$$\int_0^t v_2(s)\,ds<\infty,\qquad\int_t^\infty s^{-\frac{m'}{p}}v_2(s)\,ds<\infty,$$

$$0<\int_0^t\left(\int_s^t\left(\frac{B(y)}{y}\right)^{p'}dy\right)^{\frac{m'}{p'}}v_2(s)\,ds<\infty,\qquad t\in(0,\infty),$$

$$\int_0^1 s^{-\frac{m'}{p}}v_2(s)\,ds=\int_1^\infty v_2(s)\,ds=\infty,$$

where the function v_2 *is defined by*

$$v_2(t):=\frac{t^{\frac{m'}{p'}}v_0(t)}{v_1(t)^{m'+1}},\qquad t\in(0,\infty).$$

i) If $\max\{p,m\}\le q$*, then*

$$K\approx\sup_{t\in(0,\infty)}\left(\int_0^t\left(\int_s^t\left(\frac{B(y)}{y}\right)^{p'}dy\right)^{\frac{m'}{p'}}v_2(s)\,ds\right)^{\frac{1}{m'}}\left(\int_0^t w\right)^{\frac{1}{q}}\sup_{\tau\in[t,\infty)}\frac{u(\tau)}{B(\tau)}$$

$$+\sup_{t\in(0,\infty)}\left(\int_0^t\left(\int_s^t\left(\frac{B(y)}{y}\right)^{p'}dy\right)^{\frac{m'}{p'}}v_2(s)\,ds\right)^{\frac{1}{m'}}$$
$$\times\left(\int_t^\infty\left(\sup_{\tau\in[x,\infty)}\left(\frac{u(\tau)}{B(\tau)}\right)^q\right)w(x)\,dx\right)^{\frac{1}{q}}$$
$$+\sup_{t\in(0,\infty)}\left(\int_0^\infty\left(\frac{1}{s+t}\right)^{\frac{m'}{p}}v_2(s)\,ds\right)^{\frac{1}{m'}}\left(\int_0^t\left(\sup_{\tau\in[x,t]}u(\tau)^q\right)w(x)\,dx\right)^{\frac{1}{q}}.$$

ii) If $m\le q<p$, then

$$K\approx\sup_{t\in(0,\infty)}\left(\int_0^t\left(\int_s^t\left(\frac{B(y)}{y}\right)^{p'}dy\right)^{\frac{m'}{p'}}v_2(s)\,ds\right)^{\frac{1}{m'}}\left(\int_0^t w\right)^{\frac{1}{q}}\sup_{\tau\in[t,\infty)}\frac{u(\tau)}{B(\tau)}$$
$$+\sup_{t\in(0,\infty)}\left(\int_0^t\left(\int_s^t\left(\frac{B(y)}{y}\right)^{p'}dy\right)^{\frac{m'}{p'}}v_2(s)\,ds\right)^{\frac{1}{m'}}$$
$$\times\left(\int_t^\infty\left(\sup_{\tau\in[x,\infty)}\left(\frac{u(\tau)}{B(\tau)}\right)^q\right)w(x)\,dx\right)^{\frac{1}{q}}$$
$$+\sup_{t\in(0,\infty)}\left(\int_0^t v_2\right)^{\frac{1}{m'}}\left(\int_0^t w\right)^{\frac{1}{q}}\sup_{\tau\in[t,\infty)}\frac{u(\tau)}{B(\tau)}\left(\int_t^\tau\mathcal{B}(t,s)\,ds\right)^{\frac{p-q}{pq}}$$
$$+\sup_{t\in(0,\infty)}\left(\int_0^t v_2\right)^{\frac{1}{m'}}\left(\int_t^\infty\left(\sup_{\tau\in[s,\infty)}\sup_{x\in[\tau,\infty)}\left(\frac{u(x)}{B(x)}\right)^{\frac{pq}{p-q}}\left(\int_t^\tau\mathcal{B}(t,y)\,dy\right)\right)\right.$$
$$\left.\times\left(\int_t^s w\right)^{\frac{q}{p-q}}w(s)\,ds\right)^{\frac{p-q}{pq}}$$
$$+\sup_{t\in(0,\infty)}\left(\int_0^t v_2\right)^{\frac{1}{m'}}\left(\int_t^\infty\left(\int_s^\infty\left(\sup_{\tau\in[x,\infty)}\left(\frac{u(\tau)}{B(\tau)}\right)^q\right)w(x)\,dx\right)^{\frac{q}{p-q}}\right.$$
$$\left.\times\left(\sup_{z\in[s,\infty)}\left(\frac{u(z)}{B(z)}\right)^q\right)\left(\int_t^s\mathcal{B}(t,y)\,dy\right)w(s)\,ds\right)^{\frac{p-q}{pq}}.$$
$$+\sup_{t\in(0,\infty)}\left(\int_0^\infty\left(\frac{t}{s+t}\right)^{\frac{m'}{p}}v_2(s)\,ds\right)^{\frac{1}{m'}}\left(\int_0^\infty\left(\sup_{\tau\in[t,\infty)}u(\tau)^{\frac{pq}{p-q}}(\tau+t)^{\frac{q}{q-p}}\right)\right.$$
$$\left.\times\left(\int_0^t w\right)^{\frac{q}{p-q}}w(t)\,dt\right)^{\frac{p-q}{pq}}$$
$$+\sup_{t\in(0,\infty)}\left(\int_0^\infty\left(\frac{t}{s+t}\right)^{\frac{m'}{p}}v_2(s)\,ds\right)^{\frac{1}{m'}}\left(\int_0^\infty\left(\int_0^t\left(\sup_{y\in[x,t]}u(y)^q\right)w(x)\,dx\right)^{\frac{q}{p-q}}\right.$$

$$\times \left(\sup_{\tau\in[t,\infty)} u(\tau)^q (\tau+t)^{\frac{q}{q-p}} \right) w(t)\,dt \Bigg)^{\frac{p-q}{pq}}.$$

Proof. By Lemma 2, Theorem 1, (iv), and Fubini's theorem, we have

$$K \approx \sup_{h:\,\int_0^x h \le \int_0^x w} \left(\int_0^\infty \left(\int_t^\infty \left(\frac{1}{s} \int_0^s b(y) \left(\int_y^\infty \varphi \frac{u}{B} \right) dy \right)^{p'} ds \right)^{\frac{m'}{p'}} v_2(t)\,dt \right)^{\frac{1}{m'}}$$

$$\approx \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{\varphi\in\mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\frac{B(s)}{s} \int_s^\infty \varphi \frac{u}{B} \right)^{p'} ds \right)^{\frac{m'}{p'}} v_2(t)\,dt \right)^{\frac{1}{m'}}}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}}$$

$$+ \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{\varphi\in\mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\frac{1}{s} \int_0^s \varphi\, u \right)^{p'} ds \right)^{\frac{m'}{p'}} v_2(t)\,dt \right)^{\frac{1}{m'}}}{\|\varphi\|_{q',h^{1-q'},(0,\infty)}}$$

$$=: A + B.$$

Let $p \le q$ and $m \le q$. We first estimate A. By Theorem 9, we get

$$A \approx \sup_{h:\,\int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{p'} dy \right)^{\frac{m'}{p'}} v_2(s)\,ds \right)^{\frac{1}{m'}} \left(\int_t^\infty h \left(\frac{u}{B} \right)^q \right)^{\frac{1}{q}}.$$

By interchanging the suprema, applying Theorem 3 and Remark 1, we obtain

$$A \approx \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{p'} dy \right)^{\frac{m'}{p'}} v_2(s)\,ds \right)^{\frac{1}{m'}}$$
$$\times \sup_{h:\,\int_0^x h \le \int_0^x w} \left(\int_0^\infty h(x) \left(\frac{u(x)}{B(x)} \right)^q \chi_{[t,\infty)}(x)\,dx \right)^{\frac{1}{q}}$$
$$\approx \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{p'} dy \right)^{\frac{m'}{p'}} v_2(s)\,ds \right)^{\frac{1}{m'}}$$
$$\times \left(\int_0^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)} \right)^q \chi_{[t,\infty)}(\tau) \right) w(x)\,dx \right)^{\frac{1}{q}}$$
$$\approx \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{p'} dy \right)^{\frac{m'}{p'}} v_2(s)\,ds \right)^{\frac{1}{m'}} \left(\int_0^t w \right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)}$$
$$+ \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{p'} dy \right)^{\frac{m'}{p'}} v_2(s)\,ds \right)^{\frac{1}{m'}}$$

$$\times \left(\int_t^\infty \left(\sup_{\tau \in [x,\infty)} \left(\frac{u(\tau)}{B(\tau)} \right)^q \right) w(x)\, dx \right)^{\frac{1}{q}}.$$

Now we estimate B. Note that, by Theorem 8, the following equivalency holds:

$$\begin{aligned} B &\approx \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t \in (0,\infty)} \left(\int_t^\infty s^{-\frac{m'}{p}} v_2(s)\, ds \right)^{\frac{1}{m'}} \left(\int_0^t h\, u^q \right)^{\frac{1}{q}} \\ &\quad + \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t \in (0,\infty)} t^{-\frac{1}{p}} \left(\int_0^t v_2(s)\, ds \right)^{\frac{1}{m'}} \left(\int_0^t h\, u^q \right)^{\frac{1}{q}} \\ &\approx \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t \in (0,\infty)} \left(\int_0^\infty \left(\frac{1}{s+t} \right)^{\frac{m'}{p}} v_2(s)\, ds \right)^{\frac{1}{m'}} \left(\int_0^t h u^q \right)^{\frac{1}{q}}. \end{aligned}$$

By interchanging suprema, applying Theorem 3 and Remark 1, we have

$$\begin{aligned} B &\approx \sup_{t \in (0,\infty)} \left(\int_0^\infty \left(\frac{1}{s+t} \right)^{\frac{m'}{p}} v_2(s)\, ds \right)^{\frac{1}{m'}} \left(\sup_{h:\, \int_0^x h \le \int_0^x w} \int_0^\infty h(x) u(x)^q \chi_{(0,t]}(x)\, dx \right)^{\frac{1}{q}}. \\ &\approx \sup_{t \in (0,\infty)} \left(\int_0^\infty \left(\frac{1}{s+t} \right)^{\frac{m'}{p}} v_2(s)\, ds \right)^{\frac{1}{m'}} \left(\int_0^\infty \left(\sup_{\tau \in [x,\infty)} u(\tau)^q \chi_{(0,t]}(\tau) \right) w(x)\, dx \right)^{\frac{1}{q}} \\ &\approx \sup_{t \in (0,\infty)} \left(\int_0^\infty \left(\frac{1}{s+t} \right)^{\frac{m'}{p}} v_2(s)\, ds \right)^{\frac{1}{m'}} \left(\int_0^t \left(\sup_{\tau \in [x,t]} u(\tau)^q \right) w(x)\, dx \right)^{\frac{1}{q}}. \end{aligned}$$

Combining the estimates for A and B, we obtain the result.

Let $m \le q < p$. By Theorem 9, we get

$$\begin{aligned} A &\approx \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t \in (0,\infty)} \left(\int_0^t v_2(s) \left(\int_s^t \left(\frac{B(y)}{y} \right)^{p'} dy \right)^{\frac{m'}{p'}} ds \right)^{\frac{1}{m'}} \left(\int_t^\infty h \left(\frac{u}{B} \right)^q dx \right)^{\frac{1}{q}} \\ &\quad + \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t \in (0,\infty)} \left(\int_0^t v_2 \right)^{\frac{1}{m'}} \left(\int_t^\infty \mathcal{B}(t,s) \left(\int_s^\infty h \left(\frac{u}{B} \right)^q dx \right)^{\frac{p}{p-q}} ds \right)^{\frac{p-q}{pq}} \\ &=: A_1 + A_2. \end{aligned}$$

Since in the previous case A_1 was estimated as

$$A_1 \approx \sup_{t \in (0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{p'} dy \right)^{\frac{m'}{p'}} v_2(s)\, ds \right)^{\frac{1}{m'}} \left(\int_0^t w \right)^{\frac{1}{q}} \sup_{\tau \in [t,\infty)} \frac{u(\tau)}{B(\tau)}$$

$$+\sup_{t\in(0,\infty)}\left(\int_0^t\left(\int_s^t\left(\frac{B(y)}{y}\right)^{p'}dy\right)^{\frac{m'}{p'}}v_2(s)\,ds\right)^{\frac{1}{m'}} \times\left(\int_t^\infty\left(\sup_{\tau\in[x,\infty)}\left(\frac{u(\tau)}{B(\tau)}\right)^q\right)w(x)\,dx\right)^{\frac{1}{q}},$$

we continue with the estimate of A_2.

By duality and Fubini's theorem, we have

$$A_2=\sup_{h:\int_0^x h\le\int_0^x w}\ \sup_{t\in(0,\infty)}\left(\int_0^t v_2\right)^{\frac{1}{m'}}\left(\sup_{\psi\in\mathfrak{M}^+[t,\infty)}\frac{\int_t^\infty\psi(s)\left(\int_s^\infty h\left(\frac{u}{B}\right)^q\right)ds}{\left(\int_t^\infty\psi(s)^{\frac{p}{q}}\mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\right)^{\frac{q}{p}}}\right)^{\frac{1}{q}}$$

$$=\sup_{h:\int_0^x h\le\int_0^x w}\ \sup_{t\in(0,\infty)}\left(\int_0^t v_2\right)^{\frac{1}{m'}}\left(\sup_{\psi\in\mathfrak{M}^+[t,\infty)}\frac{\int_t^\infty h(x)\left(\frac{u(x)}{B(x)}\right)^q\left(\int_t^x\psi\right)dx}{\left(\int_t^\infty\psi(s)^{\frac{p}{q}}\mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\right)^{\frac{q}{p}}}\right)^{\frac{1}{q}}.$$

By interchanging suprema, duality and Theorem 3, in view of Remark 1, we arrive at

$$A_2=\sup_{t\in(0,\infty)}\left(\int_0^t v_2\right)^{\frac{1}{m'}}\left(\sup_{\psi\in\mathfrak{M}^+[t,\infty)}\frac{\sup_{h:\int_0^x h\le\int_0^x w}\int_0^\infty h(x)\left(\frac{u(x)}{B(x)}\right)^q\chi_{[t,\infty)}(x)\int_t^x\psi\,dx}{\left(\int_t^\infty\psi(s)^{\frac{p}{q}}\mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\right)^{\frac{q}{p}}}\right)^{\frac{1}{q}}$$

$$=\sup_{t\in(0,\infty)}\left(\int_0^t v_2\right)^{\frac{1}{m'}}\left(\sup_{\psi\in\mathfrak{M}^+[t,\infty)}\frac{\int_0^\infty\left(\sup_{\tau\in[x,\infty)}\left(\frac{u(\tau)}{B(\tau)}\right)^q\chi_{[t,\infty)}(\tau)\int_t^\tau\psi\right)w(x)\,dx}{\left(\int_t^\infty\psi(s)^{\frac{p}{q}}\mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\right)^{\frac{q}{p}}}\right)^{\frac{1}{q}}$$

$$\approx\sup_{t\in(0,\infty)}\left(\int_0^t v_2\right)^{\frac{1}{m'}}\left(\int_0^t w\right)^{\frac{1}{q}}\sup_{\tau\in[t,\infty)}\frac{u(\tau)}{B(\tau)}\left(\sup_{\psi\in\mathfrak{M}^+[t,\infty)}\frac{\int_t^\tau\psi(s)\,ds}{\left(\int_t^\infty\psi(s)^{\frac{p}{q}}\mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\right)^{\frac{q}{p}}}\right)^{\frac{1}{q}}$$

$$+\sup_{t\in(0,\infty)}\left(\int_0^t v_2\right)^{\frac{1}{m'}}\left(\sup_{\psi\in\mathfrak{M}^+[t,\infty)}\frac{\int_t^\infty\left(\sup_{\tau\in[x,\infty)}\left(\frac{u(\tau)}{B(\tau)}\right)^q\int_t^\tau\psi\right)w(x)\,dx}{\left(\int_t^\infty\psi(s)^{\frac{p}{q}}\mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\right)^{\frac{q}{p}}}\right)^{\frac{1}{q}}$$

$$\approx \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\int_0^t w\right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)} \left(\int_t^\infty \mathcal{B}(t,s)\chi_{[t,\tau]}(s)\,ds\right)^{\frac{p-q}{pq}}$$
$$+ \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\sup_{\psi\in\mathfrak{M}^+[t,\infty)} \frac{\int_t^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)}\right)^q \int_t^\tau \psi\right) w(x)\,dx}{\left(\int_t^\infty \psi(s)^{\frac{p}{q}} \mathcal{B}(t,s)^{\frac{q-p}{q}}\,ds\right)^{\frac{q}{p}}} \right)^{\frac{1}{q}}.$$

Applying Theorem 6 yields

$$A_2 \approx \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\int_0^t w\right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)} \left(\int_t^\tau \mathcal{B}(t,s)\,ds\right)^{\frac{p-q}{pq}}$$
$$+ \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\int_t^\infty \left(\sup_{\tau\in[s,\infty)} \sup_{x\in[\tau,\infty)} \left(\frac{u(x)}{B(x)}\right)^{\frac{pq}{p-q}} \left(\int_t^\tau \mathcal{B}(t,y)\,dy\right)\right)\right.$$
$$\left. \times \left(\int_t^s w\right)^{\frac{q}{p-q}} w(s)\,ds\right)^{\frac{p-q}{pq}}$$
$$+ \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\int_t^\infty \left(\int_s^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)}\right)^q\right) w(x)\,dx\right)^{\frac{q}{p-q}}\right.$$
$$\left. \times \left(\sup_{z\in[s,\infty)} \left(\frac{u(z)}{B(z)}\right)^q\right) \left(\int_t^s \mathcal{B}(t,y)\,dy\right) w(s)\,ds\right)^{\frac{p-q}{pq}}.$$

Combining the estimates for A_1 and A_2, we arrive at

$$A \approx \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y}\right)^{p'} dy\right)^{\frac{m'}{p'}} v_2(s)\,ds\right)^{\frac{1}{m'}} \left(\int_0^t w\right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)}$$
$$+ \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y}\right)^{p'} dy\right)^{\frac{m'}{p'}} v_2(s)\,ds\right)^{\frac{1}{m'}}$$
$$\times \left(\int_t^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)}\right)^q\right) w(x)\,dx\right)^{\frac{1}{q}}$$
$$+ \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\int_0^t w\right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{u(\tau)}{B(\tau)} \left(\int_t^\tau \mathcal{B}(t,s)\,ds\right)^{\frac{p-q}{pq}}$$
$$+ \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\int_t^\infty \left(\sup_{\tau\in[s,\infty)} \sup_{x\in[\tau,\infty)} \left(\frac{u(x)}{B(x)}\right)^{\frac{pq}{p-q}} \left(\int_t^\tau \mathcal{B}(t,y)\,dy\right)\right)\right.$$
$$\left. \times \left(\int_t^s w\right)^{\frac{q}{p-q}} w(s)\,ds\right)^{\frac{p-q}{pq}}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\int_t^\infty \left(\int_s^\infty \left(\sup_{\tau\in[x,\infty)} \left(\frac{u(\tau)}{B(\tau)}\right)^q\right) w(x)\,dx\right)^{\frac{q}{p-q}} \times \left(\sup_{z\in[s,\infty)} \left(\frac{u(z)}{B(z)}\right)^q\right) \left(\int_t^s \mathcal{B}(t,y)\,dy\right) w(s)\,ds\right)^{\frac{p-q}{pq}}.$$

Now we estimate B. By Theorem 8 we have

$$B \approx \sup_{h:\int_0^x h\le\int_0^x w}\ \sup_{t\in(0,\infty)} \left(\int_t^\infty s^{-\frac{m'}{p}} v_2(s)\,ds\right)^{\frac{1}{m'}} \left(\int_0^t h\,u^q\right)^{\frac{1}{q}} + \sup_{h:\int_0^x h\le\int_0^x w}\ \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\int_t^\infty s^{\frac{q}{q-p}} \left(\int_0^s h\,u^q\right)^{\frac{q}{p-q}} h(s)u(s)^q\,ds\right)^{\frac{p-q}{pq}}.$$

Since

$$\left(\int_0^t hu^q\right)^{\frac{p}{p-q}} = \int_0^t d\left(\int_0^s hu^q\right)^{\frac{p}{p-q}} \approx \int_0^t \left(\int_0^s hu^q\right)^{\frac{q}{p-q}} h(s)u(s)^q\,ds,$$

we get

$$B \approx \sup_{h:\int_0^x h\le\int_0^x w}\ \sup_{t\in(0,\infty)} \left(\int_t^\infty s^{-\frac{m'}{p}} v_2(s)\,ds\right)^{\frac{1}{m'}} \left(\int_0^t \left(\int_0^s hu^q\right)^{\frac{q}{p-q}} h(s)u(s)^q\,ds\right)^{\frac{p-q}{pq}} + \sup_{h:\int_0^x h\le\int_0^x w}\ \sup_{t\in(0,\infty)} \left(\int_0^t v_2\right)^{\frac{1}{m'}} \left(\int_t^\infty s^{\frac{q}{q-p}} \left(\int_0^s h\,u^q\right)^{\frac{q}{p-q}} h(s)u(s)^q\,ds\right)^{\frac{p-q}{pq}}.$$

By Lemma 1, we arrive at

$$B \approx \sup_{h:\int_0^x h\le\int_0^x w}\ \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{t}{s+t}\right)^{\frac{m'}{p}} v_2(s)\,ds\right)^{\frac{1}{m'}} \times \left(\int_0^\infty \left(\frac{1}{s+t}\right)^{\frac{q}{p-q}} \left(\int_0^s hu^q\right)^{\frac{q}{p-q}} h(s)u(s)^q\,ds\right)^{\frac{p-q}{pq}}.$$

Applying Theorem 10 to the last integral, we obtain

$$\int_0^\infty \left(\frac{1}{s+t}\right)^{\frac{q}{p-q}} \left(\int_0^s hu^q\right)^{\frac{q}{p-q}} h(s)u(s)^q\,ds \approx \int_0^\infty \left(\int_0^s hu^q\right)^{\frac{p}{p-q}} d\left(-\left(\frac{1}{s+t}\right)^{\frac{q}{p-q}}\right)$$

$$\approx -\int_0^\infty \left(\int_0^s hu^q\right)^{\frac{p}{p-q}} \left(\frac{1}{s+t}\right)^{\frac{2q-p}{p-q}} \left(-\frac{1}{(s+t)^2}\right) ds$$
$$= \int_0^\infty \left(\int_0^s hu^q\right)^{\frac{p}{p-q}} \left(\frac{1}{s+t}\right)^{\frac{p}{p-q}} ds.$$

By duality, we get

$$B \approx \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{t}{s+t}\right)^{\frac{m'}{p}} v_2(s)\, ds\right)^{\frac{1}{m'}}$$
$$\times \left(\int_0^\infty \left(\int_0^s hu^q\right)^{\frac{p}{p-q}} \left(\frac{1}{s+t}\right)^{\frac{p}{p-q}} ds\right)^{\frac{p-q}{pq}}$$
$$\approx \sup_{h:\, \int_0^x h \le \int_0^x w} \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{t}{s+t}\right)^{\frac{m'}{p}} v_2(s)\, ds\right)^{\frac{1}{m'}}$$
$$\times \left(\sup_{\psi\in\mathfrak{M}^+} \frac{\int_0^\infty \psi(s)\left(\int_0^s hu^q\right) ds}{\left(\int_0^\infty \psi(s)^{\frac{p}{q}} (s+t)^{\frac{p}{q}}\, ds\right)^{\frac{q}{p}}}\right)^{\frac{1}{q}}.$$

Interchanging suprema, Fubini theorem and Theorem 3 imply

$$B \approx \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{t}{s+t}\right)^{\frac{m'}{p}} v_2(s)\, ds\right)^{\frac{1}{m'}}$$
$$\times \left(\sup_{\psi\in\mathfrak{M}^+} \frac{\sup_{h:\, \int_0^x h \le \int_0^x w} \int_0^\infty h(\tau)u(\tau)^q \left(\int_\tau^\infty \psi\right) d\tau}{\left(\int_0^\infty \psi(s)^{\frac{p}{q}} (s+t)^{\frac{p}{q}}\, ds\right)^{\frac{q}{p}}}\right)^{\frac{1}{q}}$$
$$\approx \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{t}{s+t}\right)^{\frac{m'}{p}} v_2(s)\, ds\right)^{\frac{1}{m'}}$$
$$\times \left(\sup_{\psi\in\mathfrak{M}^+} \frac{\int_0^\infty \left(\sup_{\tau\in[x,\infty)} u(\tau)^q \left(\int_\tau^\infty \psi\right)\right) w(x)\, dx}{\left(\int_0^\infty \psi(s)^{\frac{p}{q}} (s+t)^{\frac{p}{q}}\, ds\right)^{\frac{q}{p}}}\right)^{\frac{1}{q}}.$$

Applying Theorem 7 yields

$$B \approx \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{t}{s+t} \right)^{\frac{m'}{p}} v_2(s)\,ds \right)^{\frac{1}{m'}} \left(\int_0^\infty \left(\sup_{\tau\in[t,\infty)} u(\tau)^{\frac{pq}{p-q}} (\tau+t)^{\frac{q}{q-p}} \right) \right.$$

$$\left. \times \left(\int_0^t w \right)^{\frac{q}{p-q}} w(t)\,dt \right)^{\frac{p-q}{pq}}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{t}{s+t} \right)^{\frac{m'}{p}} v_2(s)\,ds \right)^{\frac{1}{m'}} \left(\int_0^\infty \left(\int_0^t \left(\sup_{y\in[x,t]} u(y)^q \right) w(x)\,dx \right)^{\frac{q}{p-q}} \right.$$

$$\left. \times \left(\sup_{\tau\in[t,\infty)} u(\tau)^q (\tau+t)^{\frac{q}{q-p}} \right) w(t)\,dt \right)^{\frac{p-q}{pq}}.$$

Finally, if we combine the estimates of A and B, then we get the result and the proof is completed. ◀

4. Boundedness of $M_{\phi,\Lambda^\alpha(b)}$ from $G\Gamma(p,m,v)$ into $\Lambda^q(w)$

In this section we formulate and prove the reduction theorem for the boundedness of $M_{\phi,\Lambda^\alpha(b)}$ from $G\Gamma(p,m,v)$ into $\Lambda^q(w)$ and calculate the best constant in the inequality

$$\left(\int_0^\infty \left[\left(M_{\phi,\Lambda^\alpha(b)} f \right)^*(x) \right]^q w(x)\,dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left(\int_0^x [f^*(\tau)]^p\,d\tau \right)^{\frac{m}{p}} v(x)\,dx \right)^{\frac{1}{m}}, \tag{14}$$

which is required to hold for all $f \in \mathfrak{M}(\mathbb{R}^n)$.

A function $\phi : (0,\infty) \to (0,\infty)$ is said to satisfy the Δ_2-condition, denoted $\phi \in \Delta_2$, if for some $C > 0$

$$\phi(2t) \le C\,\phi(t) \quad \text{for all} \quad 0 < t < \infty.$$

A function $\phi : (0,\infty) \to (0,\infty)$ is said to be quasi-increasing, if for some $C > 0$

$$\phi(t_1) \le C\phi(t_2),$$

whenever $0 < t_1 \le t_2 < \infty$.

A function $\phi : (0,\infty) \to (0,\infty)$ is said to satisfy the Q_r-condition, $0 < r < \infty$, denoted $\phi \in Q_r(0,\infty)$, if for some constant $C > 0$

$$\phi\left(\sum_{i=1}^n t_i \right) \le C \left(\sum_{i=1}^n \phi(t_i)^r \right)^{1/r}$$

for every finite set of non-negative real numbers $\{t_1, \ldots, t_n\}$.

Theorem 15. *Let $0 < p, m, q < \infty$, $0 < \alpha \le r < \infty$ and $v, w \in \mathcal{W}(0,\infty)$. Assume that $\phi \in Q_r(0,\infty)$ is a quasi-increasing function. Suppose that $b \in \mathcal{W}(0,\infty)$ is such that $0 < B(t) < \infty$ for all $t > 0$, $B \in \Delta_2$, $B(\infty) = \infty$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then inequality* (14) *holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality*

$$\bigg(\int_0^\infty \big[T_{B/\phi^\alpha,b} h^*(x)\big]^{\frac{q}{\alpha}} w(x)\,dx \bigg)^{\frac{1}{q}} \le C \bigg(\int_0^\infty \bigg(\int_0^x [h^*(\tau)]^{\frac{p}{\alpha}}\,d\tau \bigg)^{\frac{m}{p}} v(x)\,dx \bigg)^{\frac{1}{m}}$$

holds for all $h \in \mathfrak{M}(\mathbb{R}^n)$.

Proof. Assume that the inequality

$$\bigg(\int_0^\infty \big[\big(M_{\phi,\Lambda^\alpha(b)} f\big)^*(x)\big]^q w(x)\,dx \bigg)^{\frac{1}{q}} \le C \bigg(\int_0^\infty \bigg(\int_0^x [f^*(\tau)]^p\,d\tau \bigg)^{\frac{m}{p}} v(x)\,dx \bigg)^{\frac{1}{m}}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$.

Denote by $\mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n)$ the set of all measurable, non-negative, radially decreasing functions on $\mathbb{R}^n$, that is,

$$\mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n) := \{ f \in \mathfrak{M}(\mathbb{R}^n) : \ f(x) = h(|x|),\ x \in \mathbb{R}^n \text{ with } h \in \mathfrak{M}^+((0,\infty);\downarrow) \}.$$

Recall that the inequality

$$\big(M_{\phi,\Lambda^\alpha(b)} g\big)^*(x) \ge C \sup_{\tau \in [x,\infty)} \phi(\tau)^{-1} \bigg(\int_0^\tau [g^*(y)]^\alpha b(y)\,dy \bigg)^{\frac{1}{\alpha}}$$

holds for all $g \in \mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n)$ with constant $C > 0$ independent of g and x (cf. [38, Lemma 3.12]).

Thus the inequality

$$\bigg(\int_0^\infty \bigg[\sup_{\tau \in [x,\infty)} \phi(\tau)^{-1} \bigg(\int_0^\tau [g^*(y)]^\alpha b(y)\,dy \bigg)^{\frac{1}{\alpha}} \bigg]^q w(x)\,dx \bigg)^{\frac{1}{q}} \le C \bigg(\int_0^\infty \bigg(\int_0^x [g^*(\tau)]^p\,d\tau \bigg)^{\frac{m}{p}} v(x)\,dx \bigg)^{\frac{1}{m}}$$

holds for all $g \in \mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n)$, which evidently can be rewritten as follows:

$$\bigg(\int_0^\infty \big[(T_{B/\phi^\alpha,b} g^*)(x)\big]^{\frac{q}{\alpha}} w(x)\,dx \bigg)^{\frac{1}{q}} \le C \bigg(\int_0^\infty \bigg(\int_0^x [g^*(\tau)]^{\frac{p}{\alpha}}\,d\tau \bigg)^{\frac{m}{p}} v(x)\,dx \bigg)^{\frac{1}{m}}.$$

Since for any $h \in \mathfrak{M}(\mathbb{R}^n)$ there exists $g \in \mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n)$ such that $g^* = h^*$, the inequality

$$\left(\int_0^\infty \left[(T_{B/\phi^\alpha,b} h^*)(x) \right]^{\frac{q}{\alpha}} w(x)\, dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left(\int_0^x [h^*(\tau)]^{\frac{p}{\alpha}}\, d\tau \right)^{\frac{m}{p}} v(x)\, dx \right)^{\frac{1}{m}}$$

holds for all $h \in \mathfrak{M}(\mathbb{R}^n)$, as well.

Now assume that the inequality

$$\left(\int_0^\infty \left[(T_{B/\phi^\alpha,b} h^*)(x) \right]^{\frac{q}{\alpha}} w(x)\, dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left(\int_0^x [h^*(\tau)]^{\frac{p}{\alpha}}\, d\tau \right)^{\frac{m}{p}} v(x)\, dx \right)^{\frac{1}{m}}$$

holds for all $h \in \mathfrak{M}(\mathbb{R}^n)$.

Obviously, the last inequality is equivalent to the inequality

$$\left(\int_0^\infty \left[\sup_{\tau \in [x,\infty)} \phi(\tau)^{-1} \left(\int_0^\tau [f^*(y)]^\alpha b(y)\, dy \right)^{\frac{1}{\alpha}} \right]^q w(x)\, dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left(\int_0^x [f^*(\tau)]^p\, d\tau \right)^{\frac{m}{p}} v(x)\, dx \right)^{\frac{1}{m}}$$

for all $f \in \mathfrak{M}(\mathbb{R}^n)$.

Recall that the inequality

$$(M_{\phi,\Lambda^\alpha(b)} f)^*(t) \le C \sup_{\tau \in [t,\infty)} \phi(\tau)^{-1} \left(\int_0^\tau [f^*(y)]^\alpha b(y)\, dy \right)^{\frac{1}{\alpha}}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ (cf. [38, Corollary 3.6]).

Consequently, the inequality

$$\left(\int_0^\infty \left[\left(M_{\phi,\Lambda^\alpha(b)} f \right)^*(x) \right]^q w(x)\, dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left(\int_0^x [f^*(\tau)]^p\, d\tau \right)^{\frac{m}{p}} v(x)\, dx \right)^{\frac{1}{m}}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$, as well.

The proof is completed. ◄

Combining Theorem 15 with Theorems 11, 12, 13 and 14, respectively, we get the following four statements.

Theorem 16. *Let* $0 < m \le \alpha \le r < \infty$*,* $0 < p \le \alpha < q < \infty$ *and* $b \in \mathcal{W}(0,\infty) \cap \mathfrak{M}^+((0,\infty);\downarrow)$ *be such that the function* $B(t)$ *satisfies* $0 < B(t) < \infty$ *for every* $t \in (0,\infty)$*,* $B \in \Delta_2$*,* $B(\infty) = \infty$ *and* $B(t)/t^{\alpha/r}$ *is quasi-increasing.*

Moreover, let $\phi \in \mathcal{W}(0,\infty) \cap C(0,\infty)$ *be such that* $\phi \in Q_r(0,\infty)$ *is a quasi-increasing function. Assume that* $v \in \mathcal{W}_{m,p}(0,\infty)$ *and* $w \in \mathcal{W}(0,\infty)$*. Then*

$$\begin{aligned}
&\|M_{\phi,\Lambda^\alpha(b)}\|_{\mathrm{G}\Gamma(p,m,v)\to\Lambda^q(w)} \\
&\quad\approx \sup_{t\in(0,\infty)} \frac{1}{v_1(t)^{\frac{1}{m}}} \bigg(\int_0^t \bigg(\sup_{\tau\in[x,t]} \frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)} \bigg)^q w(x)\,dx \bigg)^{\frac{1}{q}} \\
&\qquad + \sup_{t\in(0,\infty)} \frac{B(t)^{\frac{1}{\alpha}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_0^t w \bigg)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{1}{\phi(\tau)} \\
&\qquad + \sup_{t\in(0,\infty)} \frac{B(t)^{\frac{1}{\alpha}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_t^\infty \bigg(\sup_{\tau\in[x,\infty)} \frac{1}{\phi(\tau)} \bigg)^q w(x)\,dx \bigg)^{\frac{1}{q}},
\end{aligned}$$

where v_1 *is defined by* (5)*.*

Theorem 17. *Let* $0 < m \le \alpha \le r < \infty$, $\alpha < \min\{p, q\} < \infty$ *and* $b \in \mathcal{W}(0,\infty) \cap \mathfrak{M}^+((0,\infty);\downarrow)$ *be such that the function* $B(t)$ *satisfies* $0 < B(t) < \infty$ *for every* $t \in (0,\infty)$*,* $B \in \Delta_2$*,* $B(\infty) = \infty$ *and* $B(t)/t^{\alpha/r}$ *is quasi-increasing. Moreover, let* $\phi \in \mathcal{W}(0,\infty) \cap C(0,\infty)$ *be such that* $\phi \in Q_r(0,\infty)$ *is a quasi-increasing function. Assume that* $v \in \mathcal{W}_{m,p}(0,\infty)$ *and* $w \in \mathcal{W}(0,\infty)$*.*

i) If $p \le q$*, then*

$$\begin{aligned}
&\|M_{\phi,\Lambda^\alpha(b)}\|_{\mathrm{G}\Gamma(p,m,v)\to\Lambda^q(w)} \\
&\quad\approx \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} s^{-\frac{1}{p}} \bigg(\int_0^s \bigg(\sup_{\tau\in[x,s]} \frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)} \bigg)^q w(x)\,dx \bigg)^{\frac{1}{q}} \\
&\qquad + \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[(t,\infty)} \bigg(\int_t^s \bigg(\frac{B(\tau)}{\tau} \bigg)^{\frac{p}{p-\alpha}} d\tau \bigg)^{\frac{p-\alpha}{p\alpha}} \bigg(\int_0^s w \bigg)^{\frac{1}{q}} \sup_{\tau\in[s,\infty)} \frac{1}{\phi(\tau)} \\
&\qquad + \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \sup_{s\in[t,\infty)} \bigg(\int_t^s \bigg(\frac{B(\tau)}{\tau} \bigg)^{\frac{p}{p-\alpha}} d\tau \bigg)^{\frac{p-\alpha}{p\alpha}} \\
&\qquad\qquad \times \bigg(\int_s^\infty \bigg(\sup_{\tau\in[x,\infty)} \frac{1}{\phi(\tau)} \bigg)^q w(x)\,dx \bigg)^{\frac{1}{q}};
\end{aligned}$$

ii) If $q < p$*, then*

$$\begin{aligned}
&\|M_{\phi,\Lambda^\alpha(b)}\|_{\mathrm{G}\Gamma(p,m,v)\to\Lambda^q(w)} \\
&\quad\approx \sup_{t\in(0,\infty)} \frac{1}{v_1(t)^{\frac{1}{m}}} \bigg(\int_0^t \bigg(\sup_{\tau\in[x,t]} \frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)} \bigg)^q w(x)\,dx \bigg)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_0^t w \bigg)^{\frac{1}{q}} \bigg(\sup_{\tau\in[t,\infty)} \frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)} \tau^{-\frac{1}{p}} \bigg) \\
&+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_t^\infty \bigg(\sup_{\tau\in[s,\infty)} \bigg(\frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)} \bigg)^{\frac{pq}{p-q}} \tau^{\frac{q}{q-p}} \bigg) \\
&\qquad\qquad \times \bigg(\int_t^s w \bigg)^{\frac{q}{p-q}} w(s)\,ds \bigg)^{\frac{p-q}{pq}} \\
&+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_t^\infty \bigg(\int_t^s \bigg(\sup_{y\in[x,s]} \frac{B(y)^{\frac{1}{\alpha}}}{\phi(y)} \bigg)^q w(x)\,dx \bigg)^{\frac{q}{p-q}} \\
&\qquad\qquad \times \bigg(\sup_{y\in[s,\infty)} \bigg(\frac{B(y)^{\frac{1}{\alpha}}}{\phi(y)} \bigg)^q \tau^{\frac{q}{q-p}} \bigg) w(s)\,ds \bigg)^{\frac{p-q}{pq}} \\
&+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_0^t w(x)\,dx \bigg)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{1}{\phi(\tau)} \bigg(\int_t^\tau \tilde{\mathcal{B}}(t,s)\,ds \bigg)^{\frac{p-q}{pq}} \\
&+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_t^\infty \bigg(\sup_{\tau\in[x,\infty)} \sup_{y\in[\tau,\infty)} \bigg(\frac{1}{\phi(y)} \bigg)^{\frac{pq}{p-q}} \int_t^\tau \tilde{\mathcal{B}}(t,s)\,ds \bigg) \\
&\qquad\qquad \times \bigg(\int_t^x w \bigg)^{\frac{q}{p-q}} w(x)\,dx \bigg)^{\frac{p-q}{pq}} \\
&+ \sup_{t\in(0,\infty)} \frac{t^{\frac{1}{p}}}{v_1(t)^{\frac{1}{m}}} \bigg(\int_t^\infty \bigg(\int_x^\infty \bigg(\sup_{\tau\in[s,\infty)} \frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)} \bigg)^q w(s)\,ds \bigg)^{\frac{q}{p-q}} \\
&\qquad \times \bigg(\sup_{y\in[x,\infty)} \frac{B(y)^{\frac{1}{\alpha}}}{\phi(y)} \bigg)^q \bigg(\int_t^x \tilde{\mathcal{B}}(t,s)\,ds \bigg) w(x)\,dx \bigg)^{\frac{p-q}{pq}},
\end{aligned}
$$

where

$$
\tilde{\mathcal{B}}(t,s) := \bigg(\int_t^s \bigg(\frac{B(y)}{y} \bigg)^{\frac{p}{p-\alpha}} dy \bigg)^{\frac{p(q-\alpha)}{\alpha(p-q)}} \bigg(\frac{B(s)}{s} \bigg)^{\frac{p}{p-\alpha}}, \qquad 0 < t < s < \infty.
$$

Theorem 18. *Let $0 < p \le \alpha \le r < \infty$, $\alpha < \min\{m, q\} < \infty$ and $b \in \mathcal{W}(0,\infty) \cap \mathfrak{M}^+((0,\infty);\downarrow)$ be such that the function $B(t)$ satisfies $0 < B(t) < \infty$ for every $t \in (0,\infty)$, $B \in \Delta_2$, $B(\infty) = \infty$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Moreover, let $\phi \in \mathcal{W}(0,\infty) \cap C(0,\infty)$ be such that $\phi \in Q_r(0,\infty)$ is a quasi-increasing function. Assume that $v \in \mathcal{W}_{m,p}(0,\infty)$ and $w \in \mathcal{W}(0,\infty)$.*

i) If $m \le q$, then

$\|M_{\phi,\Lambda^\alpha(b)}\|_{\mathrm{G}\Gamma(p,m,v)\to\Lambda^q(w)}$

$$\approx \sup_{t\in(0,\infty)} \bigg(\int_t^\infty \frac{v_0}{v_1^{\frac{2m-\alpha}{m-\alpha}}}\bigg)^{\frac{m-\alpha}{m\alpha}} \bigg(\int_0^t \bigg(\sup_{\tau\in[x,t]} \frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)}\bigg)^q w(x)\,dx\bigg)^{\frac{1}{q}}$$
$$+ \sup_{t\in(0,\infty)} \bigg(\int_0^t \frac{B^{\frac{m}{m-\alpha}} v_0}{v_1^{\frac{2m-\alpha}{m-\alpha}}}\bigg)^{\frac{m-\alpha}{m\alpha}} \bigg(\int_0^t w\bigg)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{1}{\phi(\tau)}$$
$$+ \sup_{t\in(0,\infty)} \bigg(\int_0^t \frac{B^{\frac{m}{m-\alpha}} v_0}{v_1^{\frac{2m-\alpha}{m-\alpha}}}\bigg)^{\frac{m-\alpha}{m\alpha}} \bigg(\int_t^\infty \bigg(\sup_{\tau\in[x,\infty)} \frac{1}{\phi(\tau)}\bigg)^q w(x)\,dx\bigg)^{\frac{1}{q}};$$

ii) If $q < m$, then

$$\|M_{\phi,\Lambda^\alpha(b)}\|_{G\Gamma(p,m,v)\to\Lambda^q(w)}$$
$$\approx \bigg(\int_0^\infty \bigg(\sup_{\tau\in[t,\infty)} \bigg(\frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)}\bigg)^{\frac{mq}{m-q}} \bigg(\int_\tau^\infty \tilde{\mathfrak{B}}_1\bigg)\bigg)\bigg(\int_0^t w\bigg)^{\frac{q}{m-q}} w(t)\,dt\bigg)^{\frac{m-q}{mq}}$$
$$+ \bigg(\int_0^\infty \bigg(\int_0^t \bigg(\sup_{y\in[x,t]} \frac{B(y)^{\frac{1}{\alpha}}}{\phi(y)}\bigg)^q w(x)\,dx\bigg)^{\frac{q}{m-q}}$$
$$\times \bigg(\sup_{\tau\in[t,\infty)} \bigg(\frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)}\bigg)^q \int_\tau^\infty \tilde{\mathfrak{B}}_1\bigg) w(t)\,dt\bigg)^{\frac{m-q}{mq}}$$
$$+ \bigg(\int_0^\infty \bigg(\sup_{\tau\in[t,\infty)} \sup_{s\in[\tau,\infty)} \bigg(\frac{1}{\phi(s)}\bigg)^{\frac{mq}{m-q}} \int_0^\tau \tilde{\mathfrak{B}}_2\bigg)\bigg(\int_0^t w\bigg)^{\frac{q}{m-q}} w(t)\,dt\bigg)^{\frac{m-q}{mq}}$$
$$+ \bigg(\int_0^\infty \bigg(\int_t^\infty \bigg(\sup_{\tau\in[x,\infty)} \frac{1}{\phi(\tau)}\bigg)^q w(x)\,dx\bigg)^{\frac{q}{m-q}}$$
$$\times \bigg(\sup_{y\in[t,\infty)} \frac{1}{\phi(y)}\bigg)^q \bigg(\int_0^t \tilde{\mathfrak{B}}_2\bigg) w(t)\,dt\bigg)^{\frac{m-q}{mq}},$$

where the functions $\tilde{\mathfrak{B}}_1$ and $\tilde{\mathfrak{B}}_2$ are defined for all $s \in (0,\infty)$ by

$$\tilde{\mathfrak{B}}_1(s) := \bigg(\int_s^\infty \frac{v_0(t)}{v_1(t)^{\frac{2m-\alpha}{m-\alpha}}}\,dt\bigg)^{\frac{m(q-\alpha)}{\alpha(m-q)}} \frac{v_0(s)}{v_1(s)^{\frac{2m-\alpha}{m-\alpha}}}$$

and

$$\tilde{\mathfrak{B}}_2(s) := \bigg(\int_0^s \frac{B(t)^{\frac{m}{m-\alpha}} v_0(t)}{v_1(t)^{\frac{2m-\alpha}{m-\alpha}}}\,dt\bigg)^{\frac{m(q-\alpha)}{\alpha(m-q)}} \frac{B(s)^{\frac{m}{m-\alpha}} v_0(s)}{v_1(s)^{\frac{2m-\alpha}{m-\alpha}}},$$

respectively.

Theorem 19. *Let $0 < \alpha \le r < \infty$, $\alpha < \min\{m, p, q\} < \infty$ and $b \in \mathcal{W}(0,\infty) \cap \mathfrak{M}^+((0,\infty);\downarrow)$ be such that the function $B(t)$ satisfies $0 < B(t) < \infty$ for every $t \in (0,\infty)$, $B \in \Delta_2$, $B(\infty) = \infty$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Moreover, let $\phi \in \mathcal{W}(0,\infty) \cap C(0,\infty)$ be such that $\phi \in Q_r(0,\infty)$ is a quasi-increasing function. Assume that $v \in \mathcal{W}_{m,p}(0,\infty)$ and $w \in \mathcal{W}(0,\infty)$. Suppose that*

$$\int_0^t \tilde{v}_2(s)\, ds < \infty, \qquad \int_t^\infty s^{-\frac{m\alpha}{p(m-\alpha)}} \tilde{v}_2(s)\, ds < \infty,$$

$$0 < \int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{\frac{p}{p-\alpha}} dy \right)^{\frac{m(p-\alpha)}{p(m-\alpha)}} \tilde{v}_2(s)\, ds < \infty, \qquad t \in (0,\infty),$$

$$\int_0^1 s^{-\frac{m\alpha}{p(m-\alpha)}} \tilde{v}_2(s)\, ds = \int_1^\infty \tilde{v}_2(s)\, ds = \infty,$$

where the function $\tilde{v}_2$ is defined by

$$\tilde{v}_2(t) := \frac{t^{\frac{m(p-\alpha)}{p(m-\alpha)}} v_0(t)}{v_1(t)^{\frac{2m-\alpha}{m-\alpha}}}, \qquad t \in (0,\infty).$$

i) If $\max\{p, m\} \le q$, then

$$\|M_{\phi,\Lambda^\alpha(b)}\|_{G\Gamma(p,m,v)\to\Lambda^q(w)}$$

$$\approx \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{\frac{p}{p-\alpha}} dy \right)^{\frac{m(p-\alpha)}{p(m-\alpha)}} \tilde{v}_2(s)\, ds \right)^{\frac{m-\alpha}{m\alpha}} \left(\int_0^t w \right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{1}{\phi(\tau)}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{\frac{p}{p-\alpha}} dy \right)^{\frac{m(p-\alpha)}{p(m-\alpha)}} \tilde{v}_2(s)\, ds \right)^{\frac{m-\alpha}{m\alpha}}$$

$$\times \left(\int_t^\infty \left(\sup_{\tau\in[x,\infty)} \frac{1}{\phi(\tau)} \right)^q w(x)\, dx \right)^{\frac{1}{q}}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{1}{s+t} \right)^{\frac{m\alpha}{p(m-\alpha)}} \tilde{v}_2(s)\, ds \right)^{\frac{m-\alpha}{m\alpha}} \left(\int_0^t \left(\sup_{\tau\in[x,t]} \frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)} \right)^q w(x)\, dx \right)^{\frac{1}{q}};$$

ii) If $m \le q < p$, then

$$\|M_{\phi,\Lambda^\alpha(b)}\|_{G\Gamma(p,m,v)\to\Lambda^q(w)}$$

$$\approx \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{\frac{p}{p-\alpha}} dy \right)^{\frac{m(p-\alpha)}{p(m-\alpha)}} \tilde{v}_2(s)\, ds \right)^{\frac{m-\alpha}{m\alpha}} \left(\int_0^t w \right)^{\frac{1}{q}} \sup_{\tau\in[t,\infty)} \frac{1}{\phi(\tau)}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_0^t \left(\int_s^t \left(\frac{B(y)}{y} \right)^{\frac{p}{p-\alpha}} dy \right)^{\frac{m(p-\alpha)}{p(m-\alpha)}} \tilde{v}_2(s)\, ds \right)^{\frac{m-\alpha}{m\alpha}}$$

$$
\begin{aligned}
&\times \left(\int_t^\infty \left(\sup_{\tau \in [x,\infty)} \frac{1}{\phi(\tau)} \right)^q w(x)\,dx \right)^{\frac{1}{q}} \\
&+ \sup_{t\in(0,\infty)} \left(\int_0^t \tilde{v}_2 \right)^{\frac{m-\alpha}{m\alpha}} \left(\sup_{\tau\in[t,\infty)} \frac{1}{\phi(\tau)} \left(\int_t^\tau \tilde{\mathcal{B}}(t,y)\,dy \right)^{\frac{p-q}{pq}} \right) \left(\int_0^t w \right)^{\frac{1}{q}} \\
&+ \sup_{t\in(0,\infty)} \left(\int_0^t \tilde{v}_2 \right)^{\frac{m-\alpha}{m\alpha}} \left(\int_t^\infty \left(\sup_{\tau\in[s,\infty)} \left(\sup_{x\in[\tau,\infty)} \left(\frac{1}{\phi(x)} \right)^{\frac{pq}{p-q}} \right) \right.\right. \\
&\qquad \left.\left. \times \left(\int_t^\tau \tilde{\mathcal{B}}(t,y)\,dy \right) \right) \left(\int_t^s w \right)^{\frac{q}{p-q}} w(s)\,ds \right)^{\frac{p-q}{pq}} \\
&+ \sup_{t\in(0,\infty)} \left(\int_0^t \tilde{v}_2 \right)^{\frac{m-\alpha}{m\alpha}} \left(\int_t^\infty \left(\int_s^\infty \left(\sup_{\tau\in[x,\infty)} \frac{1}{\phi(\tau)} \right)^q w(x)\,dx \right)^{\frac{q}{p-q}} \right. \\
&\qquad \left. \times \left(\sup_{y\in[s,\infty)} \frac{1}{\phi(y)} \right)^q \left(\int_t^s \tilde{\mathcal{B}}(t,y)\,dy \right) w(s)\,ds \right)^{\frac{p-q}{pq}} \\
&+ \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{t}{s+t} \right)^{\frac{m\alpha}{p(m-\alpha)}} \tilde{v}_2(s)\,ds \right)^{\frac{m-\alpha}{m\alpha}} \left(\int_0^\infty \left(\sup_{\tau\in[t,\infty)} \left(\frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)} \right)^{\frac{pq}{p-q}} \right.\right. \\
&\qquad \left.\left. \times (\tau+t)^{\frac{q}{q-p}} \right) \left(\int_0^t w \right)^{\frac{q}{p-q}} w(t)\,dt \right)^{\frac{p-q}{pq}} \\
&+ \sup_{t\in(0,\infty)} \left(\int_0^\infty \left(\frac{t}{s+t} \right)^{\frac{m\alpha}{p(m-\alpha)}} \tilde{v}_2(s)\,ds \right)^{\frac{m-\alpha}{m\alpha}} \\
&\qquad \times \left(\int_0^\infty \left(\int_0^t \left(\sup_{y\in[x,t]} \frac{B(y)^{\frac{1}{\alpha}}}{\phi(y)} \right)^q w(x)\,dx \right)^{\frac{q}{p-q}} \right. \\
&\qquad \left. \times \left(\sup_{\tau\in[t,\infty)} \left(\frac{B(\tau)^{\frac{1}{\alpha}}}{\phi(\tau)} \right)^q (\tau+t)^{\frac{q}{q-p}} \right) w(t)\,dt \right)^{\frac{p-q}{pq}}.
\end{aligned}
$$

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