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On a Boundary Value Problem for Operator-Differential Equations in Hilbert Space

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Abstract. In this paper, we study regular and Fredholm solvability of a Neumann type boundary value problem for a second order elliptic type equation with operator coefficients for a separable Hilbert space on a finite domain. The conditions of regular and Fredholm solvability for the given problem in terms of only the coefficients of the equation are found. The estimates for intermediate derivatives operators are obtained. These estimates determine the solvability conditions for our problem. Note that the considered operator equations have variable coefficients.

Key Words and Phrases: operator-differential equation, Hilbert space, linear operator, spectrum, self adjoint operator, normal operator, regular solvability, Fredholm solvability.

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1. Introduction

Solvability of operator-differential equations has been studied by many authors, since they have significant applications in various problems of mathematical analysis, differential equations and in other fields. The Cauchy problem was first studied by E. Hille, K. Iosido, T. Kato and others. Later, the boundary value problems for elliptic operator-differential equations have been studied by A.A. Dezin [6], V.I. Gorbachuk and M.L. Gorbachuk [11], M.G. Krein [12], S.Ya. Yakubov [21] and others. Boundary value problems for operator-differential equations on a semi-axis have been considered by M.G.Gasymov [8], A. Dubinsky [7], S.S. Mirzoev [19], A.A. Shkalikov [20] and other authors. Boundary value problems in an infinite domain with discontinuous coefficients A.R. Aliyev [4, 5], G.M. Gasymova [9, 10], S.S. Mirzoev, A.R. Aliev, L.M. Rustamova [17, 18], S.S. Mirzoev, A.R. Aliyev, G.M. Gasimova [16].

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In a finite domain, the boundary value problems have been studied, for example, by S.S. Mirzoev and G.A. Agaeva [14, 15], G.A. Agaeva [1, 2, 3].

Let H be a separable Hilbert space with scalar derivatives (x,y), C be a positive-definite operator in H with domain of definition $D(C)$. Then the domain of definition of the operator C^{γ} becomes a Hilbert space H_{γ} with scalar product $(x,y)_{\gamma}=(C^{\gamma}x,C^{\gamma}y)$, $\gamma\geq 0$. For $\gamma=0$ we assume that $H_0=H$ and $(x,y)_0=(x,y)$

Denote by $L_2((0,1):H)$ a Hilbert space of functions determined almost everywhere in $(0,1)$ with

$$\|f\|_{L_2((0,1):H)}=\left(\int_0^1\|f(t)\|^2\,dt\right)^{1/2}<\infty.$$

Following [13], we determine the Hilbert space

$$W_2^2((0,1):H)=u:C^2u\in L_2((0,1):H),\ u''\in L_2((0,1):H)$$

with the norm

$$\|u\|_{W_2^2((0,1):H)}=\left(\left\|u''\right\|^2_{L_2((0,1):H)}+\|Cu\|^2_{L_2((0,1):H)}\right)^{1/2}.$$

Note that the following assertions are true for the functions from $W_2^2((0,1):H)$ [13]:

1° For any $u\in W_2^2((0,1):H)$ we have the following inequality (theorem on intermediate derivatives):

$$||Cu'||_{L_2((0,1):H)}\leq const||u||_{W_2^2((0,1):H)}.$$

2° For any $t_0\in[0,1]$, there exist $u(t_0)$ and $u'(t_0)$. Moreover, $u(t_0)\in H_{3/2}$, $u'(t_0)\in H_{1/2}$ and we have the inequality (the trace theorem)

$$||u(t_0)||_{3/2}\leq const||u||_{W_2^2((0,1):H)},$$

and

$$||u'(t_0)||_{1/2}\leq const||u||_{W_2^2((0,1):H)}.$$

We determine the subspace $\overset{\circ}{W}{}_2^2((0,1):H)$ of the space $W_2^2((0,1):H)$ as follows:

$$\overset{\circ}{W}{}_2^2((0,1):H)=\{u:u\in W_2^2((0,1):H)\,,\ u'(0)=0,\ u'(1)=0\}.$$

It follows from the trace theorem that $\overset{\circ}{W}{}_2^2((0,1):H)$ is a complete Hilbert space. In the Hilbert space H we consider the equation

$$L(d/dt)u(t) = -u''(t) + \rho(t)A^2u(t) +$$
$$+(A_1+T_1)u'(t) + (A_2+T_2)u(t) = f(t),\ t \in (0,1) \tag{1}$$

with boundary conditions

$$u'(0) = 0 \quad , \quad u'(1) = 0, \tag{2}$$

where the operator coefficients satisfy the following conditions:
1) A is a normal operator with completely continuous inverse A^{-1}, whose spectrum is contained in the angular sector

$$S_\varepsilon = \ \{\lambda : |\arg\lambda| \le \varepsilon,\ 0 \le \varepsilon < \pi/2\ \};$$

2) $\rho(t)$ is a numeric function determined almost everywhere in the interval $(0,1)$, is measurable and bounded. Moreover, $\alpha \le \rho(t) \le \beta$, where $\alpha > 0$, $\beta > 0$;
3) the operators $B_1 = A_1A^{-1}$ and $B_2 = A_2A^{-2}$ are bounded in H;
4) the operators $K_1 = T_1A^{-1}$ and $K_2 = T_2A^{-2}$ are completely continuous in H.

Condition 1) implies that the operator can be represented in the form of $A = UC$, where C is a positive-definite operator, and U is a unitary operator. Moreover,

$$Cx = \sum_{k=1}^{\infty} \mu_k(x,e_k)e_k,\ \ Ux = \sum_{k=1}^{\infty} e^{i\varphi_k}(x,e_k)e_k,$$

where

$$Ae_k = \lambda_k e_k,\ \ \lambda_k = \mu_k e^{i\varphi_k},\ \ |\lambda_k| = \mu_k,\ \ \lambda_k = \ \mu_k e^{i\varphi_k},\ \varphi_k = \arg\lambda_k \in S_\varepsilon,$$
$$k = 1,2,...,\ \mu_1 \le \mu_2 \le ... \le \mu_k... .$$

Definition 1. *If for $f(t) \in L_2((0,1):H)$ there exists $u(t) \in W_2^2((0,1):H)$, satisfying the equation* (1) *almost everywhere in the interval* $(0,1)$*, then $u(t)$ is called a regular solution of the equation* (1)*.*

Definition 2. *If for any $f(t) \in L_2((0,1):H)$ there exists a regular solution $u(t)$ of the equation* (1) *satisfying boundary conditions* (2) *in the sense of convergence*

$$\lim_{t\to 0}||u'(t)||_{1/2} = 0\ ,\quad \lim_{t\to +0}||u'(1-t)||_{1/2} = 0 \tag{3}$$

and the estimate

$$||u(t)||_{W_2^2((0,1):H)}\ \le\ const||f||_{L_{2((0,1):H)}}, \tag{4}$$

then the problem (1)–(2) *is called regularly solvable.*

Definition 3. *If there exist finite-dimensional spaces* $\tilde{W}_2^2((0,1):H) \subset \overset{\circ}{W}{}_2^2((0,1):H)$ *and* $\tilde{L}_2((0,1):H) \subset L_2((0,1):H)$*, moreover, if* $\dim \tilde{W}_2^2((0,1):H) = \dim \tilde{L}_2((0,1):H)$ *and for any* $f(t) \in L_2((0,1):H)\Theta\tilde{L}_2((0,1):H)$ *there exists regular solution of equation* (1) $u(t) \in \tilde{W}_2^2((0,1):H)$ *satisfying the boundary conditions in the sense of* (3) *and the estimate* (4) *holds, then the problem* (1), (2) *is called Fredholm solvable.*

In the space $\overset{\circ}{W}{}_2^2((0,1):H)$ we define the following operators that act in $L_2((0,1)$:

$$Lu = P_0u + P_1u + Tu, \;\; u \in \overset{0}{W}{}_2^2((0,1):H),$$

where

$$P_0u = -u'' + \rho(t)A^2u, \; u \in \overset{0}{W}{}_2^2((0,1):H),$$

$$P_1u = A_1u' + A_2u, \; u \in \overset{0}{W}{}_2^2((0,1):H),$$

$$Tu = T_1u' + T_2u, \; , u \in \overset{0}{W}{}_2^2((0,1):H).$$

Note that from the theorem on intermediate derivatives it follows that each of these operators is continuous in $u \in \overset{0}{W}{}_2^2((0,1):H)$. Indeed,

$$\|P_0u\|_{L_2((0,1):H)} \leq \|u^{''}\| + \beta\|A^2u\|_{L_2((0,1):H)} \leq const\|u\|_{W_2^2((0,1):H)}$$

$$\|P_1u\|_{L_2((0,1):H)} \leq \|A_1u^{'}\| + \|A_2u^{'}\|_{L_2((0,1):H)} \leq$$

$$\leq \|A_1A^{-1}\| \, \|Au^{'}\|_{L_2((0,1):H)} + \|A_2A^{-2}\| \, \|A^2u\|_{L_2((0,1):H)} =$$

$$= \|B_1\| \, \|Cu'\|_{L_2((0,1):H)} + ||B_2|| \, ||C^2u||_{L_2((0,1):H)} \leq const\|u\|_{W_2^2((0,1):H)}$$

$$\|Tu\|_{L_2((0,1):H)} \leq \|K_1\| \, \|Cu'\|_{L_2((0,1):H)} + \|K_2\|||C^2u||_{L_2(0,1):H)} \leq const\|u\|_{W_2^2((0,1):H)}.$$

Thus, the solvability of the problem (1),(2) is reduced to the solvability of the equation

$$Lu = P_0u + P_1u + Tu = f,$$

where $f(t) \in L_2((0,1):H)$, while $u(t) \in \overset{0}{W}{}_2^2((0,1):H)$.

2. Some results

Theorem 1. *Let the conditions 1) be fulfilled. Then for all* $u \in \overset{0}{W}{}_2^2((0,1):H)$ *we have the inequalities*

$$\left\|Au'\right\|_{L_2((0,1):H)} \leq d_1(\varepsilon)\left\|P_0 u\right\|_{L_2((0,1):H)} \tag{5}$$

and

$$\left\|A^2 u\right\|_{L_2((0,1):H)} \leq d_0(\varepsilon)\left\|P_0 u\right\|_{L_2((0,1):H)}, \tag{6}$$

where

$$d_1(\varepsilon)=\frac{1}{2\sqrt{\alpha}}\,\frac{1}{cos\varepsilon}\quad (0\leq\varepsilon<\pi/2)\ \ ,\ \ d_2(\varepsilon)=\begin{cases}\frac{1}{\alpha}, & 0\leq\varepsilon\leq\pi/4\\ \frac{1}{\alpha\sqrt{2}}\,\frac{1}{cos\varepsilon}, & \pi/4\leq\varepsilon<\pi/2.\end{cases} \tag{7}$$

Proof. Denote $f=P_0u$. Then

$$||\rho^{-1/2}f||^2_{L_2((0,1):H)} = ||-\rho^{-1/2}u''+\rho^{1/2}A^2u||^2_{L_2((0,1):H)} =$$

$$=||-\rho^{-1/2}u''||^2_{L_2((0,1):H)}+||\rho^{1/2}A^2u||^2_{L_2((0,1):H)}-2Re(u'',A^2u)_{L_2((0,1):H)}. \tag{8}$$

On the other hand, considering $u \in \overset{0}{W}{}_2^2((0,1):H)$ $(u'(0)=u'(1)=0)$, after integrating by parts we have

$$(u'',A^2u)_{L_2((0,1):H)} = \int_0^1 (u''(t),A^2u(t))dt =$$

$$=(C^{1/2}u'(t)^2,U^2C^{3/2}u(t))|_0^1-(A^*u',Au')_{L_2((0,1):H)} = -(A^*u',Au')_{L_2((0,1):H)}.$$

Then it follows from the equality (8) that

$$||\rho^{-1/2}f||^2_{L_2((0,1):H)} = ||\rho^{-1/2}u''||^2_{L_2((0,1):H)}+$$

$$+||\rho^{1/2}A^2u||^2_{L_2((0,1):H)}+2Re(A^*u',Au')_{L_2((0,1):H)}.$$

Now, using spectral expansion of the operator A , we obtain

$$Re(A^*u',Au')_{L_2((0,1):H)} \geq \cos 2\varepsilon||Cu'||^2_{L_2((0,1):H)} = \cos 2\varepsilon||Au'||^2_{L_2((0,1):H)}.$$

Then the equality (8) yields

$$\|\rho^{-1/2}f\|^2_{L_2((0,1):H)} \geq \|\rho^{-1/2}u''\|^2_{L_2((0,1):H)}+$$

$$+\|\rho^{1/2}A^2u\|_{L_2((0,1):H)}+2\cos 2\varepsilon\|Au'\|_{L_2((0,1):H)}. \qquad (9)$$

Since $u'(0)=u'(1)=0$, after integration by parts we have:

$$\|Au'\|^2_{L_2((0,1):H)}=\|Cu'\|^2_{L_2((0,1):H)}=(Cu',Cu')_{L_2((0,1):H)}=$$

$$=\left(C^{3/2}u(t),C^{1/2}u'(t)|_0^1-(u'',C^2u)_{L_2((0,1):H)}\right)=$$

$$=-\left(u'',C^2u\right)_{L_2((0,1):H)}=-(\rho^{-1/2}u'',\rho^{1/2}C^2u)_{L_2((0,1):H)}\leq$$

$$\leq\frac{1}{2}\left(\|\rho^{-1/2}u''\|^2_{L_2((0,1):H)}+\|\rho^{1/2}C^2u\|^2_{L_2((0,1):H)}\right)=$$

$$=\frac{1}{2}\left(\|\rho^{-1/2}u''\|^2_{L_2((0,1):H)}+\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)}\right).$$

Taking into account (9) in the last inequality, we obtain

$$\|Au'\|^2_{L_2((0,1):H)}\leq\frac{1}{2}\left(\|\rho^{-1/22}f\|^2_{L_2((0,1):H)}-2\cos 2\varepsilon\|Au'\|^2_{L_2((0,1):H)}\right)$$

or

$$(1+\cos 2\varepsilon)||Au'||^2_{L_2((0,1):H)}\leq\frac{1}{2}(\|\rho^{-1/2}f\|^2_{L_2((0,1):H)},$$

i.e.

$$||Au'||^2_{L_2((0,1):H)}\leq\frac{1}{4\cos^2\varepsilon}\|\rho^{-1/2}f\|^2_{L_2((0,1):H)} \qquad (10)$$

or

$$||Au'||_{L_2((0,1):H)}\leq\frac{1}{2\cos\varepsilon}||\rho^{-1/2}f||_{L_2((0,1):H)}\leq\frac{1}{2\sqrt{\alpha}\cos\varepsilon}\|f\|=$$

$$=\frac{1}{2\sqrt{\alpha}\cos\varepsilon}\|P_0u\|_{L_2((0,1):H)}\ \ (0\leq\varepsilon<\pi/2).$$

Inequality (5) is proved. ◄

Now we prove the inequality (6).

a) Let $0\leq\varepsilon\leq\pi/4$ $(\cos 2\varepsilon\geq 0)$. Then from (9) we obtain

$$\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)}\leq||\rho^{-1/2}f||^2_{L_2((0,1):H)},$$

i.e.

$$\|A^2u\|_{L_2((0,1):H)}=\|\rho^{-1/2}\rho^{1/2}A^2u\|_{L_2((0,1):H)}\leq\frac{1}{\sqrt{\alpha}}\|\rho^{1/2}A^2u\|_{L_2((0,1):H)}\leq$$

$$\leq\frac{1}{\sqrt{\alpha}}\|\rho^{-1/2}f\|_{L_2((0,1):H)})\leq\frac{1}{\alpha}\|f\|_{L_2((0,1):H)},$$

i.e. inequality (6) for $0 \le \varepsilon \le \pi/4$ is proved.

b) Let $\pi/4 < \varepsilon \le \pi/2$ $(\cos 2\varepsilon \le 0)$. Then inequality (9) yields

$$\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)} \le \|\rho^{-1/2}f\|^2_{L_2((0,1):H)} - 2\cos 2\varepsilon \|Au'\|^2_{L_2((0,1):H)}.$$

Since $\alpha||A^2u||^2_{L_2((0,1):H)} \le ||\rho^{1/2}A^2u||^2_{L_2((0,1):H)}$ and $\cos 2\varepsilon \le 0$, from inequality (5) we obtain

$$\alpha\|A^2u\|^2_{L_2((0,1):H)} \le \|\rho^{-1/2}f\|^2_{L_2((0,1):H)} - \frac{2\cos 2\varepsilon}{4\alpha\cos^2\varepsilon}\|f\|^2_{L_2((0,1):H)} \le$$

$$\le \frac{1}{\alpha}\|f\|^2_{L_2((0,1):H)} - \frac{\cos 2\varepsilon}{2\alpha\cos^2\varepsilon}\|f\|^2_{L_2((0,1):H)} = \frac{1}{2\alpha\cos^2\varepsilon}\|f\|^2_{L_2((0,1):H)}$$

or

$$||A^2u||^2_{L_2((0,1):H)} \le \frac{1}{2\alpha^2\cos^2\varepsilon}||f||^2_{L_2((0,1):H)},$$

i.e.

$$||A^2u||_{L_2((0,1):H)} \le \frac{1}{\alpha\sqrt{2}\cos\varepsilon}||f||_{L_2((0,1):H)}.$$

The theorem is proved.

3. Basic results

Here we show the conditions for regular and Fredholm solvability of problem (1),(2).

Theorem 2. *The operator* $P_0 : \overset{0}{W}{}_2^2((0,1):H) \to L_2((0,1):H)$ *is isomorphic.*

Show that $KerP_0 = \{0\}$. Indeed, if $P_0u = 0$, then from the inequality (6) it follows that $A^2u = 0$, i.e. $u = 0$. We now show that for any $f \in L_2((0,1):H)$ the equation $P_0u = f$ has a solution. If we consider the operator P_0 in $L_2((0,1):H)$, then it is obvious that $D(P_0) = \overset{0}{W}{}_2^2((0,1):H)$ and its adjoint operator has the domain of definition $\overset{0}{W}{}_2^2((0,1):H)$, and for $u \in \overset{0}{W}{}_2^2((0,1):H)$

$$P_0^*u = -u'' + \rho(t)A^{*2}u, \quad u \in \overset{0}{W}{}_2^2((0,1):H).$$

Since the operator A^* possesses all the properties of the operator A, we have $KerP_0^* = \{0\}$. Then ImP_0 is an everywhere dense set in $L_2((0,1):H)$. On the other hand,

$$\|P_0u\|^2_{L_2((0,1):H)} = \|f\| \ge \alpha\|\rho^{-1/2}f\|^2_{L_2((0,1):H)}) \ge \|\rho^{-1/2}u''\|^2_{L_2((0,1):H)})+$$

$$+||\rho^{1/2}A^2u||^2_{L_2((0,1):H)}) + 2\cos 2\varepsilon ||Au'||^2_{L_2((0,1):H)}.$$

Obviously, for $0 \le \varepsilon \le \pi/4$

$$\|P_0u\|^2_{L_2((0,1):H)} \ge \alpha \|\rho^{-1/2}u''\|^2_{L_2((0,1):H)} +$$

$$+\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)}) \ge \alpha(\frac{1}{\beta}\|u''\|^2_{L_2((0,1):H)} +$$

$$+\alpha\|A^2u\|_{L_2((0,1):H)} \ge const\|u\|_{W_2((0,1):H)} \ge const\|u\|_{L_2((0,1):H)}.$$

Let now $\pi/4 \le \varepsilon < \pi/2$. Then $\cos 2\varepsilon \le 0$. By (9), from equality (10) we obtain

$$\|P_0u\|^2_{L_2((0,1):H)} \ge \|\rho^{-1/2}u''\|^2_{L_2((0,1):H)} + \|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)} + \frac{2\cos 2\varepsilon}{4\cos^2\varepsilon}\ \frac{1}{\alpha}\|f\|^2,$$

i.e.

$$\|f\|^2_{L_2((0,1):H)} \ge \alpha\|\rho^{-1/2}f\|^2_{L_2((0,1):H)} \ge \alpha\left(\|\rho^{-1/2}u''\|^2_{L_2((0,1):H)} +\right.$$

$$\left.+\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)} + \frac{1}{\alpha}\frac{\cos 2\varepsilon}{2\cos^2\varepsilon}\ \|f\|^2_{L_2((0,1):H)}\right) = \alpha\left(\|\rho^{-1/2}u''\|^2_{L_2((0,1):H)} +\right.$$

$$\left.+\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)}\right) + \frac{\cos 2\varepsilon}{2\cos^2\varepsilon}\ \|f\|^2,$$

i.e.

$$\left(1 - \frac{\cos 2\varepsilon}{2\cos^2\varepsilon}\right)\ \|f\|^2) \ge \alpha\left(\frac{1}{\beta}\|u''\|^2_{L_2((0,1):H)} + \alpha||A^2u||^2_{L_2((0,1):H)}\right)$$

Hence we have

$$\frac{1}{2\cos^2\varepsilon}\ ||f||^2) \ge \min(1,\alpha^2)const\|u''\|^2_{W_2^2((0,1):H)},$$

i.e.

$$\|P_0u\|_{L_2((0,1):H)} \ge const\|u\|_{W_2^2((0,1):H)} \ge const\|u\|_{L_2((0,1):H)}.$$

Consequently, the image of the operator P_0 is closed, i.e. $ImP_0 = L_2((0,1):H)$. Then there exists a bounded operator ${P_0}^{-1}$, i.e. P_0 is an isomorphism.

The theorem is proved.

We have

Theorem 3. *Let the conditions 1)-3) be fulfilled. If*

$$q(\varepsilon) = d_1(\varepsilon)||B_1|| + d_2||B_2|| < 1, \tag{11}$$

where the numbers $d_1(\varepsilon)$ and $d_2(\varepsilon)$ are determined from Theorem 1 by the equalities (7), then the operator $P = P_0 + P_1$ isomorphically maps the space $\overset{0}{W}{}^2_2((0,1):H)$ onto $L_2((0,1):H)$.

Proof. Show that the operator P isomorphically maps the space $\overset{0}{W}{}_2^2((0,1):H)$ onto $L_2((0,1):H)$ subject to the theorem conditions. For any $f \in L_2((0,1):H)$ we consider the equation

$$Pu = P_0u + P_1u = f, \quad f \in L_2((0,1):H) \;\; u \in \overset{0}{W}{}_2^2((0,1):H). \tag{12}$$

Since by Theorem 2 the operator $P_0 : \overset{0}{W}{}_2^2((0,1):H) \to L_2((0,1):H)$ is an isomorphisim, the inverse operator

$P_0^{-1} : L_2((0,1):H) \to \overset{0}{W}{}_2^2((0,1):H)$ is bounded. Then, denoting $\omega = P_0u$, we obtain $u = P_0^{-1}\omega$. Obviously, for any $\omega \in L_2((0,1):H)$ there exists $u \in \overset{0}{W}{}_2^2((0,1):H)$, for which $u = P_0^{-1}\omega$.

Then from (12) we obtain the following equation in the space $L_2((0,1):H)$:

$$\omega + P_1\,P_0^{-1}\omega = f \quad , \quad \omega,\, f \in L_2((0,1):H).$$

For any $\omega \in L_2((0,1):H)$

$$||P_1P_0^{-1}\omega||_{L_2((0,1):H)} = ||P_1u||_{L_2((0,1):H)} \le ||A_1u'||_{L_2((0,1):H)}+$$

$$+||A_2u'||_{L_2((0,1):H)} \le ||A_1A^{-1}||\,||Au'||_{L_2((0,1):H)} + ||A_2A^{-2}||\,||A^2u||_{L_2((0,1):H)}.$$

Taking into account the results of Theorem 1, we have

$$\|P_1P_0^{-1}\omega\|_{L_2((0,1):H)} \le ||B_1||d_1(\varepsilon)||P_0u||_{L_2((0,1):H)}+$$

$$+||B_2||d_2(\varepsilon)||P_0u||_{L_2((0,1):H)} = q(\varepsilon)||P_0u||_{L_2((0,1):H)} = q(\varepsilon)||\omega||_{L_2((0,1):H)}.$$

Since $q(\varepsilon) < 1$, the operator $E + P_1P_0^{-1}$ is invertible in H, and

$$u = P_0^{-1}(E + P_1P_0^{-1})^{-1}f.$$

Obviously, we have the inequality

$$||u||_{W_2^2((0,1):H)} \le const||f||_{L_2((0,1):H)}.$$

The isomorphism of the operator P is proved. ◀

Note that if we replace the condition (11) by

$$d_1(\varepsilon)||B_1 + K_1|| + d_2(\varepsilon)||B_2 + K_2|| < 1,$$

then the operator $L = P + T$ is also acting isomorphically from $\overset{0}{W}{}_2^2((0,1):H)$ into $L_2((0,1):H)$. But we do not use complete continuity of the operators K_i $(i = 1,2)$. Their continuity suffices here.

We now prove the Fredholm property of the operator $L:\ \overset{0}{W}{}_2^2((0,1):H) \to L_2((0,1):H)$.

We have

Theorem 4. *Let the conditions 1)-4) be fulfilled, and the inequality (11) hold. Then the operator L is a Fredholm operator, i.e.*

a) $\dim KerL = \dim\ KerL^* < \infty$,

b) ImL *is a closed set in* $L_2((0,1):H)$.

Proof. We rewrite the equation $Lu = f$, $f \in L_2((0,1):H)$, $u \in \overset{0}{W}{}_2^2((0,1):H)$ as

$$Lu = Pu + Tu = f,$$

where $P = P_0 + P_1$, while T is determined as $Tu = T_1 u' + T_2 u$, $u \in \overset{0}{W}{}_2^2((0,1):H)$.

We proved that, subject to the condition of the theorem, the operator P is an isomorphism between the spaces $\overset{0}{W}{}_2^2((0,1):H)$ and $L_2((0,1):H)$. We write the operator L in the form

$$Lu = Pu + Tu,\ u \in \overset{0}{W}{}_2^2((0,1):H).$$

Since for $u \in \overset{0}{W}{}_2^2((0,1):H)$

$$Tu = T_1 u' + T_2 u = T_1 A^{-1} A u' + T_2 A^{-2} A u = K_1 A u' + K_2 A^2 u,$$

where K_1 and K_2 are completely continuous operators in H, it follows from the results of [21, p. 83-84] that for any $\varepsilon > 0$

$$\|K_1 A u'\|_{L_2((0,1):H)} \le \varepsilon \|u\|_{W_2^2((0,1):H)} + \eta(\varepsilon)||u||_{L_2((0,1):H)}$$

and

$$||K_2 A^2 u||_{L_2((0,1):H)} \le \varepsilon ||u||_{W_2^2((0,1):H)} + \eta(\varepsilon)||u||_{L_2((0,1):H)}.$$

Hence for rather small $\varepsilon_1 > 0$ we have:

$$||Tu||_{L_2((0,1):H)} \le ||T_1 A^{-1}||||K_1 A u'||_{L_2((0,1):H)} + ||T_2 A^{-2}||||K_2 A^2 u||_{L_2((0,1):H)} \le$$

$$\leq \varepsilon ||u||_{W_2^2((0,1):H)} + \eta(\varepsilon_1)||u||_{L_2((0,1):H)} \quad (\varepsilon_1 = 2\varepsilon). \tag{13}$$

Let us show that from inequality (13) it follows that the operator $T : \overset{0}{W}{}_2^2((0,1) : H) \to L_2((0,1) : H)$ is completely continuous. Since A^{-1} is a completely continuous operator, the imbedding $W_2^2((0,1) : H) \hookrightarrow L_2((0,1) : H)$ is compact. Then the set $Q = \{u \; : \; u \in W_2^2((0,1) : H), \, ||u||_{W_2^2((0,1):H)} \leq c\}$ is compact in $L_2((0,1) : H)$.

Therefore, from this set we can select the sequence $\{u\}_{n=1}^{\infty} \in Q$, that converges in the norm of $L_2((0,1) : H)$, i.e. $||u_n - u_m||_{L_2((0,1):H)} \to 0$ as $n, m \to \infty$.

As $u \in Q$, we have $||u_n - u_m||_{W_2^2((0,1):H)} \leq 2c$. Then

$$||Tu_n - Tu_m||_{L_2((0,1):H)} \leq 2\varepsilon_1 c + \eta(\varepsilon_1)||u_n - u_m||_{L_2((0,1):H)},$$

since we can find n_0 such that $||u_n - u_m||_{L_2((0,1):H)} < \varepsilon_2$ for $n > n_0\,, m > n_0$, where the numbers $\varepsilon > 0$, $\varepsilon_1 > 0\,, \varepsilon_2 > 0$ are rather small. For $n > n_0$ and $m > n_0$ with rather small $\delta > 0$

$$||Tu_n - Tu_m|| \leq (2\varepsilon_1 c + \eta(\varepsilon_1)\varepsilon) < \delta, \; n, m > n_0,$$

i.e. the sequence $\{Tu_n\}$ converges in the space $L_2((0,1) : H)$. This means that the operator $T : \overset{0}{W}{}_2^2((0,1) : H) \to L_2((0,1) : H)$ is compact and $E + P^{-1}T$ is a Fredholm operator in $\overset{0}{W}{}_2^2((0,1) : H)$. Therefore the image of the operator $E + P^{-1}T$ is closed . From Theorem 2 it follows that the operator

$$L = P + T = P(E + P^{-1}T)$$

is a Fredholm operator since P is an isomorphic operator from $\overset{0}{W}{}_2^2((0,1) : H)$ to $L_2((0,1) : H)$.

The theorem is proved. ◀

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