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A General Result on (ϕ, δ)-Monotone Sequences

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Abstract. In this paper, a theorem dealing with the $|\bar{N}, p_n|_k$ summability factors of infinite series has been generalized to $|A, p_n, \beta; \gamma|_k$ summability method by using (ϕ, δ)-monotone sequences. This new theorem also includes some new results.

Key Words and Phrases: absolute matrix summability, infinite series, summability factors, (ϕ, δ) monotone sequences, Hölder's inequality, Minkowski's inequality.

2010 Mathematics Subject Classifications: 26D15, 40A05, 40C05, 40D15, 40G99

1. Introduction

A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. A sequence (μ_n) is said to be (ϕ, δ)-monotone if and only if $\mu_n \geq 0$, $\mu_n \to 0$ ultimately and $\Delta\mu_n \geq -\delta_{n+1}$, where (δ_n) is a sequence of non-negative numbers, (ϕ_n) is a positive monotone increasing sequence and $\sum \phi_n\delta_n < \infty$ (see [15]). Let $\sum a_n$ be an infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-m} = p_{-m} = 0, m \geq 1).$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$$

defines the sequence (σ_n) of the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [2]). The series $\sum a_n$ is said to be summable

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$\left|\bar{N},p_n\right|_k$, $k\geq 1$, if (see [1])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sigma_n-\sigma_{n-1}\right|^k<\infty.$$

Let $A=(a_{nv})$ be a normal matrix, i.e. a lower triangular matrix of non-zero diagonal entries. The series $\sum a_n$ is said to be summable $|A,p_n,\beta;\gamma|_k$, ($k\geq 1$, $\gamma\geq 0$ and β is a real number), if (see [10])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}|A_n(s)-A_{n-1}(s)|^k<\infty,$$

where

$$A_n(s)=\sum_{v=0}^{n}a_{nv}s_v,\quad n=0,1,\dots$$

If we take $\beta=1$, then $|A,p_n,\beta;\gamma|_k$ summability reduces to $|A,p_n;\gamma|_k$ summability method (see [5]). If we take $\beta=1$ and $\gamma=0$, then $|A,p_n,\beta;\gamma|_k$ summability reduces to $|A,p_n|_k$ summability method (see [16]).

2. Known Result

In [13], the following theorem dealing with $|\bar{N},p_n|_k$ summability factors of infinite series has been proved.

Theorem 1. *Let (p_n) be a sequence of positive numbers such that*

$$P_n=O(np_n)\quad as\quad n\to\infty. \tag{1}$$

Suppose that there exists a sequence of numbers (μ_n), which is (ϕ,δ) monotone with $\sum\mu_n\Delta\phi_n$ is convergent. If the conditions

$$\sum_{n=1}^{m}n|\Delta^2\mu_n|\phi_n=O(1)\quad as\quad m\to\infty, \tag{2}$$

and

$$\sum_{n=1}^{m}\frac{p_n}{P_n}|t_n|^k=O(\phi_m)\quad as\quad m\to\infty, \tag{3}$$

where $t_n=\dfrac{1}{n+1}\sum_{v=0}^{n}va_v$, are satisfied, then the series $\sum a_n\mu_n$ is summable $|\bar{N},p_n|_k$, $k\geq 1$.

If we take $\mu_n = \dfrac{2^{(-1)^n}}{n^4}$ and $\phi_n = \log n$, the conditions of Theorem 1 are satisfied. But the sequence (μ_n) does not satisfy the conditions of the theorem of Mazhar [4] on $|C,1|_k$ summability.

Lemma 1. *[13] Under the conditions of Theorem 1, we get*

$$n\phi_n|\Delta\mu_n| = O(1) \quad as \quad n \to \infty. \tag{4}$$

Lemma 2. *[15] If the sequence (μ_n) is (ϕ, δ) monotone and $\sum \mu_n \Delta\phi_n$ converges, then*

$$\mu_n\phi_n = o(1) \quad as \quad n \to \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} \phi_{n+1}|\Delta\mu_n| < \infty. \tag{6}$$

3. Main Result

There are many papers on absolute matrix summability (see [3, 6, 7, 8, 9, 11, 12]). The aim of this paper is to generalize Theorem 1 to $|A, p_n, \beta; \gamma|_k$ summability. Before stating the main theorem, we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are given as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \tag{7}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots \tag{8}$$

Note that $\bar{A}$ and $\hat{A}$ are well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{9}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{10}$$

Now let us prove the following theorem.

Theorem 2. *Let $A=(a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0}=1, \quad n=0,1,..., \tag{11}$$

$$a_{n-1,v}\geq a_{nv} \quad for \quad n\geq v+1, \tag{12}$$

$$a_{nn}=O\left(\frac{p_n}{P_n}\right), \tag{13}$$

$$|\hat{a}_{n,v+1}|=O\left(v\left|\Delta_v\hat{a}_{nv}\right|\right), \tag{14}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|=O\left(\left(\frac{P_v}{p_v}\right)^{\beta(\gamma k+k-1)-k}\right) \quad as \quad m\to\infty, \tag{15}$$

where $\Delta_v(\hat{a}_{nv})=\hat{a}_{nv}-\hat{a}_{n,v+1}$. If all conditions of Theorem 1 are satisfied with the condition (3) *replaced by*

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k}|t_n|^k=O(\phi_m) \quad as \quad m\to\infty \tag{16}$$

then the series $\sum a_n\mu_n$ is summable $|A,p_n,\beta;\gamma|_k$, $k\geq 1$, $\gamma\geq 0$ and $-\beta(\gamma k+k-1)+k>0$.

Proof. Let (Θ_n) denote A-transform of the series $\sum a_n\mu_n$. Then, by (7) and (8), we have

$$\bar{\Delta}\Theta_n = \sum_{v=1}^{n}\frac{\hat{a}_{nv}\mu_v}{v}va_v.$$

By Abel's transformation, we get

$$\begin{aligned}
\bar{\Delta}\Theta_n &= \sum_{v=1}^{n-1}\Delta_v\Big(\frac{\hat{a}_{nv}\mu_v}{v}\Big)\sum_{r=1}^{v}ra_r+\frac{\hat{a}_{nn}\mu_n}{n}\sum_{r=1}^{n}ra_r\\
&= \sum_{v=1}^{n-1}\Delta_v\Big(\frac{\hat{a}_{nv}\mu_v}{v}\Big)(v+1)t_v+\frac{\hat{a}_{nn}\mu_n}{n}(n+1)t_n\\
&= \sum_{v=1}^{n-1}\frac{v+1}{v}\Delta_v(\hat{a}_{nv})\mu_v t_v+\sum_{v=1}^{n-1}\frac{v+1}{v}\hat{a}_{n,v+1}\Delta\mu_v t_v\\
&+ \sum_{v=1}^{n-1}\hat{a}_{n,v+1}\mu_{v+1}\frac{t_v}{v}+\frac{n+1}{n}a_{nn}\mu_n t_n\\
&= \Theta_{n,1}+\Theta_{n,2}+\Theta_{n,3}+\Theta_{n,4}.
\end{aligned}$$

To prove Theorem 2, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}|\Theta_{n,r}|^k<\infty \quad for \quad r=1,2,3,4.$$

First, using Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}|\Theta_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_v|^k|t_v|^k\right)\\
&\times \left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_v|^k|t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}|\mu_v||\mu_v|^{k-1}|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\gamma k+k-1)-k}|\mu_v||t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\mu_v|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\beta(\gamma k+k-1)-k}|t_r|^k\\
&+ O(1)|\mu_m|\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{\beta(\gamma k+k-1)-k}|t_r|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\mu_v|\phi_{v+1}+O(1)|\mu_m|\phi_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 2.

Now, since $v|\Delta\mu_v| = O(1/\phi_v) = O(1)$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}|\Theta_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\mu_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\mu_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\mu_v||t_v|^k\right)\\
&\times \left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\mu_v|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\mu_v||t_v|^k\right)\\
&\times \left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\mu_v||t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}v|\Delta\mu_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}v|\Delta\mu_v||t_v|^k\left(\frac{P_v}{p_v}\right)^{\beta(\gamma k+k-1)-k}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\mu_v|)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\beta(\gamma k+k-1)-k}|t_r|^k\\
&+ O(1)m\,|\Delta\mu_m|\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{\beta(\gamma k+k-1)-k}|t_r|^k\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta^2\mu_v|\phi_v+\sum_{v=1}^{m-1}|\Delta\mu_{v+1}|\phi_{v+1}+O(1)m\,|\Delta\mu_m|\,\phi_m\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2, Lemma 1 and Lemma 2.

Again using Hölder's inequality and by (14), we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}|\Theta_{n,3}|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\mu_{v+1}|\frac{|t_v|}{v}\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_{v+1}||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_{v+1}|^k|t_v|^k\right)\\
&\times \left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_{v+1}||t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}|\mu_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\gamma k+k-1)-k}|\mu_{v+1}||t_v|^k\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

as in $\Theta_{n,1}$.
Finally, we get

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}|\Theta_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}a_{nn}^k|\mu_n|^k|t_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k}|\mu_n||t_n|^k\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

as in $\Theta_{n,1}$.
Therefore, we obtain

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}|\Theta_{n,r}|^k = O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2,3,4.$$

This completes the proof of theorem. ◀

4. Conclusion

If we take $\beta = 1$, $\gamma = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 1. If we take $\beta = 1$, then we get a known result on $|A, p_n; \gamma|_k$ summability method (see [14]). Also, if we take $\beta = 1$ and $\gamma = 0$, then we get a new theorem involving $|A, p_n|_k$ summability. Moreover, if we take $\beta = 1$, $\gamma = 0$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a theorem of Mazhar [4] on $|C, 1|_k$ summability.

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