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Coefficient Estimates for Functions with Respect to Symmetric Points Based on Shell-Like Curves Defined by Convolution

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Abstract. Making use of the Hadamard product (or convolution), we find some estimates for symmetric points, related to shell-like curves connected with Fibonacci numbers. We determine the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class of Bi-univalent functions of the Bazilevič type of order γ. We also obtain the Fekete-Szegö result for the function class.

Key Words and Phrases: analytic functions, bi-univalent, Bazilevič, Fekete-Szegö, coefficient inequalities, starlike functions and convex functions, subordination.

2010 Mathematics Subject Classifications: 30C45, 30C80

1. Introduction and Motivation

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

analytic in the open unit disk $\mathcal{U}$. Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. It is well known that every function $f \in S$ has a function f^{-1}, defined by

$$f^{-1}[f(z)] = z\,;\,(z \in \mathcal{U})$$

and

$$f[f^{-1}(w)] = w\,; \qquad (\,|w| < r_0(f)\,;\, r_0(f) \geq \frac{1}{4}\,).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2 w^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \quad (2)$$

A function $f \in \mathcal{A}$ is said to be biunivalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$. Let f and g be analytic in the open unit disk $\mathcal{U}$. The function f is subordinate to g written as $f \prec g$ in $\mathcal{U}$, if there exists a function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathcal{U})$ such that $f(z) = g(w(z)), (z \in \mathcal{U})$.

Many derivative operators can be written in terms of convolution of some analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to better understand the geometric properties of such operators. The convolution or Hadamard product of two functions $f, g \in \mathcal{A}$ is denoted by $f * g$ and is defined as follows:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (3)$$

where $f(z)$ is given by (1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$.
Alamoush and Darus [1] introduced differential operator $D^k_{\alpha,\beta,\delta,\lambda} : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} D^k_{\alpha,\beta,\delta,\lambda} f(z) &= z + \sum_{n=2}^{\infty} [\lambda(\alpha+\beta-1)(n-1)]^k \mathcal{C}(\delta, n) z^n, \\ &= z + \sum_{n=2}^{\infty} \Upsilon_n^k \mathcal{C}(\delta, n) z^n, \end{aligned} \quad (4)$$

for $k = 0, 1, 2, \cdots, 0 < \alpha \le 1, 0 < \beta \le 1, \lambda \ge 0, \delta \ge 0, z \in \mathcal{U}, \mathcal{C}(\delta, n) = \binom{n+\delta-1}{\delta}$ and $\Upsilon_n^k = [\lambda(\alpha+\beta-1)(n-1)]^k$.

Also, they discussed several interesting geometrical properties exhibited by the operator $D^k_{\alpha,\beta,\delta,\lambda}$. Even though the parameters family of operators $D^k_{\alpha,\beta,\delta,\lambda}$ λ is a very specialized case of the one widely-(and extensively-) investigated by some other authors (see [2]-[3]), it is also easily seen that $D^k_{\alpha,\beta,\delta,\lambda}$ provides a generalization of the convolution between Ruscheweyh derivative operator [4] and Salagean derivative operator [5].

Recently, many authors investigated bounds for various subclasses of biunivalent function class Σ (see [6], [7]) and obtained non-sharp coefficient estimates for the first two coefficients $|a_2|$ and $|a_3|$ of (1). For $n \ge 4$, this problem is yet to be solved ([8]-[9]). In 1955, Bazilevič [10] introduced the following class of univalent functions in $\mathcal{U}$:

$$B_1(\mu) = \left(f \in \mathcal{A} : \Re\left(\frac{z^{1-\mu}(f'(z))}{[f(z)]^{1-\mu}} \right) > 0, \mu \ge 0, z \in \mathcal{U} \right). \quad (5)$$

This can be generalized as follows:

$$B_{\alpha,\mu} = \left(f \in \mathcal{A} : \Re\left(\frac{z^{1-\mu}\left(f'(z)\right)}{\left[f(z)\right]^{1-\mu}} \right) > \alpha, 0 < \alpha \leq 1, \mu \geq 0, z \in \mathcal{U} \right). \tag{6}$$

Several authors have discussed various subfamilies of Bazilevič functions of type Υ. In this paper, we find estimates for the coefficients $\mid a_2 \mid$ and $\mid a_3 \mid$ for functions in the new subclasses of function class Σ involving the operator $D^k_{\alpha,\beta,\delta,\lambda}$. Several closely-related classes are also considered and some relevant connections to earlier known results are pointed out.

In [11], Raina and J. Sokół showed that

$$\begin{aligned} p(z) &= \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} = \left(t+\frac{1}{t}\right)\frac{t}{1-t-t^2} \\ &= \frac{1}{\sqrt{5}}\left(t+\frac{1}{t}\right)\left(\frac{1}{1-(1-\tau)t}-\frac{1}{1-\tau t}\right) \\ &= \left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty} u_n t^n = 1+\sum_{n=2}^{\infty}\left(u_{n-1}+u_{n+1}\right)\tau^2 z^n, \end{aligned} \tag{7}$$

where

$$u_n = \frac{(1-\tau)^n-\tau^n}{\sqrt{5}}, \tau = \frac{1-\sqrt{5}}{2}, t = \tau z \qquad (n = 1, 2, \cdots). \tag{8}$$

This shows the relevant connection between p and the sequence of Fibonacci numbers u_n, such that $u_0 = 0; u_1 = 1; u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \cdots$. Raina and J. Sokół also got

$$\begin{aligned} p(z) &= 1+\sum_{n=1}^{\infty} p_n z^n \\ &= 1+(u_0+u_2)\tau z+(u_1+u_3)\tau^2 z^2+\sum_{n=3}^{\infty}(u_{n-3}+u_{n-2}+u_{n-1}+u_n)\tau_n z_n \\ &= 1+\tau z+3\tau^2 z^2+4\tau^3 z^3+7\tau^4 z^4+11\tau^5 z^5+\cdots . \end{aligned} \tag{9}$$

2. Function class $B_{\Sigma}^{k,\alpha,\beta,\delta,\lambda}$

In this section, we introduce a new subclass of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $\mid a_2 \mid$ and $\mid a_3 \mid$ for the function class by subordination.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then there exists an analytic function u such that $\mid u(z) \mid < 1$ in $\mathcal{U}$ and $p(z) = \overline{p}(u(z))$. Therefore, the function

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{10}$$

is in the class $\mathcal{P}$. It follows that

$$u(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_2 + c_1 c_3 - \frac{c_1^3}{4}\right)\frac{z^3}{4} + \cdots, \tag{11}$$

and

$$\begin{aligned}\overline{p}(u(z)) = 1 + \frac{\overline{p}_1 c_1 z}{2} + \left[\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\overline{p}_1 + \frac{c_1^2}{4}\overline{p}_2\right] z^2 + \\ \left[\frac{1}{2}\left(c_2 + c_1 c_3 - \frac{c_1^3}{4}\right)\overline{p}_1 + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\overline{p}_2 + \frac{c_1^3}{8}\overline{p}_3\right] z^3 + \cdots .\end{aligned} \tag{12}$$

And similarly, there exists an analytic function v such that $\mid v(w) \mid < 1$ in $\mathcal{U}$ and $p(w) = \overline{p}(v(w))$. Therefore, the function

$$k(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1 w + d_2 w^2 + \cdots \tag{13}$$

is in the class $\mathcal{P}(0)$. It follows that

$$v(w) = \frac{d_1 w}{2} + \left(d_2 - \frac{d_1^2}{2}\right)\frac{w^2}{2} + \left(d_2 + d_1 d_3 - \frac{d_1^3}{4}\right)\frac{w^3}{4} + \cdots, \tag{14}$$

and

$$\begin{aligned}\overline{p}(v(w)) = 1 + \frac{\overline{p}_1 d_1 w}{2} + \left[\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\overline{p}_1 + \frac{d_1^2}{4}\overline{p}_2\right] w^2 + \\ \left[\frac{1}{2}\left(d_2 + d_1 d_3 - \frac{d_1^3}{4}\right)\overline{p}_1 + \frac{1}{2}d_1\left(d_2 - \frac{d_1^2}{2}\right)\overline{p}_2 + \frac{d_1^3}{8}\overline{p}_3\right] w^3 + \cdots .\end{aligned} \tag{15}$$

Definition 1. *A function $f \in \Sigma$ given by (1) is said to be in the class $B_{\Sigma}^{k,\alpha,\beta,\delta,\lambda}$ if the following conditions are satisfied:*

$$\frac{z^{1-\gamma}\left(D^k_{\alpha,\beta,\delta,\lambda} f(z)\right)'}{\left[D^k_{\alpha,\beta,\delta,\lambda} f(z)\right]^{1-\gamma}} \prec \overline{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z - \tau^2 z^2} \tag{16}$$

and

$$\frac{z^{1-\gamma}\left(D^k_{\alpha,\beta,\delta,\lambda}w(z)\right)'}{\left[D^k_{\alpha,\beta,\delta,\lambda}w(z)\right]^{1-\gamma}} \prec \overline{p}(w) = \frac{1+\tau^2w^2}{1-\tau z-\tau^2w^2}, \tag{17}$$

where $\gamma \geq 0, \tau = \frac{1-\sqrt{5}}{2} \approx -0.618$, $z, w \in \mathcal{U}$ and g is given by (2).

Theorem 1. *[12] The function* $p(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} = \left(t+\frac{1}{t}\right)\frac{t}{1-t-t^2}$ *belongs to the class* $\mathcal{P}(B)$ *with* $\beta = \frac{\sqrt{5}}{10} \approx 0.2236$.

Now we give the following lemma which will be used later:

Lemma 1. *[13] Let* $p = \mathcal{P}$ *with* $p(z) = 1 + c_1z + c_2z^2 + \cdots$ *. Then*

$$| C_n | \leq 2 \qquad for\ n \geq 0. \tag{18}$$

We begin by finding the estimates for the coefficients $| a_2 |$ and $| a_3 |$ for functions in the class $B^{k,\alpha,\beta,\delta,\lambda}_{\Sigma}(\gamma,\overline{p})$.

3. Coefficient Bounds for the Function Class $B^{k,\alpha,\beta,\delta,\lambda}_{\Sigma}(\gamma,\overline{p})$

Theorem 2. *Let the function* $f(z)$ *given by (1) be in the class* $B^{k,\alpha,\beta,\delta,\lambda}_{\Sigma}(\gamma,\overline{p})$. *Then*

$$| a_2 | \leq \frac{| \tau |}{\sqrt{2(\gamma+2)\Upsilon^k_3\mathcal{C}(\delta,3) + ((\gamma-1)(\gamma+2)+(2-6\tau)(\gamma+1)^2)\left[\Upsilon^k_2\mathcal{C}(\delta,2)\right]^2}} \tag{19}$$

and

$$|a_3| \leq$$

$$\leq \frac{|\tau|\left[\begin{array}{r}2(\gamma+2)\Upsilon^k_3\mathcal{C}(\delta,3)+\left((\gamma-1)(\gamma+2)+(2-6\tau)(\gamma+1)^2\right)\left[\Upsilon^k_2\mathcal{C}(\delta,2)\right]^2+\\ \tau\left[(\gamma+2)\Upsilon^k_3\mathcal{C}(\delta,3)\right]\end{array}\right]}{\begin{array}{l}(\gamma+2)\Upsilon^k_3\mathcal{C}(\delta,3)\\ \left[2(\gamma+2)\Upsilon^k_3\mathcal{C}(\delta,3)+((\gamma-1)(\gamma+2)+(2-6\tau)(\gamma+1)^2)\left[\Upsilon^k_2\mathcal{C}(\delta,2)\right]^2\right]\end{array}} \tag{20}$$

Proof. Let $f \in B^{k,\alpha,\beta,\delta,\lambda}_{\Sigma}(\gamma,\overline{p})$ and $g = f - 1$. Then there are analytic functions $u, v : \mathcal{U} \to \mathcal{U}$ with $u(0) = 0 = v(0)$, satisfying

$$\frac{z^{1-\gamma}\left(D^k_{\alpha,\beta,\delta,\lambda}f(z)\right)'}{\left[D^k_{\alpha,\beta,\delta,\lambda}f(z)\right]^{1-\gamma}} = \overline{p}(u(z)) \tag{21}$$

and

$$\frac{z^{1-\gamma}\left(D^k_{\alpha,\beta,\delta,\lambda}w(z)\right)'}{\left[D^k_{\alpha,\beta,\delta,\lambda}w(z)\right]^{1-\gamma}}=\overline{p}(v(w)), \tag{22}$$

where $\tau=\frac{1-\sqrt{5}}{2}\approx -0.618$, $z,w\in\mathcal{U}$ and g is given by (2). Since

$$\frac{z^{1-\gamma}\left(D^k_{\alpha,\beta,\delta,\lambda}f(z)\right)'}{\left[D^k_{\alpha,\beta,\delta,\lambda}f(z)\right]^{1-\gamma}}=1+\frac{\overline{p}_1c_1z}{2}+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\overline{p}_1+\frac{c_1^2}{4}\overline{p}_2\right]z^2+$$

$$+\left[\frac{1}{2}\left(c_3+c_1c_2-\frac{c_1^3}{4}\right)\overline{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\overline{p}_2+\frac{c_1^3}{8}\overline{p}_3\right]z^3+\cdots, \tag{23}$$

and

$$\frac{z^{1-\gamma}\left(D^k_{\alpha,\beta,\delta,\lambda}w(z)\right)'}{\left[D^k_{\alpha,\beta,\delta,\lambda}w(z)\right]^{1-\gamma}}=1+\frac{\overline{p}_1d_1w}{2}+\left[\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\overline{p}_1+\frac{d_1^2}{4}\overline{p}_2\right]w^2+$$

$$\left[\frac{1}{2}\left(d_3+d_1d_2-\frac{d_1^3}{4}\right)\overline{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\overline{p}_2+\frac{d_1^3}{8}\overline{p}_3\right]w^3+\cdots, \tag{24}$$

it is evident that

$$1+(\gamma+1)\Upsilon_2^k\mathcal{C}(\delta,2)a_2z+$$

$$\left[(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)a_3+\frac{(\gamma-1)(\gamma+2)}{2}\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2a_2^2\right]z^2+\cdots=$$

$$1+\frac{\overline{p}_1c_1z}{2}+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\overline{p}_1+\frac{c_1^2}{4}\overline{p}_2\right]z^2+$$

$$\left[\frac{1}{2}\left(c_2+c_1c_3-\frac{c_1^3}{4}\right)\overline{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\overline{p}_2+\frac{c_1^3}{8}\overline{p}_3\right]z^3+\cdots, \tag{25}$$

and

$$1-(\gamma+1)\Upsilon_2^k\mathcal{C}(\delta,2)a_2w+\left[\left(2(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)+\frac{(\gamma-1)(\gamma+2)}{2}\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2\right)a_2^2-\right.$$

$$\left.(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)a_3\right]w^2+\cdots=1+\frac{\overline{p}_1d_1w}{2}+\left[\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\overline{p}_1+\frac{d_1^2}{4}\overline{p}_2\right]w^2+$$

$$\left[\frac{1}{2}\left(d_2 + d_1 d_3 - \frac{d_1^3}{4}\right)\overline{p}_1 + \frac{1}{2}d_1\left(d_2 - \frac{d_1^2}{2}\right)\overline{p}_2 + \frac{d_1^3}{8}\overline{p}_3\right] w^3 + \cdots, \tag{26}$$

which yield the following relations:

$$(\gamma+1)\Upsilon_2^k \mathcal{C}(\delta,2) a_2 = \frac{\tau c_1}{2}, \tag{27}$$

$$(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) a_3 + \frac{(\gamma-1)(\gamma+2)}{2}\left[\Upsilon_2^k \mathcal{C}(\delta,2)\right]^2 a_2^2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tau + \frac{3}{4}c_1^2\tau^2, \tag{28}$$

and

$$-(\gamma+1)\Upsilon_2^k \mathcal{C}(\delta,2) a_2 = \frac{\tau d_1}{2}, \tag{29}$$

$$\begin{aligned}\left(2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) + \frac{(\gamma-1)(\gamma+2)}{2}\left[\Upsilon_2^k \mathcal{C}(\delta,2)\right]^2\right) a_2^2 - (\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) a_3 \\ = \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tau + \frac{3}{4}d_1^2\tau^2.\end{aligned} \tag{30}$$

It follows from (9), (27) and (29) that

$$c_1 = -d_1, \tag{31}$$

and

$$a_2^2 = \frac{\left(c_1^2 + d_1^2\right)}{8(\gamma+1)^2\left(\Upsilon_2^k \mathcal{C}(\delta,2)\right)^2}\tau^2. \tag{32}$$

Now, by summing (28) and (30), we obtain

$$\begin{aligned}\left(2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) + (\gamma-1)(\gamma+2)\left[\Upsilon_2^k \mathcal{C}(\delta,2)\right]^2\right) a_2^2 = \\ = \frac{1}{2}(c_2+d_2)\tau - \frac{1}{4}\left(c_1^2+d_1^2\right)\tau + \frac{3}{4}\left(c_1^2+d_1^2\right)\tau^2.\end{aligned} \tag{33}$$

By putting (30) in (31), we have

$$\begin{aligned}\left[2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) + \left((\gamma-1)(\gamma+2) + (2-6\tau)(\gamma+1)^2\right)\left[\Upsilon_2^k \mathcal{C}(\delta,2)\right]^2\right] a_2^2 = \\ = \frac{1}{2}(c_2+d_2)\tau^2.\end{aligned} \tag{34}$$

Therefore, using Lemma 1 we obtain

$$| a_2 | \leq \frac{| \tau |}{\sqrt{2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) + ((\gamma-1)(\gamma+2) + (2-6\tau)(\gamma+1)^2)\left[\Upsilon_2^k \mathcal{C}(\delta,2)\right]^2}}. \tag{35}$$

Now, to find the bound for $| a_3 |$, let us subtract (30) from (28). Then we obtain

$$2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3)a_3 - 2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3)a_2^2 = \frac{1}{2}\left(c_2 - d_2\right)\tau. \tag{36}$$

Hence, we get

$$2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3)\left|a_3\right| \leq 2\left|\tau\right| + 2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3)\left|a_2^2\right|. \tag{37}$$

Then, in view of (33), we obtain

$$|a_3| \leq \frac{|\tau|\left[\begin{array}{r} 2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) + \left((\gamma-1)(\gamma+2) + (2-6\tau)(\gamma+1)^2\right)\left[\Upsilon_2^k \mathcal{C}(\delta,2)\right]^2 + \\ \tau\left[(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3)\right] \end{array}\right]}{\begin{array}{l}(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) \\ \left[2(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) + ((\gamma-1)(\gamma+2) + (2-6\tau)(\gamma+1)^2)\left[\Upsilon_2^k \mathcal{C}(\delta,2)\right]^2\right]\end{array}}. \tag{38}$$

◀

4. Fekete-Szegö inequality for the function of $B_{\Sigma}^{k,\alpha,\beta,\delta,\lambda}(\gamma,\overline{p})$

Fekete-Szegö [14] introduced the generalized functional $| a_3 - \mu a_2^2 |$, where μ is some real number. Due to Zaprawa [15], in the following theorem we define the Fekete-Szegö functional for$f \in B_{\Sigma}^{k,\alpha,\beta,\delta,\lambda}(\gamma,\overline{p})$.

Theorem 3. *Let $f \in B_{\Sigma}^{k,\alpha,\beta,\delta,\lambda}(\gamma,\overline{p})$ be given by 1. Then for all $\mu \in \mathbb{R}$ we have*

$$\left|a_3 - \mu a_2^2\right| = \begin{cases} \frac{|\tau|}{4(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3)} & 0 \leq \left|h\left(\mu\right)\right| \leq \frac{|\tau|}{4(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3)} \\ 4\left|h\left(\mu\right)\right|, & \left|h\left(\mu\right)\right| \geq \frac{|\tau|}{4(\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3)}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[(\gamma+2)^2\left[\Upsilon_2^k \mathcal{C}(\delta,2)\right]^2 + \left((\gamma+2)\Upsilon_3^k \mathcal{C}(\delta,3) + \left(\frac{(\gamma-1)(\gamma+2)}{2} - 3(\gamma+1)^2\right)\left[\Upsilon_2^k \mathcal{C}(\delta,2)\right]^2\right)\tau\right]}. \tag{39}$$

Proof.
From (34) and (36) we obtain

$$a_3-\mu a_2^2=\frac{(1-\mu)(c_2+d_2)\tau^2}{4\left[(\gamma+1)^2\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2+\left((\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)+\left(\frac{(\gamma-1)(\gamma+2)}{2}-3(\gamma+1)^2\right)\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2\right)\tau\right]}+\frac{(c_2-d_2)\tau}{4(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)} \tag{40}$$

$$a_3-\mu a_3=$$

$$\left[\frac{(1-\mu)\tau^2}{4\left[(\gamma+2)^2\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2+\left((\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)+\left(\frac{(\gamma-1)(\gamma+2)}{2}-3(\gamma+1)^2\right)\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2\right)\tau\right]}+\frac{\tau}{4(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)}\right]c_2+$$

$$\left[\frac{(1-\mu)\tau^2}{4\left[(\gamma+2)^2\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2+\left((\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)+\left(\frac{(\gamma-1)(\gamma+2)}{2}-3(\gamma+1)^2\right)\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2\right)\tau\right]}-\frac{\tau}{4(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)}\right]d_2. \tag{41}$$

So we have

$$a_3-\mu a_3=\left[h(\mu)+\frac{\tau}{4(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)}\right]c_2+\left[[h(\mu)-\frac{\tau}{4(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)}\right]d_2, \tag{42}$$

where

$$h(\mu)=\frac{(1-\mu)\tau^2}{4\left[(\gamma+2)^2\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2+\left((\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)+\left(\frac{(\gamma-1)(\gamma+2)}{2}-3(\gamma+1)^2\right)\left[\Upsilon_2^k\mathcal{C}(\delta,2)\right]^2\right)\tau\right]}. \tag{43}$$

Then, by taking modulus of (41), we conclude that

$$\left|a_3-\mu a_2^2\right|=\begin{cases}\frac{|\tau|}{4(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)} & 0\le|h(\mu)|\le\frac{|\tau|}{4(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)}\\ 4\,|h(\mu)|, & |h(\mu)|\ge\frac{|\tau|}{4(\gamma+2)\Upsilon_3^k\mathcal{C}(\delta,3)}.\end{cases}$$

Hence the proof of the theorem is complete. ◄

5. Conclusion

The aim of this paper is to get many interesting and fruitful usages of a wide variety of Fibonacci numbers in Geometric Function Theory. By defining a subclass of starlike functions with respect to symmetric points of Σ related to shell-like curves connected with Fibonacci numbers. we were able to unify and extend the various classes of analytic bi-univalent function, and new extensions were discussed in detail. The results are new and better improvement to initial Taylor-Maclaurin coefficients $\mid a_2 \mid$ and $\mid a_3 \mid$.

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