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Introduction of Inner Distributions and Their Approximation with Blaschke Distributions. Characterization of Blaschke Distributions

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Abstract. We define inner distributions on the unit disk as boundary values of classical inner functions. Also, we introduce Blaschke distributions on the unit disk. We give properties of the introduced inner and Blaschke distributions. We prove that an inner distribution on the unit disk can be uniformly approximated by a sequence of Blaschke distributions. We characterize the Blaschke distributions in the spirit of Fatou's theorem.

Key Words and Phrases: boundary values of distributions, inner distributions, Blaschke product.

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1. Introduction

At the end of the last century, the distributions were introduced by L. Schwartz [14] which shed new light on mathematics and different areas of applied mathematics, physics, etc. He defined new concepts, called generalized functions or distributions, and created a new theory of distributions. Using this new concept, he managed to solve many problems which existed for many years. The concept was authentic and unconventional, even for the mathematicians at that time. The field of distributions has many subfields and is still developing, cf. [1, 16, 17]. For almost a century, many authors [5, 14, 2, 15] did research in different areas of distributions, boundary values of distributions [14, 12]. These fields of research are still attractive and interesting for mathematicians, physicists, and scientists in general. For more results concerning distributions and boundary values of distributions we refer to [2, 12, 7, 8, 9, 10, 6]. Our main concern in this article is

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boundary values of holomorphic functions in the sense of distributions which is a continuation of numerous previous works in this field [2, 12, 7, 8, 9, 10, 6].

This article is organized as follows. Section 2 is dedicated to notations and basic results concerning holomorphic functions such as inner functions and Blaschke products. In Section 3 we introduce our inner and Blaschke distributions on a unit disk and give some properties. We embed the classical inner functions on the unit disk and Blaschke products in a convenient way (the way we define the corresponding distributions) into the space of distributions and prove that this embedding is continuous. In Section 4 we give a distributional analog of the famous Frostman theorem concerning the uniform approximation of inner functions with Blaschke products, and the distributional analog of Fatou's theorem concerning Blaschke distributions.

2. Notation and preliminaries

We use the following notations and preliminaries. U stands for the open unit disk in $\mathbb{C}$ with ∂U its boundary, i.e. $U = \{z \in \mathbb{C} | |z| < 1\}$ and $\partial U = \{z \in \mathbb{C} | |z| = 1\}$. Π^+ denotes the upper half plane, i.e. $\Pi^+ = \{z \in \mathbb{C} | Imz > 0\}$. For a function holomorphic on a domain Ω, we write $f \in H(\Omega)$. Inner functions on the disk are functions $I(z) \in H(U)$ continuous on ∂U such that $|I(z)| \leq 1$ for $z \in U$, and $|I(z)| = 1$ a.e. on ∂U. Examples of inner functions which have zeroes are Blaschke products and functions without zeroes such as $\exp \frac{z-1}{z+1}$. Next we collect properties and results of Blaschke products and inner functions known in the literature [3, 7, 13]. A finite Blaschke product on $D = \bar{U}$ is a function of the form

$$B(z) = e^{i\varphi} \prod_{j=1}^{n} \frac{z - z_j}{1 - z\bar{z}_j},$$

where $\varphi \in \mathbb{R}$ and $|z_j| < 1$ for $j = 1, \ldots, n$. Such $B = B(z)$ possesses the following properties:

1. is holomorphic in U and continuous on ∂U;
2. is an inner function;
3. has zeroes at $z_1, z_2, ..., z_n$ only, and poles at $1/\bar{z}_1, 1/\bar{z}_2, ...1/\bar{z}_n$ only.

For a function $f \in H^\infty(U)$ which is an inner function, for the nontangential limit (which always exists) the following holds: $|f(e^{i\theta})| = 1$ a.e. on ∂U. Other nontrivial examples of inner functions are functions of the form

$$f(z) = e^{i\theta} z^m \prod_{j=1}^{\infty} \frac{-z_j}{|z_j|} \frac{z - z_j}{1 - z\bar{z}_j},$$

where the convergence of the infinite product is assured by requiring that $\sum_{j=1}^{\infty}(1-|z_j|) < \infty$, where each zero z_j is counted with its multiplicity. There are non-constant inner functions with no zeros in U called singular functions. They are functions of the form

$$f(z) = \exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(e^{i\theta})\right),$$

where $d\mu$ denotes a positive measure on ∂U which is singular with respect to the Lebesgue measure.

For the completeness of this article, in the sequel, we state classical results which we prove in a distributional setting. This is our main concern in this article. The first result is a characterization result about Blaschke products via some boundary condition on the inner function.

Theorem 1. *(Fatou [4]) Let f be an inner function on the unit disk. Then the boundary value*

$$\lim_{r\to 1}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta$$

exists. Moreover, f is a Blaschke product if and only if

$$\lim_{r\to 1}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta = 0.$$

The second result shows that the Blaschke products are uniformly dense in the class of inner functions on U.

Theorem 2. *(Frostman [4]). Let $I(z)$ be a non-constant inner function on the unit disk. Then for all $|z_j| < 1$, except possibly for a set of logarithmic capacity zero, the function*

$$B(z) = \frac{I(z) - z_j}{1 - \bar{z}_j I(z)}$$

is a Blaschke product.

The following remarks refer to the upper half space Π^+. The Blaschke product in the upper plane Π^+ with zeros z_n is

$$B(z) = (\frac{z-i}{z+i})^m \prod_{n=1}^{\infty} \frac{|z_n^2+1|}{z_n^2+1}\frac{z-z_n}{z-\bar{z}_n}, z \in \Pi^+,$$

where $\sum_{n=1}^{\infty} \frac{y_n}{1+|z_n|^2} < \infty, z_n = x_n + y_n \in \Pi^+$. If f is holomorphic in Π^+ and bounded, thus if $|f(z)| < C$, for every $z \in \Pi^+$, then the boundary value function

$f^*(x) = \lim_{y\to 0^+} f(x+iy)$ exists almost everywhere on $\mathbb{R}$ and $f^* \in L^\infty(\mathbb{R})$. We call the holomorphic function $f(z), z \in \Pi^+$, an inner function if: $|I(z)| < 1$, for all $z \in \Pi^+$, and its boundary function satisfies the condition $I^*(x)| = 1$ a.e. on $\mathbb{R}$, $I^*(x) = \lim_{y\to 0^+} I(x+iy)$ a.e. on $\mathbf{R}$.

Distributional background: $C^\infty(\mathbb{R}^n)$ denotes the set of all complex-valued functions infinitely differentiable on $\mathbb{R}^n$; $C_0^\infty(\mathbb{R}^n)$ is the subset of $C^\infty(\mathbb{R}^n)$ which contains compactly supported functions. Support of the function f, denoted by supp$f$, is the closure of the set $\{x \in \mathbb{R}^n | f(x) \neq 0\}$ in $\mathbb{R}^n$. $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ denotes the space $C_0^\infty(\mathbb{R}^n)$ in which the convergence is defined in the following way: the sequence $\{\varphi_j\}$ of functions $\varphi_j \in \mathcal{D}$ converges to φ as $j \to j_0$ if and only if there exists a compact subset of $\mathbb{R}^n$ such that supp$\varphi_j \subseteq K$ for all j, supp$\varphi \subseteq K$, and for every n-tuple α of nonnegative integers the sequence $\{D_x^\alpha \varphi_j(x)\}$ converges to $D_x^\alpha(x)$ uniformly on K, as $j \to j_0$. By $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$ we denote the space of all continuous, linear functionals on $\mathcal{D}$, where the continuity is understood in the following sense: from $\varphi_j \to \varphi$ in $\mathcal{D}$ it follows that $\langle T, \varphi_j\rangle \to \langle T, \varphi\rangle$ in $\mathbb{C}$, as $j \to j_0$. The space $\mathcal{D}'$ is called the space of distributions. We use the notation $\langle T, \varphi\rangle = T(\varphi)$ for the value of the functional T acting on the function φ. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then the functional T_f on $\mathcal{D}$ defined by

$$\langle T, \varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \varphi \in \mathcal{D},$$

is an element of $\mathcal{D}'$ and is called the regular distribution generated by the function f.

We say that the sequence $\{T_j\}$, $T_j \in \mathcal{D}'$ converges weakly to the distribution T if $\langle T, \varphi\rangle = \lim_{j\to\infty}\langle T_j, \varphi\rangle$ for every φ in the test space $\mathcal{D}$. The sequence $\{T_j\}$, $T_j \in \mathcal{D}'$, converges in a strong topology to the distribution T if $\langle T, \varphi\rangle = \lim_{j\to\infty}\langle T_j, \varphi\rangle$ uniformly with respect to φ in a bounded subset of the test space $\mathcal{D}$.

3. Definition of Blaschke distributions and introduction of inner distributions

This section is dedicated to the definitions of inner distributions and their properties. We mentioned previously known Blaschke distributions in the literature and here we give them on the unit disk and prove important properties. Let $B(x)$ be the Blaschke product, $z \in \Pi^+$, with zeros z_n that belong to the upper half-plane. In [11], a distributional version of B denoted by B^+ was introduced in the following way:

$$\langle B^+, \varphi\rangle = \lim_{y\to 0^+} \int_{-\infty}^{\infty} B(z)\varphi(x)dx, \ \ z = x+iy \in \Pi^+, \varphi \in \mathcal{D}(\mathbb{R}).$$

The authors in [11] proved that B^+ is a distribution and named its upper half plane Blaschke distribution generated by B. Similarly, using the reflection principle, they defined Blaschke distribution on the lower half plane and obtained similar results for the lower half plane Blaschke distribution, i.e.

$$\langle B^-, \varphi\rangle = \lim_{y\to 0^+} \int_{-\infty}^{\infty} B(z)\varphi(x)dx, \;\; z = x - iy \in \Pi^-, \varphi \in \mathcal{D}(\mathbb{R}).$$

Our intention is to use similar arguments to define Blaschke distribution version of a Blaschke product on the unit disk. The procedure follows. For the rest of the article we will frequently use the following lemma.

Lemma 1. *Let* $\omega(z) = \dfrac{z-i}{z+i}$*. The mapping* ω *has the following properties:*

1. *ω maps the real numbers into the circle ∂U;*
2. *ω maps Π^+ into U.*

Proof. Let $x \in \mathbb{R}$. Then $|\omega(x)| = |\dfrac{x-i}{x+i}| = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$. The latter implies the first claim is satisfied. For the second, we take $z = x + iy, y > 0$. Then

$$|z + i| = |x + i(y + 1)| = \sqrt{x^2 + (y + 1)^2} > \sqrt{x^2 + (y - 1)^2} = |z - i|.$$

Hence, $|\omega(z)| < 1$, i.e. $\omega(z) \in U$. ◄

Now we define our distribution version of Blaschke product and inner function.

Definition 1. *Let B be a Blaschke product on U. Blaschke distribution associated with B is a distribution T_B with*

$$\langle T_B, \varphi\rangle = \lim_{y\to 0^+} \int_{-\infty}^{\infty} B\left(\frac{x+iy-i}{x+iy+i}\right)\varphi(x)dx$$

for arbitrary $\varphi \in \mathcal{D}$.

Definition 2. *Let I be an inner function on U. Inner distribution associated with I is a distribution*

$$\langle I_B, \varphi\rangle = \lim_{y\to 0^+} \int_{-\infty}^{\infty} I\left(\frac{x+iy-i}{x+iy+i}\right)\varphi(x)dx$$

for arbitrary $\varphi \in \mathcal{D}$.

Remark 1. *The boundary values of $B(\omega(z))$ and $I(\omega(z))$, as $z \to x \in \mathbb{R}$, exist almost everywhere as shown in Section 2.*

Remark 2. *The integrals in Definition 1 and Definition 2 are convergent when $\varphi \in \mathcal{D}$. Indeed, because of the previous lemma the argument in the Blaschke product and in the inner function, respectively, $\frac{z-i}{z+i} \in U$ for every $z \in \Pi^+$, and hence $|B(\omega(x))| \leq 1$ and $|I(\omega(x))| \leq 1$. Because of the dominant convergence, one can use the limit-free notation in the definitions.*

Clearly, T_B, T_I, given in Definition 1 and Definition 2, are linear functionals on $\mathcal{D}$. We will prove the linearity and continuity only for T_I. The linearity and continuity of T_B can be proved analogously.

T_I is linear: Let $\varphi_1, \varphi_2 \in \mathcal{D}$ and $\alpha, \beta \in \mathbb{C}$. We have

$$\begin{aligned}\langle T_I, \alpha\varphi_1 + \beta\varphi_2\rangle &= \lim_{y\to 0^+} \int_{-\infty}^{\infty} I\left(\frac{z-i}{z+i}\right)(\alpha\varphi_1(x) + \beta\varphi_2(x))dx \\ &= \alpha \lim_{y\to 0^+} \int_{-\infty}^{\infty} I\left(\frac{z-i}{z+i}\right)\varphi_1(x)dx + \beta \lim_{y\to 0^+} \int_{-\infty}^{\infty} I\left(\frac{z-i}{z+i}\right)\varphi_2(x))dx \\ &= \alpha\langle T_I, \varphi_1\rangle + \beta\langle T_i, \varphi_2\rangle.\end{aligned}$$

T_I is continuous: let $\varphi_j \in \mathcal{D}$, $\varphi_j \to \varphi$, in $\mathcal{D}$ as $j \to j_0$. That means that there is a compact set K such that supp$\varphi_j \subseteq K$ for each j, supp$\varphi \subseteq K$. Then:

$$\begin{aligned}\lim_{j\to j_0} \langle T_I, \varphi_j\rangle &= \lim_{j\to j_0} \lim_{y\to 0^+} \int_{-\infty}^{\infty} I(\omega(z))\varphi_j(x)dx \\ &= \lim_{j\to j_0} \lim_{y\to 0^+} \int_K I(\omega(z))\varphi_j(x)dx \\ &= \lim_{y\to 0^+} \int_K I(\omega(z)) \lim_{j\to j_0} \varphi_j(x)dx \\ &= \langle T_I, \varphi\rangle.\end{aligned}$$

Interchange of limiting operations is valid, since I is bounded on U. Denote by $\mathbf{B}(U)$ the algebra of Blaschke products on U. We have proved the following theorem:

Theorem 3. *Let B be a Blaschke product on U and f be an inner function on U. Then T_B, T_I, defined by Definitions 1 and 2, respectively, are distributions. The mappings*

$$B \mapsto T_B, i \mapsto T_I, \mathbf{B}(U) \to \mathcal{D}',$$

are continuous.

Proof. All claims have been proved above except the continuity of the mappings. Our next concern is the continuity of the same mappings. By our definition,

$$\langle T_I, \varphi\rangle = \lim_{y\to 0^+} \int_{-\infty}^{\infty} I(\omega(z))\varphi(x)dx.$$

Hence,

$$|\langle T_I, \varphi\rangle| = \int_{-\infty}^{\infty} |I(\omega(z))||\varphi(x)|dx = \int_{-\infty}^{\infty} |\varphi(x)|dx.$$

The integral $\int_{-\infty}^{\infty} |\varphi(x)|dx$ can be majorized by the seminorms p_K. Then we obtain

$$\langle T_I, \varphi\rangle \leq C p_K(\varphi)$$

for arbitrary $\varphi \in \mathcal{D}$. ◀

Corollary 1. *The mappings T_B, T_I can be given by the equalities*

$$\langle T_B, \varphi\rangle = \int_{-\infty}^{\infty} B\left(\frac{x-i}{x+i}\right)\varphi(x)dx,$$
$$\langle T_I, \varphi\rangle = \int_{-\infty}^{\infty} I\left(\frac{x-i}{x+i}\right)\varphi(x)dx.$$

Proof. Follows from the first remark. ◀

To state our main results, we give a result that guarantees the existence of boundary value in the distributional sense.

Theorem 4. *Let I be an inner function on the unit disk U. Then the boundary value*

$$\lim_{y\to 0^+} \int_{-\infty}^{\infty} \log^+\left(I\left(\frac{x+iy-i}{x+iy+i}\right)\right)\varphi(x)dx$$

exists for every $\varphi \in \mathcal{D}$.

Proof. It is obvious that for an arbitrary complex number z, we have $\frac{z-i}{z+i} \in D$. Let $\varphi \in \mathcal{D}$ be arbitrary but fixed. Then $\sup_{x\in\mathbb{R}} |\varphi(x)| = C$ for some constant $C < \infty$, which implies

$$\int_{-\infty}^{\infty} |\log^+\left(I\left(\frac{x+iy-i}{x+iy+i}\right)\right)||\varphi(x)|dx \leq C \int_{-\infty}^{\infty} |\log^+\left(I\left(\frac{x+iy-i}{x+iy+i}\right)\right)|dx.$$

Arbitrariness of $y > 0$ implies that

$$\lim_{y\to 0^+} \int_{-\infty}^{\infty} |\log^+\left(I\left(\frac{x+iy-i}{x+iy+i}\right)\right)|dx \leq \int_{-\infty}^{\infty} \log^+\left(I\left(\frac{x-i}{x+i}\right)\right)dx \leq \lim_{r\to 1^-} \int_0^{2\pi} \log^+(I(re^{i\theta}))d\theta < \infty.$$

The last integral converges because of the Fatou theorem given in Section 2.
◀

The following theorem is one of our main results.

Theorem 5. *The boundary condition for the inner function B*

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x+iy-i}{x+iy+i}\right)\right)\varphi(x)dx=0,$$

for all $\varphi\in\mathcal{D}$, is fulfilled if and only if B is a Blaschke product.

Proof. Let B be a Blaschke product. Then

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x+iy-i}{x+iy+i}\right)\right)dx=0.$$

We choose $\varphi\in\mathcal{D}$ and fix it. We have

$$\int_{-\infty}^{\infty}|\log^+\left(B\left(\frac{x-i}{x+i}\right)\right)\varphi(x)|\le C\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x-i}{x+i}\right)\right)dx.$$

Arbitrariness of $y>0$ implies that

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x+iy-i}{x+iy+i}\right)\right)\varphi(x)dx$$
$$\le C\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x+iy-i}{x+iy+i}\right)\right)dx=0,$$

where the constant C is the one chosen in the previous theorem. Hence

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x+iy-i}{x+iy+i}\right)\right)\varphi(x)dx=0.$$

For the opposite direction, for arbitrary $\varphi\in\mathcal{D}$ let

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x+iy-i}{x+iy+i}\right)\right)\varphi(x)dx=0. \tag{1}$$

We use the dominant convergence theorem to continue as follows:

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x+iy-i}{x+iy+i}\right)\right)dx$$
$$=\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x-i}{x+i}\right)\right)\sum_{i=1}^{\infty}\varphi_i(x)dx,$$

where $\{\varphi_i\}$ is a partition of unity consisting of nonnegative compactly supported functions. The limit

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x+iy-i}{x+iy+i}\right)\right)dx$$

exists (Fatou theorem). Use of dominant convergence theorem implies

$$\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x-i}{x+i}\right)\right)dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x-i}{x+i}\right)\right)\sum_{i=1}^{n}\varphi_i(x)dx$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\log^+\left(B\left(\frac{x-i}{x+i}\right)\right)\varphi_i(x)dx = 0.$$

In the last summation we used the assumption (1). ◄

Corollary 2. *Let T_I be an inner distribution on the unit disk. Then T_I is a zero distribution if and only if $I = 0$ or is a Blaschke product.*

Proof. The proof follows from the last theorem. ◄

4. Approximation of inner distributions with a sequence of Blaschke distributions

The theorem below is a natural distributional analog of the classical result concerning inner functions [4]. More precisely, we intend to use uniform approximation result which is a consequence of Frostman theorem. Let us state that result.

Theorem 6. *[7] Let $\varepsilon > 0$. For arbitrary inner function f there exists a Blaschke product B such that $\|f - B\|_\infty < \varepsilon$.*

And here is our distributional version:

Theorem 7. *Let T_I be an inner distribution. There exists a sequence of Blaschke distributions $\{T_{B_n}\}$ converging to T_I in $\mathcal{D}'$.*

Proof. Let $\varepsilon > 0$ be arbitrary and T_I be an inner distribution. Then for the associated inner function $I(z)$ on D we have

$$\langle T_I, \varphi\rangle = \lim_{y\to 0^+}\int_{-\infty}^{\infty} I(\omega(z))\varphi(x)dx, \varphi \in \mathcal{D}.$$

Using Theorem 6, we obtain a sequence of Blaschke products $\{B_n\}$ converging uniformly on U to the inner function I. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$|I(z) - B_n(z)| < \varepsilon,$$

for all $n \geq n_0$ and every z in U. Let $\varphi \in \mathcal{D}$ and $\mathrm{supp}\varphi = K$. Let for $n \in \mathbb{N}$, T_{B_n} be the inner distribution associated with B_n. For $n \geq n_0$, we obtain

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} I(\omega(z))\varphi(x)dx - \int_{-\infty}^{\infty} B_n(\omega(z))\varphi(x)dx \right| \\
&= \left| \int_{-\infty}^{\infty} \left(I(\omega(z)) - B_n(\omega(z))\right)\varphi(x)dx \right| \leq \varepsilon \int_K |\varphi(x)|dx = \varepsilon p_K(\varphi).
\end{aligned}$$

Hence $|\langle T_I, \varphi\rangle - \langle T_{B_n}, \varphi\rangle| < \varepsilon p_K(\varphi)$, so the sequence $\{T_{B_n}\}$ converges weakly to the inner distribution T_I in $\mathcal{D}'$. The strong convergence is proved similarly. ◄

5. Conclusions

We can associate distribution with every Blaschke product or inner function on a unit disk. We gave an approximation result for such inner distributions with Blaschke distributions. And we gave characterization result for Blaschke distributions.

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