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Solvability of Riemann Boundary Value Problems and Applications to Approximative Properties of Perturbed Exponential System in Orlicz Spaces

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Abstract. This work deals with the Orlicz space and the Hardy-Orlicz classes of analytic functions, generated by its norm. Non-homogeneous Riemann boundary value problem with piecewise Hölderian coefficient is considered in these classes. Based on N-function, we introduce new characteristic of Orlicz space and establish its relationship with the Boyd indices of considered Orlicz space. In terms of this characteristic and corresponding Boyd indices, we find a sufficient condition on the jumps of the argument of coefficient for solvability of Riemann boundary value problem in Hardy-Orlicz space and we construct a general solution. The obtained results are applied to establish the approximative properties (completeness, minimality, basicity) of a linear phase exponential system for corresponding Orlicz space.

Key Words and Phrases: Orlicz space, Hardy-Orlicz classes, Riemann boundary value problems, approximative properties.

2010 Mathematics Subject Classifications: 33B10, 46E30, 54D70

1. Introduction

Consider the following perturbed exponential system

$$E_\lambda = \left\{ e^{i\lambda_n t} \right\}_{n \in Z},$$

where $\lambda = \{\lambda_n\} \subset C$ is in general some sequence of complex numbers. Investigation of approximative properties of the system E_λ in Lebesgue spaces has a long and deep history(see, e.g. [9, 22]). In the case of $\lambda_n = n - \beta signn, n \in Z$, where $\beta \in C$ is some parameter, criterion for the basicity of the system E_λ for

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$L_p(-\pi,\pi), 1 < p < +\infty$, has been found in [11](see also [4]). This direction of approximation theory has been further developed in [12, 15, 13, 16, 19, 14, 17]. Note that there are in general two methods for investigation of approximative properties of a system: the method of function theory and the method of boundary value problems for analytic functions. The idea of using boundary value problems in the study of approximative properties of perturbed trigonometric systems belongs to A.V. Bitsadze [23]. Then this method has been successfully used by S. M. Ponomarev [24, 25] and E. I. Moiseev [26, 4] to establish the basicity of linear phase trigonometric systems for Lebesgue spaces. Note that the works [27, 29, 28, 30, 6, 31, 32, 8, 33, 34] are directly related to this topic.

In this work, our aim is to extend the method of Riemann boundary value problem for investigation approximative properties of perturbed trigonometric systems to the case of Orlicz spaces.

We consider the Orlicz space and the Hardy-Orlicz classes of analytic functions, generated by its norm. The non-homogeneous Riemann boundary value problem with piecewise Hölderian coefficient is treated in these classes. Based on N-function, we introduce new characteristic of Orlicz space and establish its relationship with the Boyd indices of considered Orlicz space. In terms of this characteristic and corresponding Boyd indices, we find a sufficient condition on the jumps of the argument of coefficient for solvability of Riemann boundary value problem in Hardy-Orlicz space and we construct a general solution. The obtained results are applied to establish the approximative properties (completeness, minimality, basicity) of a linear phase exponential system for corresponding Orlicz space.

We will use the following standard notations:

- N will denote natural numbers, $Z_+ = \{0\} \bigcup N, Z = \{-N\} \bigcup Z_+$;
- R will stand for the set of real numbers, $R_+ = \{x \in R : x \geq 0\}$ and by C we will denote the set of complex numbers;
- $\chi_M(\cdot)$ will be the characteristic function of the set M;
- $\omega = \{z \in C : |z| < 1\}$ will denote a unit disk in C and $\gamma = \partial\omega$ will be a unit circle;
- $\bar{M}$ will stand for the closure of the set M in the corresponding norm;
- $(\bar{\cdot})$ will denote the complex conjugation;
- $[X]$ will denote the algebra of linear bounded operators acting in the Banach space X;

- $|M|$ will be the Lebesgue measure of the set M;
- p' will denote the conjugate of the number p: $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Auxiliary Facts

Let us define the concept of so called N-function.

Definition 1. *Continuous convex function $M(\cdot)$ on R is called an N-function if it is even and satisfies the conditions*

$$\lim_{u \to 0} \frac{M(u)}{u} = 0,$$

and

$$\lim_{u \to \infty} \frac{M(u)}{u} = \infty.$$

The set of all N-functions is denoted by $\mathfrak{N}$.

Definition 2. *Let $M \in \mathfrak{N}$. The function*

$$M^*(v) = \max_{u \geq 0} [u|v| - M(u)] \ , \forall v \in R,$$

is called an N-function complementary to $M(\cdot)$.

Further characterization of function $M^*(\cdot)$ is following. Let the function $p : R_+ \to R_+$ have the following properties:

i) $p(t) \geq 0, \forall t \geq 0$;

ii) $p(\cdot)$ is right continuous;

iii) $p(\cdot)$ is nondecreasing;

iv) $p(0) = 0$ and $p(\infty) = \lim_{t \to \infty} p(t) = \infty$.

Define

$$q(s) = \sup_{p(t) \leq s} t, \ \ \forall s \geq 0.$$

The function $q(\cdot)$ has the same properties as the function $p(\cdot)$. The functions

$$M(u) = \int_0^{|u|} p(t)\, dt \text{ and } M^*(v) = \int_0^{|v|} q(t)\, dt$$

are called N-functions complementary to each other.

We also need the concept of Δ_2-condition.

Definition 3. *The function $M \in \mathfrak{N}$ satisfies Δ_2-condition for large values of u, if* $\exists k > 0 \wedge \exists u_0 \geq 0$:

$$M(2u) \leq kM(u), \forall u \geq u_0.$$

The set of all N-functions satisfying Δ_2-condition is denoted by (Δ_2).

Δ_2-condition is equivalent to requiring that $\forall l > 1, \exists k(l) > 0 \wedge \exists u_0 \geq 0$:

$$M(lu) \leq k(l) M(u), \forall u \geq u_0.$$

We also need the following

Definition 4. *We will say that $M \in (\nabla_2)$, if $M \in \mathfrak{N}$ and $\exists k > 2 \wedge \exists u_0 > 0$:*

$$M(2u) \geq kM(u), \forall u \geq u_0,$$

i.e.

$$\lim_{u\to\infty} \inf \frac{M(2u)}{M(u)} > 2.$$

Denote by $\mathfrak{F}(G)$ the set of all functions measurable (in Lebesgue sense) on G. Let's define the Orlicz space $L_M(G)$ on measurable (in Lebesgue sense) set $G \subset R$.

Let

$$\rho_M(u) = \int_G M[u(x)]\,dx,$$

and define

$$L_M(G) = \{u \in \mathfrak{F}(G) : \rho_M(u) < +\infty\}.$$

$L_M(G)$ is called an Orlicz class. Let $M; M^* \in \mathfrak{N}$ be N-functions complementary to each other. Let's consider

$$L_M^*(G) = \{u \in \mathfrak{F}(G) : |(u,v)| < +\infty, \forall v \in L_{M^*}(G)\},$$

where

$$(u,v) = \int_G u(x)\overline{v(x)}dx.$$

$L_M^*(G)$ is called an Orlicz space with the norm $\|\cdot\|_M$:

$$\|u\|_M = \sup_{\rho_{M^*}(\nu)\leq 1} |(u;v)|.$$

With this norm, $L_M^*(G)$ becomes a Banach space. In $L_M^*(G)$, the following Luxemburg norm

$$\|u\|_{(M)} = \inf\left\{\lambda > 0 : \rho_M\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

is equivalent to the norm $\|\cdot\|_M$. The following statement is valid.

Statement 1. *If* $M \in (\Delta_2)$, *then* $L_M^*(G) = L_M(G)$ *and the closure of the set of bounded (including continuous) functions coincides with* $L_M^*(G)$.

More information on the above facts can be found in [20, 21].

Let's introduce the following characteristic of the space $L_M(-\pi,\pi) \equiv L_M$:

$$\gamma_M = \inf\{\alpha : |t|^\alpha \in L_M\}.$$

For further presentation, we need the concept of Boyd indices of Orlicz spaces. So, let $M \in \mathfrak{N}$ and $M^{-1}(\cdot)$ be its inverse on R_+. Assume

$$h(t) = \limsup_{x\to\infty} \frac{M^{-1}(x)}{M^{-1}(tx)},\ t > 0,$$

and define the following numbers

$$\alpha_M = -\lim_{t\to\infty} \frac{\log h(t)}{\log t};\ \ \beta_M = -\lim_{t\to 0+} \frac{\log h(t)}{\log t}.$$

The numbers α_M and β_M are called upper and lower Boyd indices, respectively, for the Orlicz space L_M. The following relations are true for these indices

$$0 \le \alpha_M \le \beta_M \le 1;$$

$$\alpha_M + \beta_{M^*} \equiv 1;\ \alpha_{M^*} + \beta_M = 1,$$

where $M, M^* \in \mathfrak{N}$ are complementary to each other.

The space L_M is reflexive if and only if $0 < \alpha_M \le \beta_M < 1$. If $1 \le q < \frac{1}{\beta_M} \le \frac{1}{\alpha_M} < p \le \infty$, then the continuous embeddings

$$L_p(-\pi,\pi) \subset L_M \subset L_q(-\pi,\pi),$$

hold. More information about these facts can be found in [2, 3, 4].

The lemma below was proved in [1]:

Lemma 1. *[1] Let* $M \in (\Delta_2)$. *Then* $\gamma_M \in [-\beta_M, -\alpha_M]$. *In particular, if* $\alpha_M = \beta_M$, *then* $\gamma_M = -\alpha_M$.

From the definition of the characteristic γ_M we get the validity of the following

Statement 2. *For arbitrary points* $\{s_k\} : -\pi = s_0 < s_1 < ... < s_r < \pi$, *the finite product*

$$\mu(t) = \prod_{k=0}^{r} \left|\sin\frac{t-s_k}{2}\right|^{\alpha_k}, t \in (-\pi,\pi),$$

belongs to L_M *if* $\alpha_k > \gamma_M, \forall k = \overline{0,r}$.

To obtain our main results we will use some facts about the boundedness of singular Cauchy operators in weighted Orlicz spaces. Let's recall the definition of Muckenhoupt class A_p of weights on the unit circle γ.

We will say that the weight function $\nu : \gamma \to \overline{R}_+$ belongs to the class $A_p, 1 < p < +\infty$, if

$$[\nu]_{A_p} = \sup_{J \subset \gamma} \frac{1}{|J|} \|\nu \chi_J\|_{L_p} \left\|\nu^{-1} \chi_J\right\|_{L_{p'}} < +\infty,$$

where the sup is taken over all measurable (in Lebesgue sense) subsets $J \subset \gamma$. Before proceeding further, let us define the weighted Orlicz space $L_{M,w}$ with weight function $w(\cdot)$.

In the sequel we will identify the segment $[-\pi, \pi)$ with the circle γ by mapping $e^{it} : [-\pi, \pi) \to \gamma$. For this reason, the function $f : \gamma \to C$ will be identified with $f(t) =: f(e^{it}), t \in [-\pi, \pi)$.

So, for the weight function $w : \gamma \to \overline{R}_+$, the weighted Orlicz space $L_{M,w}$ is defined by the norm

$$\|f\|_{L_{M,w}} = \|fw\|_{L_M}, \forall f \in L_{M,w},$$

where

$$L_{M,w} = \left\{ f \in \mathfrak{F}(\gamma) : \|f\|_{L_{M,w}} < +\infty \right\}.$$

By $\mathcal{S}$ we denote the following Cauchy singular integral

$$S(f)(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - e^{ix}} d\xi \, .$$

From the results of [5] (see also [7, 6]), in particular in the case of Orlicz spaces we get the validity of the following

Theorem 1. *Let $M \in (\Delta_2)$ and α_M, β_M be the corresponding Boyd indices of Orlicz space L_M. If $w \in A_{\alpha_M^{-1}} \bigcap A_{\beta_M^{-1}}$, then the Cauchy singular operator $\mathcal{S}$ is bounded in $L_{M,w}$.*

We will need the Hardy-Orlicz class H_M^+ (mH_M^-) of analytic functions inside (outside) ω. First define the class H_M^+. For the function $f : \omega \to C$, denote $f_r(t) = f\left(re^{it}\right), t \in [-\pi, \pi)$. In the sequel, for the function $f : \gamma \to C$ by $\|f\|_{L_M}$ we will mean $\|f\|_{L_M} = \||f|\|_{L_M}$.

H_M^+ is defined by the norm

$$\|f\|_{H_M^+} = \sup_{0<r<1} \|f_r(\cdot)\|_{L_M}.$$

Let $\log^{+} u = \log \max\{1; u\}, \forall u \geq 0$. By $\mathcal{A}$ we denote the set of analytic functions $F : \omega \to C$, which satisfy

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^{+} |F_r(t)\, dt| < +\infty.$$

It is known that $F \in \mathcal{A} \iff F$ is representable in the form

$$F(z) = B(z) \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} dh(t)\right), \tag{1}$$

where $B(\cdot)$ is a Blaschke function corresponding to F and $h(\cdot)$ is a bounded variation function on $[-\pi, \pi]$. $\mathcal{A}'$ denotes the subclass of functions $F \in \mathcal{A}$ such that $h(\cdot)$ in (1) is absolutely continuous on $[-\pi, \pi)$. The following theorem is true.

Theorem 2. *[8] If the function $F : \omega \to C$ is analytic and $\|F\|_{H_M^+} < +\infty$, then $F \in \mathcal{A}'$, and conversely, if $F \in \mathcal{A}' \wedge F^{+}(\cdot) \in L_M$, then $\|F\|_{H_M^+} < +\infty$, where $F^{+}(\cdot)$ are the non-tangential boundary values of $F(\cdot)$ on γ.*

When solving Riemann boundary value problem, we will use the following Zygmund result.

Statement 3. *Let $f : [-\pi, \pi) \to R$ be a real function with $\|f\|_{L_\infty(-\pi,\pi)} < +\infty$. Then the function*

$$\Phi(z) = \exp\left(\pm \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{is}+z}{e^{is}-z} f(s)\, ds\right),$$

analytic in ω, belongs to the Hardy class H_δ^+ for $\delta > 0$ sufficiently small.

The Hardy-Orlicz class ${}_mH_M^-$ is defined as follows. Let the function f analytic outside ω, have a Laurent decomposition at a point $z = \infty$:

$$f(z) = \sum_{n=-\infty}^{m} a_n z^n, z \to \infty.$$

So, for $m > 0$ the point $z = \infty$ is a pole of order m; and for $m \leq 0$ the point $z = \infty$ is a zero of order $(-m)$. Let $f(z) = f_0(z) + f_1(z)$, where $f_0(\cdot)$ is a principal part, and $f_1(\cdot)$ is a regular part of Laurent decomposition at $z = \infty$. If $g \in H_M^+$, where $g(z) = \overline{f_0\left(\frac{1}{\bar{z}}\right)}$, $|z| < 1$, then we will say that the function $f(\cdot)$ belongs to the class ${}_mH_M^-$.

The following statement is true.

Statement 4. *Let* $M \in (\Delta_2) \bigcap (\nabla_2)$. *Then the system* $\{z^n\}_{n \in Z_+}$ $\left(\{z^n\}_{n=\overline{-\infty,m}} \right)$ *forms a basis for* H_M^+ *(for* ${}_mH_M^-$*).*

For more details see [19].

For investigation of solvability of nonhomogeneous Riemann problem we will use the results obtained in [1] for homogeneous Riemann problem in the classes $H_M^+ \times {}_mH_M^-$. Let us state those results here.

So, let the function $G : \gamma \to C$ (called the coefficient in the sequel) satisfy the following conditions:

i) $G^{\pm 1}(\cdot) \in L_\infty(-\pi, \pi)$;

ii) $\theta(t) = \arg G(e^{it})$ is a piecewise Hölder function on $[-\pi, \pi]$ with the jumps $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1, r}$, at the points of discontinuity $\{s_k\}_1^r : -\pi < s_1 < \ldots < s_r < \pi$, and let $h_0 = \theta(-\pi) - \theta(\pi)$.

Consider the homogeneous Riemann problem

$$\begin{array}{l} F^+(\tau) - G(\tau) F^-(\tau) = 0, \tau \in \gamma, \\ (F^+; F^-) \in H_M^+ \times {}_mH_M^-, \end{array} \tag{2}$$

with the coefficient $G(e^{it}) = |G(e^{it})| e^{i\theta(t)}, t \in [-\pi, \pi)$. We say that a pair of analytic functions $(F^+; F^-) \in H_M^+ \times {}_mH_M^-$ is a solution of the problem (2),if its non-tangential boundary values satisfy the equality (2) a.e. on γ.

Define the piecewise analytic functions $Z_1(\cdot), Z_2(\cdot)$ on the C with a cut γ by the following expressions

$$Z_1(z) \equiv \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \left| G(e^{it}) \right| \frac{e^{it} + z}{e^{it} - z} dt \right\},$$

$$Z_2(z) \equiv \exp\left\{ \frac{i}{4\pi} \int_{-\pi}^{\pi} \theta(t) \frac{e^{it} + z}{e^{it} - z} dt \right\}, z \notin \gamma,$$

and let

$$Z_\theta(z) = Z_1(z) Z_2(z), z \notin \gamma.$$

We will call $Z_\theta(\cdot)$ *a canonical solution of homogeneous problem* (2), *corresponding to the argument* $\theta(\cdot)$.

Let $\{n_k\}_1^r \subset Z$ be some integers. Based on θ, define the function $\theta_{1:r}(\cdot)$ by the formula

$$\theta_{1:r}(\cdot) = \begin{cases} \theta(t), \ -\pi < t < s_1, \\ \theta(t) + 2\pi n_1, \ s_1 < t < s_2, \\ \vdots \\ \theta(t) + 2\pi n_r, \ s_r < t < \pi, \end{cases}$$

and assume

$$G_{1:r}(t) = |G(t)|\, e^{i\theta_{1:r}(t)}\ ,\ \ t \in [-\pi, \pi)\,.$$

Let $Z_{\theta_{1:r}}$ be a canonical solution of the homogeneous problem (2), corresponding to the argument $\theta_{1:r}$.

So, regarding to the solvability of the homogeneous Riemann problem (2) in work[1], it is proved the following

Theorem 3. *Let $M \in \Delta_2(\infty)$ and M^* be its complementary to its functions. Suppose the coefficient $G(\cdot)$ satisfies the conditions i), ii), and $Z_\theta(\cdot)$ is a canonical solution of* (2) *corresponding to the argument $\theta(\cdot)$. Let the jumps of $\theta(\cdot)$ satisfy the inequalities*

$$\gamma_{M^*} < \frac{h_1}{2\pi} < -\gamma_M\, k = \overline{0, r}. \tag{3}$$

Then:

α) for $m \geq 0$ the problem (2) *has a general solution of the form*

$$F(Z) = Z_\theta(z)\, P_k(z)\,,$$

where $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq m$;

β) for $m < 0$ this problem has only a trivial, i.e. zero solution.

This theorem has the following direct corollary.

Corollary 1. *Let all conditions of Theorem 3 hold. Then the problem* (2) *under condition $F(\infty) = 0$ has only a trivial solution.*

3. Nonhomogeneous Riemann problem in Hardy-Orlicz classes

Consider the following nonhomogeneous Riemann problem:

$$\begin{array}{l} F^+(\tau) - G(\tau)\, F^-(\tau) = f(\arg \tau)\,, \tau \in \gamma, \\ (F^+; F^-) \in H_M^+ \times {}_m H_M^-, \end{array} \tag{4}$$

where $f(\cdot) \in L_M$ is some function. Suppose the coefficient $G(\cdot)$ satisfies the conditions i), ii), and $Z_\theta(\cdot)$ is a canonical solution of homogeneous problem corresponding to the argument $\theta(\cdot)$. In this section, we will investigate the solvability of the problem (4). First we construct the particular solution of this problem and then construct a common solution.

3.1. Particular solution of the problem (4)

Consider the following piecewise analytic function

$$F_1(z) = \frac{Z_\theta(z)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} K(t;z)\, dt, z \notin \gamma, \tag{5}$$

where $K(t;z) = \frac{e^{it}}{e^{it}-z}$ is a Cauchy kernel. Applying Sokhotski-Plemelj formulas to (5), we obtain

$$F_1^\pm(\tau) = Z_\theta^\pm(\tau) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} \frac{e^{it} dt}{e^{it}-z} \right]_\gamma^\pm =$$

$$= Z_\theta^\pm(\tau) \left(\pm\frac{1}{2} \left[Z_\theta^+(\tau) \right]^{-1} f(\arg \tau) - \left[Z_\theta^+(\tau) \right]^{-1} (Kf)(\tau) \right),$$

where $[\,\cdot\,]_\gamma^\pm$ denotes the boundary values on γ from inside (with "+") and outside (with "-"), respectively, and K is a singular Cauchy integral of the form

$$(Kf)(\tau) = \frac{Z_\theta^+(\tau)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} K(t,\tau)\, dt, \quad \tau \in \gamma.$$

We have

$$\frac{F_1^+(\tau)}{Z_\theta^+(\tau)} - \frac{F_1^-(\tau)}{Z_\theta^-(\tau)} = \frac{f(\arg \tau)}{Z_\theta^+(\tau)}, \qquad \tau \in \gamma. \tag{6}$$

Since the canonical solution $Z_\theta(\cdot)$ satisfies the relation $Z_\theta{}^+(\tau) - G(\tau) Z_\theta{}^-(\tau) = 0$, a.e. $\tau \in \gamma$, we obtain

$$\frac{Z_\theta{}^+(\tau)}{Z_\theta{}^-(\tau)} = G(\tau), \quad \text{a.e. } \tau \in \gamma, \tag{7}$$

because it is not difficult to see that $Z_\theta^\pm(\tau) \neq 0$ a.e. $\tau \in \gamma$. Considering (7) in (6), we obtain

$$F_1^+(\tau) - G(\tau) F_1^-(\tau) = f(\arg \tau) \quad \text{a.e. } \tau \in \gamma.$$

Thus, the boundary values of the function $F_1(\cdot)$ satisfy (4) a.e. on γ. Let's find out if the function $F_1(\cdot)$ belongs to the required classes, i.e. let's find the conditions under which the inclusion

$$\left(F_1^+(z); F_1^-(z) \right) \in H_+^{p,\alpha} \times_m H_-^{p,\alpha},$$

holds. Assume that the coefficient $G(\cdot)$ satisfies the conditions $i)$; $ii)$, and $\{h_k\}_0^r$ are the corresponding jumps of the function $\theta(t) = \arg G(e^{it})$. In previous section, we established the validity of representation

$$\left|Z_\theta{}^-\left(e^{it}\right)\right| = \left|Z_1^-\left(e^{it}\right)\right| u_0(t) \prod_{k=0}^{r} \left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}.$$

Consequently

$$\left|Z_\theta{}^+\left(e^{it}\right)\right|^{-1} = \left|G\left(e^{it}\right)\right|^{-1}\left|Z_\theta^-\left(e^{it}\right)\right|^{-1} \sim \prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}, \qquad t\in[-\pi,\pi]. \tag{8}$$

As $f \in L_M$, it is clear that if $\left|Z_\theta^+(\cdot)\right|^{-1} \in L_{M^*}$, then, by Hölder's inequality, the inclusion $f(\cdot)\left[Z_\theta^+(\cdot)\right]^{-1} \in L_1(-\pi,\pi)$ holds. Applying Statement 2 to (8), we see that if the inequalities

$$\frac{h_k}{2\pi} > \gamma_{M^*}, \forall k = \overline{0,r},$$

hold, then $\left|Z_\theta^+\right|^{-1} \in L_{M^*}$, and therefore $f(\cdot)\left[Z_\theta^+(\cdot)\right]^{-1} \in L_1(-\pi,\pi)$. Consequently, by classical Smirnov theorem (see, e.g., [35]), we arrive at the conclusion that the Cauchy type integral

$$F_2(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{Z_\theta\left(e^{it}\right)}K(t;z)\,dt, z\in\omega,$$

belongs to the class $H_\delta^+, \forall\delta\in(0,1)$. As established in [1], $Z_\theta(\cdot)$ belongs to the Hardy class H_δ^+ for sufficiently small $\delta > 0$. Applying Hölder's inequality, we see that the product $F_1(z) = Z_\theta(z)\,F_2(z)$ belongs to the class H_δ^+ for sufficiently small $\delta > 0$. In addition, we obtain $F_1(\cdot) \in \mathcal{A}'$. Let's find the conditions which guarantee $F_1^+(\cdot) \in L_M$ ($F_1^+(\tau)$, $\tau\in\gamma$, are non-tangential boundary values of F_1 on γ). We have

$$F_1^+(\tau) = \frac{1}{2}f(\arg\tau) - (Kf)(\tau), \ \tau\in\gamma. \tag{9}$$

It suffices to prove that $Kf \in L_M$. Let

$$g(t) = f(t)\left(Z_\theta^+\left(e^{it}\right)\right)^{-1}, \ t\in[-\pi,\pi].$$

Denote by S the following singular Cauchy operator

$$(Sg)(\tau) = \frac{1}{2\pi i}\int_\gamma\frac{g(\xi)\,d\xi}{\xi-\tau}, \ \tau\in\gamma.$$

Denote by A_M the class of weights such that the singular operator is bounded in $L_{M,\rho}$, i.e.

$$A_M = \{\rho : S \in [L_{M,\rho}]\} .$$

Consider the following weight function

$$\rho_0(t) = \left|t^2 - \pi^2\right|^{\frac{h_0}{2\pi}} \prod_{k=1}^{r} |t - s_k|^{-\frac{h_k}{2\pi}}, t \in [-\pi, \pi] . \tag{10}$$

It is not difficult to see that

$$\left|Z_\theta^+\left(e^{it}\right)\right| \sim \rho_0(t), t \in [-\pi, \pi] . \tag{11}$$

Thus

$$\|g\|_{M,\rho_0} \sim \|f\|_M .$$

These relations directly imply that the operator K acts boundedly in L_M if and only if the singular operator S acts boundedly in the weighted space L_{M,ρ_0} :

$$Kf \in L_M \Leftrightarrow Sg \in L_{M,\rho_0} .$$

So, if $\rho_0 \in A_M$, then $Sg \in L_{M,\rho_0}$, and hence $Kf \in L_M$. Then, from (9) and (11) it follows $F_1^+(\cdot) \in L_M$. Further, from Theorem 3.1 [8] (analogue of Smirnov theorem) we obtain $F_1(\cdot) \in H_M^+$. It is not difficult to see that $\exists \lim\limits_{z\to\infty} Z_\theta(z) \neq 0$, and hence $\lim\limits_{z\to\infty} F_1(z) = 0$. Taking into account this relation, we similarly prove that $F_1(\cdot) \in {}_{-1}H_M^-$. Thus, we have proved the following

Theorem 4. *Let $M \in (\Delta_2)$ be some N-function, and M^* be its complementary. Let the coefficient $G(\cdot)$ of the problem (4) satisfy the conditions $i), ii)$ and the jumps $\{h_k\}_0^r$ of the argument $\theta(\cdot)$ satisfy the relations*

$$\gamma_{M^*} < \frac{h_k}{2\pi}, k = \overline{0, r};\ \wedge \rho_0 \in A_M, \tag{12}$$

where the weight $\rho_0(\cdot)$ is defined by (10). Then the function $F_1(\cdot)$, defined by (5), is a solution of the nonhomogeneous problem (4) in the Hardy-Orlicz classes $H_M^+ \times {}_{-1}H_M^-$.

In what follows, we will seek the conditions on the jumps $\{h_k\}_0^r$, which guarantee that the weight function ρ_0 belongs to the class A_M. Let $M \in (\Delta_2)$ be some N-function, and M^* be an N-function complementary to M. Denote Boyd indices of the space L_M by α_M and β_M. From Theorem 2.2 [5] it follows that if $\rho_0 \in A_{\alpha_M^{-1}} \bigcap A_{\beta_M^{-1}}$, then $\rho_0 \in A_M$. It is known that

$$|t|^\alpha \in A_p \Leftrightarrow -\frac{1}{p} < \alpha < -\frac{1}{p} + 1, \quad 1 < p < \infty.$$

Consequently

$$\rho_0 \in A_{\alpha_M^{-1}} \Leftrightarrow -\alpha_M < -\frac{h_k}{2\pi} < -\alpha_M + 1, k = \overline{0, r};$$

$$\rho_0 \in A_{\beta_M^{-1}} \Leftrightarrow -\beta_M < -\frac{h_k}{2\pi} < -\beta_M + 1, k = \overline{0, r}.$$

As $\alpha_M \leq \beta_M$, we obtain

$$\rho_0 \in A_{\alpha_M^{-1}} \bigcap A_{\beta_M^{-1}} \Leftrightarrow \beta_M^{-1} < \frac{h_k}{2\pi} < \alpha_M, k = \overline{0, r}. \quad (13)$$

Then, as a corollary of Theorem 4, we obtain the following

Corollary 2. *Let $M \in (\Delta_2)$ be some N-function, $M^*(\cdot)$ be its complementary, the coefficient $G(\cdot)$ of the problem (4) satisfy the conditions i), ii), and the jumps $\{h_k\}_0^r$ of the argument $\theta(\cdot)$ satisfy the relations*

$$\max\left\{\gamma_{M^*}; \beta_M^{-1}\right\} < \frac{h_k}{2\pi} < \alpha_M, k = \overline{0, r}, \quad (14)$$

where α_M, β_M are the Boyd indices of the space L_M. Then the function (5) is a particular solution of the nonhomogeneous problem (4) in the Hardy-Orlicz classes $H_M^+ \times_{-1} H_M^-$.

Remark 1. *Consider the case $M(x) = \frac{1}{p}x^p$, $1 < p < +\infty$, $x \geq 0$. In this case, the complementary N-function $M^*(\cdot)$ has the form $M^*(x) = \frac{1}{q}x^q$, $\frac{1}{p} + \frac{1}{q} = 1$. It is not difficult to see that $\gamma_M = -\frac{1}{p}$; $\gamma_{M^*} = -\frac{1}{q}$. As is known (see, e.g., [36, 7]), the relation $\alpha_M = \beta_M = \frac{1}{p}$ holds in this case. Then the conditions (14) become*

$$-\frac{1}{q} < \frac{h_k}{2\pi} < \frac{1}{p}, k = \overline{0, r};$$

which coincide with the well-known conditions on the jumps $\{h_k\}$ in the case of Hardy spaces $H_p^+ \times {}_{-1}H_p^-$ (see corresponding statement in [37]).

3.2. General solution of nonhomogeneous problem

Let us construct the general solution of nonhomogeneous problem (4) in the Hardy-Orlicz classes. As the function $F_1(\cdot)$, defined by (5), is a particular solution of the problem (4), the general solution of (4) can be expressed in the form $F(\cdot) = F_0(\cdot) + F_1(\cdot)$, where $F_0(\cdot)$ is a general solution of corresponding homogeneous problem (2), and $F_1(\cdot)$ is a partial solution of the problem (4). Assume that all conditions of Theorem 3 are fulfilled. Let the conditions (3) and (12) hold.

Let's first consider the case $m \geq -1$. In this case it is clear that the particular solution $F_1(\cdot)$ belongs to the classes $H_M^+ \times_m H_M^-$. The general solution $F_0(\cdot)$ of homogeneous problem (2) in the classes $H_M^+ \times H_M^-$ has a form $F_0(\cdot) = Z_\theta(\cdot) P_k(\cdot)$, where $Z_\theta(\cdot)$ is a canonical solution of homogeneous problem corresponding to the argument $\theta(\cdot)$, and $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq m$ (for $m = -1$ we assume $P_k(\cdot) \equiv 0$). Consider the case $m < -1$. In this case, it follows from Theorem 3 that the homogeneous problem (2) has only a trivial solution in the classes $H_M^+ \times H_M^-$, i.e. $F_0(\cdot) \equiv 0$. Let us show that if the problem (4) has a solution $\Phi_1(\cdot)$, then $\Phi_1(\cdot) = F_1(\cdot)$. In fact, from ${}_mH_M^- \subset_{-1} H_M^-$ (due to $m \leq -2$) it follows that $\Phi_1(\cdot) \in H_M^+ \times_{-1} H_M^-$. As $F_1(\cdot) \in H_M^+ \times_{-1} H_M^-$, it is clear that $\Phi \in H_M^+ \times_{-1} H_M^-$, where $\Phi(\cdot) = \Phi_1(\cdot) - F_1(\cdot)$. Obviously, $\Phi(\cdot)$ is a solution of the problem (2) in the classes $H_M^+ \times_{-1} H_M^-$. By Theorem 3, the problem (2) is trivially solvable in the classes $H_M^+ \times_{-1} H_M^-$, so we obtain $\Phi(\cdot) \equiv 0 \Rightarrow \Phi_1(\cdot) \equiv F_1(\cdot)$. Clearly, $F_1(\cdot) \in H_M^+$. Let's consider the case where the inclusion $F_1(\cdot) \in_m H_M^-$ holds. From this inclusion it follows that the function $F_1(\cdot)$ has a Laurent decomposition of the form

$$F_1(z) = \sum_{k=-\infty}^{m} a_k z^k, \ z \to \infty,$$

in the vicinity of the infinitely remote point. As $\exists \lim\limits_{z\to\infty} \left|Z_\theta^-(z)\right|^{\pm 1} \neq 0$, it follows directly from the representation $F_1(\cdot) = Z_\theta(\cdot) K(\cdot)$ that the Cauchy type integral

$$K(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} K(z;t)\, dt,$$

has a decomposition of the form

$$K(z) = \sum_{k=-\infty}^{m} b_k z^k, \tag{15}$$

as $z \to \infty$. We have

$$K(z) = -\frac{z^{-1}}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} \frac{e^{it} dt}{1 - e^{it} z^{-1}} = -\sum_{k=-\infty}^{-1} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} e^{-ikt} dt\, z^k. \tag{16}$$

Comparing the decompositions (15) and (16), we see that if the orthogonality conditions

$$\int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} e^{ikt} dt = 0, \ k = \overline{1, -m-1}, \tag{17}$$

hold, then $F_1(\cdot) \in_m H_M^-$. So the following theorem is true.

Theorem 5. *Let $M \in (\Delta_2)$ be some N-function, $M^*(\cdot)$ be its complementary, the coefficient $G(\cdot)$ of the problem* (4) *satisfy the conditions i), and the jumps of the argument $\theta(\cdot)$ satisfy the relations* (3) *and* (12). *Then the non-homogeneous problem* (4) *is solvable in the Hardy-Orlicz classes $H^+_M \times_m H^-_M$ if:*

α) for $m \geq -1$ the problem (4) *has a general solution of the form $F(z) = Z_\theta(z) P_k(z) + F_1(z)$, where $Z_\theta(\cdot)$ is a canonical solution of homogeneous problem* (2) *corresponding to the argument $\theta(\cdot)$, $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq m$ (for $m = -1$ we assume $P_k(\cdot) \equiv 0$), $F_1(\cdot)$ is a particular solution of the form* (5)*:*

$$F_1(z) = \frac{Z_\theta(z)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z^+_\theta(e^{it})} K(z;t)\, dt,$$

and $f \in L_M$ is an arbitrary function;

β) for $m < -1$ the problem (4) *with the right-hand side $f(\cdot) \in L_M$ is solvable if and only if the orthogonality conditions* (17) *hold, with the unique solution representable in the form $F(z) = F_1(z)$, where $F_1(\cdot)$ is a particular solution defined by* (5).

In particular, we obtain the following

Corollary 3. *Let all the conditions of Theorem 5 hold. Then the problem* (4) *with an arbitrary right-hand side $f \in L_M$ has a unique solution $F_1(\cdot)$ of the form* (5) *in the Hardy-Orlicz classes $H^+_M \times_{-1} H^-_M$.*

In particular, taking into account Theorem 3, Corollary 2 and the relations (3) and (13), from Theorem 5 we directly obtain the following

Corollary 4. *Let $M \in (\Delta_2)$ be some $N-$function, $M^*(\cdot)$ be its complementary, $G(\cdot)$ satisfy the conditions $i)$, $ii)$, and the jumps $\{h_k\}_0^r$ satisfy the inequalities*

$$\max\left\{\gamma_{M^*};\ \beta_M^{-1}\right\} < \frac{h_k}{2\pi} < \min\left\{-\gamma_M; \alpha_M\right\}, \tag{18}$$

where α_M, β_M are the Boyd indices of the space L_M. Then the non-homogeneous problem (13) *is solvable in the Hardy-Orlicz classes $H^+_M \times_m H^-_M$ if :*

α) for $m \geq -1$ the problem (4) *has a general solution of the form $F(\cdot) = Z_\theta(\cdot) P_k(\cdot) + F_1(\cdot)$, where $Z_\theta(\cdot)$ is a canonical solution, $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq m$ (for $m = -1$ we assume $P_k(\cdot) \equiv 0$), and $F_1(\cdot)$ is a particular solution of the form* (5)*;*

β) for $m < -1$ the problem (4) *with $f(\cdot) \in L_M$ is solvable if and only if the orthogonality conditions* (17) *hold, with the unique solution in the form* (5).

Remark 2. *Let* $M(x) = \frac{x^p}{p}$, $1 < p < +\infty$, $x \geq 0$. *We have* $M^*(x) = \frac{x^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$. *In this case, the relations* $\gamma_M = -\frac{1}{p}$, $\gamma_{M^*} = -\frac{1}{q}$, *hold, and, moreover,* $\alpha_M = \beta_M = \frac{1}{p}$. *Then the conditions* (18) *become*

$$-\frac{1}{q} < \frac{h_k}{2\pi} < \frac{1}{p}, \; k = \overline{0, r},$$

which coincide with the well-known conditions for the solvability of the problem (4) *in the Hardy classes* $H_p^+ \times_{-1} H_p^-$ *(see, e.g., [37]).*

4. Basis properties of perturbed exponential system in Orlicz spaces

Consider the following perturbed model exponential system

$$E_\beta = \left\{ e^{i(n - \beta \, sign \, n)t} \right\}_{n \in Z}, \tag{19}$$

where $\beta \in R$ is a real parameter. As noted above, criterion for the basicity of the system (19) for $L_2(-\pi, \pi)$ follows from the works by N. Levinson [9] and M. Kadets [10] and consists of the inequality $|\beta| < \frac{1}{4}$. Criterion for the basicity of the system (19) for $L_p(-\pi, \pi)$, $1 < p < +\infty$, has been found by A.M. Sedletski [11], and later the same result has been obtained by E.I. Moiseev [4] using a different method in the case where β is a real parameter. Criterion for the basicity of the system (19) for Morrey type spaces has been recently obtained by B.T. Bilalov in [18].

We will consider the basis properties of the system (19) in reflexive Orlicz spaces. So throughout this section we will assume that $M \in (\Delta_2) \bigcap (\nabla_2)$ is some N-function (with complementary N-function $M^*(\cdot)$). Denote by $(z+1)_-^{-2\beta}$ the branch of multi-valued analytic function $(z+1)^{-2\beta}$ on the complex plane, cut along the negative real axis. Consider the following systems

$$h_n^+(t) = \frac{e^{i\beta t}}{2\pi} \left(e^{it} + 1\right)_-^{2\beta} \sum_{k=0}^{n} C_{2\beta}^{n-k} e^{-ikt}, \; n \in Z_+;$$

$$h_n^-(t) = -\frac{e^{i\beta t}}{2\pi} \left(e^{it} + 1\right)_-^{-2\beta} \sum_{k=1}^{n} C_{2\beta}^{n-k} e^{ikt}, \; n \in N,$$

where $C_{2\beta}^n = \frac{2\beta(2\beta-1)\ldots(2\beta-n+1)}{n!}$ are binomial coefficients. The following lemma is true.

Lemma 2. *Let $|\beta| < \frac{1}{2}$. Then the following relations are true*

$$< x_k^+, h_n^+ >=< x_{k+1}^-, h_{n+1}^- >= \delta_{nk},$$

$$< x_k^+, h_{n+1}^- >=< x_{k+1}^-, h_n^+ >= 0; \forall n, k \in Z_+,$$

where

$$< x, y >= \int_{-\pi}^{\pi} x(t) \overline{y(t)} dt; \; x_n^{\pm} = e^{\pm i(n-\beta)t}.$$

For more details on this lemma we refer the readers to [12, 4, 11]. As

$$e^{it} + 1 = -2e^{i\frac{t}{2}} \sin \frac{t-\pi}{2},$$

from Statement 2 it directly follows that the system $\left\{h_n^+; h_{n+1}^-\right\}_{n\in Z_+}$ belongs to the class L_{M^*}, when $-2\beta > \gamma_{M^*}$. Then from Lemma 2 it follows that for $-1 < 2\beta < -\gamma_{M^*}$ the system E_β is minimal in L_M. So the following lemma is true.

Lemma 3. *Let $-1 < 2\beta < -\gamma_{M^*}$. Then the system E_β is minimal in L_M.*

Now consider the basicity of the system E_β for L_M. Let $G(t) = e^{i2\beta t}$, $\theta(t) = \arg G(t) = 2\beta t$, and consider the following non-homogeneous Riemann problem:

$$F^+\left(e^{it}\right) - e^{i2\beta t} F^-\left(e^{it}\right) = e^{i\beta t} f(t), \; t \in (-\pi, \pi), \tag{20}$$

where $f(\cdot) \in L_M$ is some function. The solution is sought in the Hardy-Orlicz classes $H_M^+ \times_{-1} H_M^-$. We have $h_0 = \theta(-\pi) - \theta(\pi) = -4\beta\pi$. By (10), we have

$$\rho_0(t) = \left|t^2 - \pi^2\right|^{2\beta}, \, t \in (-\pi, \pi).$$

Applying Corollary 3 to the problem (20), we see that if the condition

$$2\beta < -\gamma_{M^*} \wedge \left|t^2 - \pi^2\right|^{2\beta} \in A_M,$$

holds, then the problem (20) has a unique solution in the classes $H_M^+ \times_{-1} H_M^-$ of the form

$$F(z) = \frac{Z_\theta(z)}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\beta t} f(t)}{Z_\theta^+\left(e^{it}\right)} K(z;t)\, dt,$$

where $f(\cdot) \in L_M$ is an arbitrary function, and $Z_\theta(\cdot)$ is a canonical solution of homogeneous problem corresponding to the argument $\theta(t) = 2\beta t$, $t \in (-\pi, \pi)$. Thus, $F^+ \in H_M^+$; $F^- \in_{-1} H_M^-$. Expanding the functions $F^+$and F^- into Taylor series with respect to the powers of z in the neighborhood of zero and with respect to the powers of $\frac{1}{z}$ in the neighborhood of the infinitely remote point, we have

$$F^+(z) = \sum_{n=0}^{\infty} a_n^+ z^n, \ F^-(z) = \sum_{n=1}^{\infty} a_n^- z^{-n},$$

where

$$a_n^{\pm} =< f, h_n^{\pm} >= \int_{-\pi}^{\pi} f(t) \overline{h_n^{\pm}(t)} \, dt.$$

It is absolutely clear that $F^+ \in H_1^+$ and $F^- \in_{-1} H_1^-$. Then, by classical Riesz theorem, we have

$$\int_{-\pi}^{\pi} \left| F^+ \left(re^{it} \right) - F^+ \left(e^{it} \right) \right| dt \to 0, \ r \to 1-0,$$

$$\int_{-\pi}^{\pi} \left| F^- \left(re^{it} \right) - F^- \left(e^{it} \right) \right| dt \to 0, \ r \to 1+0,$$

where $F^+ \left(e^{it} \right)$ and $F^- \left(e^{it} \right)$ are non-tangential boundary values of the functions $F^+(\cdot)$ and $F^-(\cdot)$ on γ from inside and outside ω, respectively. From these relations it directly follows

$$a_n^{\pm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F^{\pm} \left(e^{it} \right) e^{\mp int} dt, \ \forall n.$$

Consequently, $\{a_n^+\}$ ($\{a_n^-\}$) are Fourier coefficients of the function $F^+ \left(e^{it} \right)$ $\left(F^- \left(e^{it} \right) \right)$ with respect to the exponential system $\left\{ e^{int} \right\}_{n \in Z_+}$ $\left(\left\{ e^{-int} \right\}_{n \in N} \right)$. It is absolutely clear that $F^+ \left(e^{it} \right) \in L_M^+$, $F^- \left(e^{it} \right) \in_{-1} L_M^-$. Then, from the basicity of the system $\left\{ e^{int} \right\}_{n \in Z_+}$ $\left(\left\{ e^{-int} \right\}_{n \in N} \right)$ for L_M^+ (for ${}_{-1}L_M^-$), we obtain the decompositions

$$F^+ \left(e^{it} \right) = \sum_{n=0}^{\infty} a_n^+ e^{int}, \ F^- \left(e^{it} \right) = \sum_{n=1}^{\infty} a_n^- e^{-int}.$$

Considering these decompositions in (20), we obtain the following decomposition of $f(\cdot)$ with respect to the system E_β in L_M:

$$f(t) = \sum_{n=0}^{\infty} a_n^+ e^{i(n-\beta)t} + \sum_{n=1}^{\infty} a_n^- e^{-i(n-\beta)t}.$$

From Lemma 3 it follows that such a decomposition is unique, which proves the basicity. So the following theorem is true.

Theorem 6. *Let $M \in (\Delta_2) \bigcap (\nabla_2)$ be some N-function, and N-function $M^* (\cdot)$ be its complementary. Let the parameter $\beta \in R$ satisfy the condition*

$$2\beta < -\gamma_{M^*} \wedge \left|t^2 - \pi^2\right|^{2\beta} \in A_M.$$

Then the system E_β forms a basis for L_M.

Taking into account Corollary 4, from this theorem we obtain the following

Corollary 5. *Let the N-function $M (\cdot)$ satisfy the conditions of Theorem 2 and α_M, β_M be the Boyd indices of the space L_M. Let the parameter $\beta \in R$ satisfy the condition*

$$\max \{\gamma_{M^*};\ \beta_M - 1\} < -2\beta < \min \{-\gamma_M; \alpha_M\}. \quad (21)$$

Then the system E_β forms a basis for L_M.

Remark 3. *In case $M (\alpha) = \frac{x^p}{p}$, $x \geq 0$, $1 < p < +\infty$, we have $\alpha_M = \beta_M = \frac{1}{p}$, $q (M) = -\frac{1}{p}$, $q (M^*) = -\frac{1}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the condition (21) becomes $-\frac{1}{2p} < \beta < \frac{1}{2q}$, which coincides with the well-known condition of basicity of the system E_β for $L_p (-\pi, \pi)$.*

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