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Order Vitally Dense Injectivity and Order Vitally Essentiality

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Abstract. In this paper, we define a new closure operator on category of S-posets and study some properties of this closure operator. Also, we define the class of order vitally dense embeddings denoted by $\mathcal{M}_{ov}$, and we study the categorical properties such as product, coproduct, pushout and pullback. Next, we investigate injectivity with respect to this class of monomorphisms.

Key Words and Phrases: S-poset, order vitally dense subact, ov-injectivity.

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1. Introduction and preliminaries

The study on the category pos-S of partially ordered sets with actions of a pomonoid S can be found in many papers (see, e.g., [1], [5], [2], and [3]). Also, various kinds of closure operators are studied. In [8], Shahbaz investigates the down set closure operator in pos-S and studies injectivity with respect to the class of down closed embeddings (see [10] and [7]). We investigate the new closure operator on a commutative pomonoid, namely, an order vitally dense closure operator such that order vitally embedding emerges from this type of closure. We study some properties of this closure operator and some categorical properties of this kind of embeddings. The injectivity with respect to different classes of monomorphisms has been studied in [6], [9], [8]. In [4], in the category of S-acts, the researchers study vital dense subact and vitally dense monomorphisms and injectivity with respect to this kind of monomorphisms. We study the injectivity of S-posets with respect to the class of order vitally dense embeddings and call it ov-injectivity.

Now let us briefly give some definitions and preliminaries needed in the sequel.

Recall that a monoid S is said to be a pomonoid if it is also a poset partial order $\leq$ of which is compatible with its binary operation, i.e. $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$. A right S-poset is a poset A which is also an S-act whose $\lambda : A\times S \longrightarrow A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. An S-poset morphism is an action preserving monotone morphism between S-posets. A regular monomorphism is exactly order embedding; i.e, S-poset morphism $f : A \longrightarrow B$ for which $a \leq a'$ if and only if $f(a) \leq f(a')$, for all $a, a' \in A$.

2. Order vitally dense closure operator

In this section, we introduce a closure operator and order vitally dense monomorphism. We study some properties of this kind of closure operators and some algebraic properties of order vitally dense embeddings such as composition, colimit, coproduct, pullback, and pushout.

Let $\mathcal{C}$ be a category. Recall that a family $C = (C_B)_{B\in\mathcal{C}}$, with $C_B : \mathrm{Sub}B \to \mathrm{Sub}B$, taking any subobject $A \leq B$ to a subobject $C_B(A)$, is called a *closure operator* on $\mathcal{C}$ if it satisfies the following:

1. (*Extension*) $A \leq C_B(A)$,
2. (*Monotonicity*) $A_1 \leq A_2$ implies $C_B(A_1) \leq C_B(A_2)$,
3. (*Continuity*) $f(C_B(A)) \leq C_D(f(A))$, for all morphisms $f : B \to D$.

Moreover, a closure operator C is said to be:

(a) *Weakly hereditary* if for every S-act B and every $A \leq B$, A is C-dense in $C_B(A)$.

(b) *Hereditary* if for every S-act B and $A_1 \leq A_2 \leq B$, $C_{A_2}(A_1) = C_B(A_1)\cap A_2$.

(c) *Additive* if for every S-act B, $C_B(A_1 \cup A_2) = C_B(A_1) \cup C_B(A_2)$.

(d) *Productive* if for every family of subacts A_i of B_i, taking $A = \prod_i A_i$ and $B = \prod_i B_i$, $C_B(A) = \prod_i C_{B_i}(A_i)$.

(e) *Idempotent* if $C_B(C_B(A)) = C_B(A)$ for every S-act B and $A \leq B$.

(f) *Discrete* if $C_B(A) = A$ for every S-act B and $A \leq B$.

Definition 1. *Let S be a commutative pomonoid. A family $C^{ov} = (C^{ov}_B)_{B\in pos-S}$ with C^{ov}_B : Sub $B \longrightarrow$ Sub B is defined as $C^{ov}_B(A) = \{b \in B|\exists a \in A,\ s \in S$ such that $bs \leq a\}$, for any subact A of B.*

It is easy to show that C^v is a closure operator on **pos-**S, which is called an *order vitally closure operator*. Indeed, for any $b \in C^{ov}_B(A)$ and $t \in S$, there exists $s \in S$ and $a \in A$ with $bs \leq a$ and so $(bt)s = b(ts) = b(st) = (bs)t \leq at$ by commutativity of S. Then $bt \in C^{ov}_B(A)$, which means that $C^{ov}_B(A)$ is a sub-S-poset of B. The extension and monotonicity properties are clear. For continuity, take an S-poset morphism $f : B \to D$ and $b \in C^{0v}_B(A)$. Then $bs \leq a$ for some

$s \in S$ and $a \in A$. Then $f(b)s = f(bs) \leq f(a)$. Hence, $f(b) \in C_D^{Ov}(f(A))$, i.e. $f(C_B^v(A)) \leq C_D^v(f(A))$.

Clearly, if $A \leq B \leq D$, then $C_B^{ov}(A) \leq C_D^{ov}(A)$.

From now on, S stands for a commutative pomonoid.

A C^{ov}*-dense* sub-S-poset A of an S-act B is called *order vitally dense* or *ov-dense* for short, i.e. $C_B^{ov}(A) = B$. An S-poset morphism $f : A \to B$ is said to be *order vitally dense* or *ov-dense* if $f(A)$ is an *ov*-dense sub-S-poset of B. The class of all *ov*-dense monomorphisms of S-posets is denoted by $\mathcal{M}_{ov}$.

Definition 2. *Let A be a sub-S-poset of B. We call A vitally dense in B, if for any $b \in B$, there exists $s \in S$ such that $bs \in A$.*

Example 1. *Consider $\mathbb{Z}$ and $\mathbb{Q}$ as $(\mathbb{N}, .)$-posets with usual addition and order. Clearly, $\mathbb{Z}$ is a vitally dense sub-S-poset of $\mathbb{Q}$.*

We recall from [8] that a sub-S-poset A of S-poset B is called order dense, if for each $b \in B$ there exists $a \in A$ such that $b \leq a$. Clearly any order dense sub-S-poset is order vitally dense.

Example 2. *Let S be a group. Then there is not any non-trivial sub-S-poset of an S-poset. Moreover, on group, any order vitally dense sub-S-poset is order dense.*

Example 3. *Any vitally dense sub-S-poset of an S-poset is order vitally dense sub-S-poset, but the converse is not always true. For example, consider S-posets $A = \{\bot\}$ and $B = \{\bot, \top\}$, with trivial action and $\bot \leq \top$. Clearly, A is ov-dense in B, and it is not vitally dense in B.*

Example 4. *If A is an order vitally dense convex sub-S-poset of B, then A is vitally dense sub-S-poset of B.*

Now let us prove some properties of this closure operator.

Theorem 1. *The closure operator C^{ov} is hereditary, additive and idempotent.*

Proof. First, let us prove that it is hereditary. For this, consider sub S-poset $A_1 \leq A_2$ of B and $x \in C_{A_2}^{ov}(A_1)$. So, there exists $s \in S$ and $a_1 \in A_1$ such that $xs \leq a_1$. Obviously, $x \in C_B^{ov}(A_1) \cap A_2$. Now, suppose that $x \in C_B^{ov}(A_1) \cap A_2$. Thus, we have $x \in A_2$, and there exists $s \in S$ and $a_1 \in A_1$ such that $xs \leq a_1$. Clearly, we have $x \in\in C_{A_2}^{ov}(A_1)$. For idempotency, take any $b \in C_B^v(C_B^{ov}(A))$, where $A \leq B$. Then $bs \in C_B^{ov}(A)$ for some $s \in S$. This implies that there exists $t \in S$ and $a \in A$ such that $(bs)t \leq a$. Thus $b(st) \leq a$, which means that $b \in C_B^{ov}(A)$. For additivity, consider sub-S-poset A_1 and A_2 of B. Let us show that $C_B^{ov}(A_1 \cup A_2) = C_B^{ov}(A_1) \cup C_B^{ov}(A_2)$. Let $x \in C_B^{ov}(A_1 \cup A_2)$, so there exists $s \in S$ and $a \in A_1 \cup A_2$ such that $xs \leq a$. Since $a \in A_1 \cup A_2$, we have $x \in C_B^{ov}(A_1) \cup C_B^{ov}(A_2)$. Now, consider $x \in C_B^{ov}(A_1) \cup C_B^{ov}(A_2)$. Thus, there exists $a_1 \in A_1$ and $t \in S$ such that $xs \leq a_1$ or there exists $a_2 \in A_2$ and $t' \in S$ such that $xs \leq a_2$. In both cases, we have $x \in C_B^{ov}(A_1 \cup A_2)$. ◄

Lemma 1. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be two S-poset morphisms, where g is an embedding. Then gf is an ov-dense morphism if and only if so are f and g. In particular, $\mathcal{M}_{ov}$ is closed under composition as well as right and left cancellable.*

Proof. Assume that gf is an ov-dense morphism. We will show that $f(A)$ and $g(B)$ are ov-dense sub-S-posets of B and C, respectively. For any $b \in B$, $g(b) \in C$. Since $gf(A)$ is a v-dense sub-S-posets of C, there exist $s \in S$ and $a \in A$ for which $g(b)s \leq gf(a)$. So $g(bs) \leq gf(a)$ and then $bs \leq f(a) \in f(A)$. Now let $c \in C$. Then there exists $s \in S$ with $cs \leq gf(a)$ for some $a \in A$. So $cs \leq g(b)$, for some $b \in B$. Conversely, let f and g be ov-dense morphisms. It must be shown that $gf(A)$ is an ov-dense sub-S-poset of C. Let $c \in C$. Since $g(B)$ is ov-dense in C, there exist $s \in S$ and $b \in B$ such that $cs \leq g(b)$. Also, since $f(A)$ is ov-dense in B, there exist $t \in S$ and $a \in A$ for which $bt \leq f(a)$. Now we have $c(st) = (cs)t \leq g(b)t = g(bt) \leq gf(a) \in gf(A)$. ◄

It is clear that any epimorphism is an ov-dense morphism. So, the following is obtained by using Lemma 1.

Corollary 1. *The composition of an ov-dense morphism with an epimorphism is an ov-dense morphism.*

Proposition 1. *The class $\mathcal{M}_{ov}$ is closed under coproducts.*

Proof. Consider the diagram

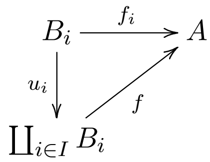
$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle u'_i} \\ \coprod_{i\in I} A_i & \xrightarrow{f} & \coprod_{i\in I} B_i \end{array}$$

where $(f_i : A_i \to B_i)_{i\in I}$ is a family of ov-dense embeddings. Let $f : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ be the S-poset morphism satisfying $f(u_i(a_i)) = u'_i f_i(a_i)$, for $a_i \in A_i$, which exists by the universal property of coproducts. In fact, $f(a_i, i) = (f_i(a_i), i)$. We claim that f is an ov-dense embedding. By [5], it is clear that f is an embedding, so it is enough to show that f is ov-dense. Let $b \in \coprod_{i\in I} B_i$. Then $b \in B_i$ for some $i \in I$ and $b = u'_i(b_i)$. Since f_i is ov-dense, there exist $a_i \in A_i$

and $s \in S$ for which $bs \leq f_i(a_i)$, and hence $bs \leq u_i' f_i(a_i) = fu_i(a_i)$, which means that f is ov-dense. ◀

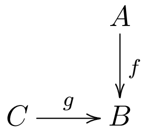
Proposition 2. *Let $(f_i : B_i \to A)_{i\in I}$ be a family of ov-dense morphisms. Then $f : \coprod_{i\in I} B_i \to A$ is an ov-dense morphism.*

Proof. Consider the diagram



where $f : \coprod_{i\in I} B_i \to A$ is the S-poset morphism obtained by the universal property of coproducts. Since f_i is ov-dense, for any $a \in A$, there exist $b_i \in B$ and $s \in S$ such that $as \leq f_i(b_i)$. So $as \leq f_i(b_i) = fu_i(b_i)$ and hence f is an ov-dense morphism. ◀

Recall that the sub pullback of a diagram

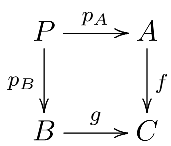


in **pos-**S is the sub S-poset $P = \{(c, a) : c \in C, a \in A, f(a) \leq g(c)\}$ of $A \times B$, and sub pullback maps $p_C : P \to C$, $p_A : P \to A$ are restrictions of the projection maps.

A class of morphisms of a category is called *pullback stable* if pullbacks transfer those morphisms. In the next result, we establish this property for ov-dense monomorphisms of S-posets.

Proposition 3. *The class $\mathcal{M}_{ov}$ is sub pullback stable.*

Proof. Consider the pullback diagram



where $P = (c, a) : c \in C, a \in A, f(a) \leq g(c)$ and $p_C : P \to C$ and $p_A : P \to A$ are restrictions of the projection maps. Assume that $f, g \in \mathcal{M}_{ov}$. Let us show that $p_A, \rho_B \in \mathcal{M}_{ov}$. By [5], it is known that p_A and ρ_B are order embeddings. Let us show that p_B is ov-dense. Let $b \in B$. Then $g(b) \in C$, and so it follows from the ov-density of f that there exist $s \in S$ and $a \in A$ such that $g(b)s = g(bs) \leq f(a)$. Now, since g is ov-dense, there exist $t \in S$ and $b' \in B$ such that $f(a)t \leq g(b')$. So, we have $g(bst) = g(b)st \leq f(a)t \leq g(b')t = g(b't)$, and since g is order embedding, we have $bst \leq b't = \rho_B(at, b't)$. It follows that ρ_B is ov-dense embedding. Similarly, we can show that ρ_A is ov-dense embedding. ◄

For a class $\mathcal{M}$ of morphisms of a category, it is said that *pushouts transfer $\mathcal{M}$-morphisms* if for a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{k} & D \end{array}$$

if $g \in \mathcal{M}$, then so is k.

The pushout of a given diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \\ B & & \end{array}$$

in **pos-**S is the factor S-poset $Q = (B \sqcup C)/\theta$, where θ is the congruence relation on $B \sqcup C$ generated by all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are the coproduct injections. Also, the pushout maps are given as $h = \gamma u_B : C \to Q, k = \gamma u_C : B \to Q$, where $\gamma : B \sqcup C \to Q$ is the canonical epimorphism.

Proposition 4. *In* **pos-**S*, pushouts transfer ov-dense embeddings.*

Proof. Consider the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{k} & Q \end{array}$$

where g is an ov-dense embedding. Let us show that k is also an ov-dense embedding. By [5], k is a regular monomorphism. So, it remains to prove that k is ov-dense. Let $[x]_\theta \in Q$. If $x = u_B(b)$ for some $b \in B$, since f is ov-dense, there exist $a \in A$ and $s \in S$ such that $bs \leq f(a)$. So, we have

$[x]_\theta s = h(bs) \leq h(f(a)) = kg(a)$. If $x = u_C(c)$ for some $c \in C$, then for any $s \in S$ we have $[x]_\theta s \leq k(cs)$, as required. ◀

For a class $\mathcal{M}$ of morphisms of a category, we say that multiple pushouts transfer $\mathcal{M}$-morphisms if in the multiple pushout $(Q, (B_i \stackrel{d'_i}{\to} Q)_{i\in I})$ of a family $\{d_i : A \to B_i \mid i \in I\}$ of $\mathcal{M}$-morphisms, $d'_i \in \mathcal{M}$, for every $i \in I$.

Similar to the pushouts, we have the following result.

Proposition 5. *Multiple pushouts transfer ov-dense embeddings.*

Proof. Let $\{d_i : A \to B_i \mid i \in I\}$ be a family of *ov*-dense embeddings. Recall that the multiple pushout of this family is $(\coprod_{i\in I} B_i)/\theta$, where θ is a congruence on $\coprod_{i\in I} B_i$ generated by all pairs $H = \{(u_i d_i(a), u_j d_j(a)) : i, j \in I, a \in A\}$, where for each $i \in I$, $u_i : B_i \to \coprod_{i\in I} B_i$ is the i-th coproduct injection map. Also, the multiple pushout maps are $d'_i = \gamma u_i : B_i \to (\coprod_{i\in I} B_i)/\theta$, where $\gamma : \coprod_{i\in I} B_i \to (\coprod_{i\in I} B_i)/\theta$ is a natural epimorphism. By [5], d'_i is an embedding for each $i \in I$. To show that each $d'_i d_i$ (and hence each d'_i by Lemma 1) is *ov*-dense, let $b \in (\coprod_{i\in I} B_i)/\theta$. Then there exist $j \in I$ and $b_j \in B_j$ such that $b = [u_j(b_j)]_\theta$. Since d_j is *ov*-dense, there exist $a \in A$ and $s \in S$ such that $b_j s \leq d_j(a)$. Now we get $bs = [u_j(b_j)]_\theta s = [u_j(b_j s)]_\theta = d'_j(b_j s) \leq d'_j d_j(a) = d'_i d_i(a) \in \mathrm{Im}(d'_i d_i)$, as required. ◀

Definition 3. For a class $\mathcal{M}$ of morphisms of category $\mathcal{A}$, we say that $\mathcal{A}$ has $\mathcal{M}$-*bounds* if for each set indexed family $\{m_i : A \to A_i \mid i \in I\}$ of $\mathcal{M}$-morphisms there is an $\mathcal{M}$-morphism $m : A \to B$ which factors over all m_i's; that is, there are $d_i : A_i \to B$ with $d_i m_i = m$. In addition, if d_i's belong to $\mathcal{M}$, it is said that $\mathcal{A}$ has $\mathcal{M}$-*amalgamation* property.

Corollary 2. *The category* **pos-**S *has* $\mathcal{M}_{ov}$*-bounds and* $\mathcal{M}_{ov}$*-amalgamation property.*

Proof. Let $\{h_i : A \to B_i \mid i \in I\}$ be a set indexed family in $\mathcal{M}_{ov}$ and $h : A \to B = (\coprod_{i\in I} B_i)/\theta$ be the multiple pushout of h_i's. Then h factors over all h_i's, and is an *ov*-dense monomorphism by Proposition 5. The second assertion also follows from Proposition 5. ◀

A *directed system* of S-posets and S-poset morphisms is a family $(A_i)_{i\in I}$ of S-posets indexed by an up-directed set I endowed by a family $(g_{ij} : A_i \to A_j)_{i\leq j\in I}$ of S-poset morphisms such that given $i \leq j \leq k \in I$, we have $g_{ij} g_{jk} = g_{ik}$, and also $g_{ii} = id$. Note that the pair $(\varinjlim A_i, \{f_i : A_i \longrightarrow \varinjlim A_i\}$, abbreviation $\varinjlim A_i$ is called the directed colimit of the directed system $((A_i)_{i\in I}, (g_{ij})_{i\leq j})$, where $i \leq j \in I$ and $f_j g_{ij} = f_i$, and for every $(B, h_i : A_i \longrightarrow B)$ with $h_j g_{ij} = h_i, i \leq j \in I$, there exists a unique map such that $\nu : \varinjlim A_i \longrightarrow B$ such that $\nu h_i = f_i$.

We recall that, by [5], the directed colimit of the directed system $((A_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ of S-posets exists, and may be represented as $(\frac{A}{\theta}, (\psi_i = \gamma_\theta u_i : A_i \longrightarrow \frac{A}{\theta})_{i\in I})$, where $\gamma_\theta : \coprod A_i \longrightarrow \frac{\coprod A_i}{\theta}$ is a natural epimorphism, u_i is a coproduct injection and
(i) $A = \coprod A_i$;
(ii) $a\theta a' (a \in A_i, a' \in A_j)$ if and only if $\exists k \geq i, j : \psi_{ik}(a) \leq \psi_{jk}(a')$;
(iii) for each $i \in I$ and $a \in A$, $\psi_I(A) = [a]_\theta$.

Theorem 2. *The category pos-S has $\mathcal{M}_{ov}$- directed colimit.*

Proof. Consider directed system $((B_\alpha)_{\alpha\in I}, (g_{\alpha\beta})_{\alpha\leq\beta\in I}$ of S-posets and S-poset morphisms, and the colimit S-poset morphisms $g_\alpha : B_\alpha \longrightarrow \underrightarrow{lim}_\alpha B_\alpha = \frac{\coprod_\alpha B_\alpha}{\rho}$. Take ov-dense monomorphisms $h_\alpha : A \longrightarrow B_\alpha, \alpha \in I$ with $g_{\alpha\beta}h_\alpha = h_\beta$ for $\alpha \leq \beta \in I$. Let $h : A \longrightarrow \underrightarrow{lim}_\alpha B_\alpha$ be the directed colimit of h_α for $\alpha \in I$. That is, $h = \underrightarrow{lim}_\alpha h_\alpha = g_\beta h_\beta$. By [5], h is an order embedding. Let us show that h is on-dense. For this, let $b \in \underrightarrow{lim}_B \beta$, so there exists $y_\alpha \in B_\alpha$ such that $b = [y_\alpha]_\rho$. Now, since h_α is ov-dense, there exist $s \in S$ and $a \in A$ such that $y_\alpha s \leq h_\alpha(a)$. Then, $bs = [y_\alpha]_\rho s = g_\alpha(y_\alpha)s \leq g_\alpha(x_\alpha) \leq g_\alpha h_\alpha(a) = h(a)$ and so h is ov-dense. ◄

Definition 4. *For a class $\mathcal{M}$ of morphisms of category $\mathcal{A}$, we say that $\mathcal{A}$ satisfies $\mathcal{M}$-chain condition if for any directed system $((A_\alpha)_{\alpha\in I}, (f_{\alpha\beta})_{\alpha\leq\beta\in I})$, whose index set I is a well-ordered chain with the least element 0 and $f_{0\alpha} \in \mathcal{M}$ for all α, there is a (so called "upper bound") family $(g_\alpha : A_\alpha \to A)_{\alpha\in I}$ with $g_0 \in \mathcal{M}$ and $g_\beta f_{\alpha\beta} = g_\alpha$.*

Proposition 6. *The category* **pos-**S *fulfills the $\mathcal{M}_{ov}$-chain condition.*

Proof. Take $A = \underrightarrow{lim}_\alpha A_\alpha$ and let $g_\alpha : A_\alpha \to A$ be a colimit map. Then, applying Proposition 2, we get the validity of the assertion. ◄

3. v-dense essentiality and v-dense injectivity

In this section, we proceed to study three different definitions of essentiality with respect to the class $\mathcal{M}_{ov}$. Also, we investigate the relationship between ov-injectivity, ov-injectivity and these three essentialities.

Consider subclass $\mathcal{M}ono$ of monomorphisms of the category pos-S and $A \overset{m}{\to} B \in \mathcal{M}_{ov}$. One usually uses one of the following definitions to say that m is called:

($\mathcal{M}_{1ov}$-essential) $A \stackrel{m}{\to} B \stackrel{f}{\to} C \in \mathcal{M}_{ov} \Rightarrow f \in \mathcal{M}_{ov}$

($\mathcal{M}_{2ov}$-essential) $A \stackrel{m}{\to} B \stackrel{f}{\to} C \in \mathcal{M}ono \Rightarrow f \in \mathcal{M}ono$ (essential monomorphism)

($\mathcal{M}_{3ov}$-essential) $A \stackrel{m}{\to} B \stackrel{f}{\to} C \in \mathcal{M}_{ov} \Rightarrow f \in \mathcal{M}ono$

Lemma 2. *The composition of $\mathcal{M}_{ov}$-essentials is a $\mathcal{M}_{ov}$-essential.*

Proof. Let $f : A \to B$ and $g : B \to C$ be $\mathcal{M}_{ov}$-essentials. We will show that so is gf. By Lemma 1, gf is an ov-dense monomorphism. Suppose that $h : C \to D$ is a homomorphism for which $h(gf) = (hg)f$ is a monomorphism. It must be shown that h is a monomorphism. Since f is a $\mathcal{M}_{ov}$-essential, hg and hence h is a monomorphism because g is a $\mathcal{M}_{ov}$-essential. ◀

Theorem 3. *The category* **pos-**S *fulfills Banaschewski's $\mathcal{M}_{ov}$-condition, in the sense that for any $\mathcal{M}_{ov}$-monomorphism $f : A \to B$ there exists an S-poset morphism $g : B \to C$ such that gf is $\mathcal{M}_{ov}$-essential.*

Proof. By [5], the category pos-S fulfills Banaschewski's condition for regular essential monomorphisms. Now consider S-poset morphism $h : C \longrightarrow D$ such that hgf is an ov-embedding. Since gf is regular essential, h is an order embedding, and so h is ov-dense. ◀

Definition 5. *Let A be an S-poset. Then A is said to be ov-dense injective if it is injective with respect to all ov-dense monomorphisms. Also, A is called an ov-dense absolute retract if A is an ov-dense retract of each of its ov-dense extensions. Clearly, an ov-dense retract of any ov-dense injective S-poset is ov-dense injective. Clearly, S-poset A is ov-dense absolute retract if and only if it is ov-dense injective.*

By [8], an S-poset A is od-injective if it is injective with respect to the class of order dense embeddings.

Theorem 4. *Let A be an S-poset. It is od-injective and vitally dense injective if and only if it is an order vitally dense injective.*

Proof. Sufficiency is clear. For necessity, let B be an order vitally dense sub-S-poset of C and $f : B \longrightarrow C$ be an S-poset morphism. We will show that there exists an S-poset morphism $h : C \longrightarrow A$ such that $h_{|B} = f$. Clearly, B is an order dense sub-S-poset of $\downarrow B$ and $\downarrow B$ is a vitally sub-S-poset of C. So, there exists S-poset morphism $g :\downarrow B \longrightarrow A$ and $h : C \longrightarrow A$ such that $g_{|B} = f$ and $h_{|\downarrow B} = g$, since A is od-injective and vitally dense injective. Thus, we have $h_{|B} = f$, as required. ◀

Theorem 5. *Let S be a pomonoid. Then the following assertions are equivalent for each S-poset A:*
(i) *A is ov-dense injective.*
(ii) *A is an ov-dense absolute retract.*
(iii) *A has no proper $\mathcal{M}_{3ov}$-essential extensions.*
(iv) *A has no proper $\mathcal{M}_{2ov}$-essential extensions.*
(v) *A has no proper $\mathcal{M}_{1ov}$-essential extensions.*

Proof. We will only prove the assertions $(ii) \Rightarrow (iii)$ and $(v) \Rightarrow (ii)$. Let B be an $\mathcal{M}_{3ov}$-essential extension of A. By (ii), there exists an *ov*-dense retraction $g : B \longrightarrow A$ such that $g_{|A} = id_A$. Since A is a $\mathcal{M}_{3ov}$-essential in B, g is a monomorphism. So, for any $b \in B$, we have $g(b) \in A$, and so $g(g(b)) = g(b)$. Thus $g(b) = b$, and we have $b \in A$. Therefore $B = A$.

Now, let us show the implication $(v) \Rightarrow (ii)$. For this, suppose that B is an *ov*-dense extension of A. By Proposition 3, there exists an S-poset morphism $g : B \longrightarrow A$ such that $g_{|A}$ is a $\mathcal{M}_{1ov}$-essential. Now, $g_{|A}$ is an isomorphism by (v). So, $g(g_{|A})^{-1}$ is a retraction, and thus A is an *ov*-dense retract of B. ◀

Definition 6. *Let A be an S-poset. Then S-poset B is called:*

(i) a maximal $\mathcal{M}_{iov}$-essential extension of A, $i = 1, 2, 3$ if B is a $\mathcal{M}_{iov}$-essential extension of A and for any S-poset morphism $h : B \longrightarrow C$, where C is a $\mathcal{M}_{iov}$-essential extension of A and $h_{|A}$ is an inclusion map, is an isomorphism.

(ii) a minimal $\mathcal{M}_{iov}$-essential extension of A, $i = 1, 2, 3$ if B is a $\mathcal{M}_{iov}$-essential extension of A and for any S-poset morphism $h : C \longrightarrow B$, where C is a $\mathcal{M}_{iov}$-essential extension of A, which maps A identically, is an isomorphism.

Lemma 3. *If B is a $\mathcal{M}_{iov}$-essential extension of an S-poset A, $i = 1, 2, 3$, and A is order vital dense embedded into some regular injective S-poset Q, then B can also be order vital dense embedded into Q.*

Proof. Straightforward. ◀

Theorem 6. *Every S-poset has a maximal $\mathcal{M}_{iov}$-essential extension, for $i = 1, 2, 3$.*

Proof. Let A be an S-poset, and P be the set of all $\mathcal{M}_{iov}$-essential extensions of an S-poset A, $i = 1, 2, 3$. Consider P as a partially ordered set under inclusion. For any chain $(A_i)_{i \in I}$ in P, $\bigcup_{i \in I} A_i \in P$ is an upper bound. By Zorn's lemma, P has a maximal element M which is in fact a maximal $\mathcal{M}_{iov}$-essential extension of an S-poset A, $i = 1, 2, 3$. ◀

Lemma 4. *Let A be an order vitally dense sub-S-poset of S-poset B and B be an order vitally dense sub-S-poset of C. Then A is a $\mathcal{M}_{iov}$-essential, $i = 1, 2, 3$ in C if and only if A is a $\mathcal{M}_{iov}$-essential in B, and B is a $\mathcal{M}_{iov}$-essential in C.*

Proof. It's clear. ◀

Definition 7. *S-poset B is a $\mathcal{M}_{iov}$-injective hull of A, for $i = 1, 2, 3$ if B is a $\mathcal{M}_{iov}$-essential extension of A as well as ov-injective. Clearly, $\mathcal{M}_{iov}$-injective hull of an S-poset, if it exists, is unique up to isomorphism.*

Theorem 7. *Let A be an S-act and B be an ov-dense extension of A. The following assertions are equivalent:*
(i) *B is a $\mathcal{M}_{1ov}$-injective hull of A.*
(ii) *B is a $\mathcal{M}_{3ov}$-injective hull of A.*
(iii) *B is a $\mathcal{M}_{2ov}$-injective hull of A.*
(iv) *B is a maximal $\mathcal{M}_{1ov}$-essential extension of A.*
(v) *B is a maximal $\mathcal{M}_{3ov}$-essential extension of A.*
(vi) *B is a maximal $\mathcal{M}_{2ov}$-essential extension of A.*
(vii) *B is a minimal ov-injective extension of A.*

Proof. Let us show the validity of the assertions $(i) \Rightarrow (iv), (iv) \Rightarrow (i), (iv) \Rightarrow (vii), (vii) \Rightarrow (i)$. The rest are clear.

$(i) \Rightarrow (iv)$, it is obtained by 4 and 5.

$(iv) \Rightarrow (i)$, let B be a maximal $\mathcal{M}_{1ov}$-essential extension of A. Then, by Lemma 4, it has no proper $\mathcal{M}_{1ov}$-essential extension, and by Theorem 3, the result holds.

$(iv) \Rightarrow (vii)$, let B be a maximal $\mathcal{M}_{1ov}$-essential extension of A. Then, by Lemma 4, it has no proper $\mathcal{M}_{1v}$-essential extension. So, it is an *ov*-dense injective, by Theorem 5. Let $f : C \longrightarrow B$ be an *ov*-dense embedding from an *ov*-dense injective order vitally dense extension C of A, which maps A identically. Since A is a $\mathcal{M}_{1ov}$-essential in B, by Lemma 4, it follows that $f(C)$ is a $\mathcal{M}_{1ov}$-essential and so $f(C) \cong C$ is an *ov*-dense injective. Now, by Theorem 5, we have $B = f(C)$. So, f is an isomorphism.

$(iv) \Rightarrow (vii)$, it is obtained by Theorem 6 and Theorem 5. ◀

The next corollary is obtained by Theorem 6 and Theorem 7.

Corollary 3. *For every S-poset, there exists a $\mathcal{M}_{iov}$-injective hull, for $i = 1, 2, 3$.*

Similar to Theorem 2.42 [9], we can prove the next Theorem.

Theorem 8. *In the category pos-S, $\mathcal{M}_{1ov}$-essential $\Rightarrow$ $\mathcal{M}_{2ov}$-essential $\Leftrightarrow$ $\mathcal{M}_{3ov}$-essential.*

Example 5. *Consider* $\mathbb{N}$ *and* $\mathbb{Z}$ *as* $(\mathbb{N}_0, +)$*-posets with usual addition and usual partial order. It is clear that* $\mathbb{Z}$ *is an ov-dense extension of* $\mathbb{N}$*. Similarly to [4], we can prove that* $\mathbb{Z}$ *is an* $\mathcal{M}_{2ov}$*-injective hulll of* $\mathbb{N}$*.*

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