Azerbaijan Journal of Mathematics
V. 14, No 1, 2024, January
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2024.1.3

Inverse Problem for a Hyperbolic Integro-Differential Equation with two Redefinition Conditions at the End of the Interval and Involution

T.K. Yuldashev*, O.Sh. Kilichev

Abstract. In this paper, we consider an inhomogeneous hyperbolic type partial integro-differential equation with degenerate kernel, two redefinition functions and involution. Intermediate data are used to find these redefinition functions. Dirichlet boundary conditions with respect to spatial variable are used. The Fourier method of separation of variables is applied. The countable system of functional-integral equations is obtained. Theorem on a unique solvability of countable system of functional-integral equations is proved. The method of successive approximations is used in combination with the method of contraction mappings. The triple of solutions of the inverse problem is obtained in the form of Fourier series. Absolute convergence of Fourier series is proved.

Key Words and Phrases: inverse problem, two redefinition functions, final conditions at the endpoint of a segment, intermediate data, three parameters, involution, unique solvability.

2010 Mathematics Subject Classifications: 35A09, 35C10, 35G30, 35L82

---

1. Problem statement

Ordinary and partial integro-differential equations of the Fredholm type are of great interest in terms of theoretical research and applications in different fields of physics, mechanics, engineering and chemistry [1-8]. Today, some new problems are posed for ordinary and partial integro-differential equations, and a large number of papers are dedicated to the boundary value and inverse problems for integro-differential equations. Problems with nonlocal conditions for differential and integro-differential equations are considered in the large number of publications, such as [9-29]. In [30-36], integro-differential equations with a degenerate kernel were considered.

*Corresponding author.

In the present paper, we study the solvability of the inverse problem for a hyperbolic type partial integro-differential equation with a degenerate kernel, three parameters, final conditions at the end of the interval and involution. This paper differs from the other relevant papers as it requires finding two unknown redefinition functions. This inverse problem also differs from the corresponding direct problem.

In the rectangular domain $\Omega = \{0 < t < T, -1 < x < 1\}$, we consider the following partial integro-differential equation:

$$U_{tt}(t,x) - \omega^2 \left[U_{xx}(t,x) + \varepsilon\, U_{xx}(t,-x)\right] = \nu \int_0^T K(t,s)\, U(s,x)\, ds + \alpha(t)\, U(t,x), \quad (1)$$

where $0 < \alpha(t) \in C[0,T]$, T is a given positive number, $|\,\varepsilon\,| < 1$, ω is a positive parameter, ν is a nonzero real parameter, $K(t,s) = \sum\limits_{r=1}^{k} a_r(t)\, b_r(s)$, $a_r(t)$, $b_r(s) \in C\,[0;T]$. It is assumed that the systems of functions $\{a_r(t)\}$ and $\{b_r(s)\}$, $r = \overline{1,k}$ are linearly independent.

To solve partial integro-differential equation (1), we use Dirichlet boundary conditions with respect to spatial variable x

$$U(t,-1) = U(t,1) = 0, \quad 0 \le t \le T. \quad (2)$$

We use also following conditions at the endpoint of the given segment with respect to time variable t:

$$U(T,x) = \varphi_1(x), \quad U_t(T,x) = \varphi_2(x), \quad -1 \le x \le 1, \quad (3)$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are redefinition functions and enough smooth on the segment $[-1,1]$. For these functions, the following conditions are fulfilled: $\varphi_i(-1) = \varphi_i(1) = 0$, $i = 1,2$.

In order to determine the redefinition functions, we use the following two intermediate conditions:

$$U(t_1,x) = \psi_1(x), \quad U_t(t_1,x) = \psi_2(x), \quad -1 \le x \le 1, \quad (4)$$

where $\psi_1(x)$ and $\psi_2(x)$ are known and enough smooth functions on the segment $[-1,1]$, $0 < t_1 < T$. For the functions $\psi_1(x)$ and $\psi_2(x)$, the following conditions are fulfilled: $\psi_i(-1) = \psi_i(1) = 0$, $i = 1,2$.

The choice of conditions (3) and (4) with the final and intermediate data is due to the fact that in practice it is not always possible to determine the initial conditions. During the process of aluminum production, before the start of the production cycle, the raw material passes through firing and the state of the raw material at the beginning of the production cycle is not known. And the final expected state of the output will be unknown in reality. We find it from a known intermediate state. So, we have to solve an inverse problem to solve the partial integro-differential equation (1).

Problem statement. Find a triple of functions

$$\left\{ U(t,x) \in C\,(\overline{\Omega}) \cap C^{2,2}_{t,x}(\Omega),\ \varphi_i(x) \in C[-1,1],\ i=1,2 \right\},$$

the first of which satisfies the partial integro-differential equation (1) and the conditions (2)–(4), where $\overline{\Omega} = \{0 \le t \le T,\ -1 \le x \le 1\}$.

Note that the problem (1)–(4) is formulated in such a way that the direct problem (1)–(3) has a unique solution for all values of the parameter ω, and the inverse problem (1)–(4) has a unique solution only for some values of ω. In addition, the second parameter ν also plays an important role in the context of solvability.

2. Formal solution of the direct problem (1)-(3)

First, consider the homogeneous differential equation

$$U_{tt}(t,x) - \omega^2 \left[U_{xx}(t,x) + \varepsilon\, U_{xx}(t,-x) \right] = 0 \tag{5}$$

with boundary conditions of the Dirichlet type

$$U(t,-1) = U(t,1) = 0, \quad 0 \le t \le T. \tag{6}$$

Problem (5), (6) will be solved by the method of separation of variables: $U(t,x) = u(t)\vartheta(x)$. Then, from this problem we arrive at the following spectral problem for an ordinary differential equation

$$\vartheta''(x) + \varepsilon\vartheta''(-x) + \lambda\vartheta(x) = 0 \tag{7}$$

with boundary conditions

$$\vartheta(-1) = 0, \quad \vartheta(1) = 0. \tag{8}$$

In case of even eigenfunctions, equation (7) takes the form of

$$(1+\varepsilon)\vartheta_1''(x) + \lambda_1\vartheta_1(x) = 0. \tag{9}$$

Solving the differential equation (9) with conditions (8), we find the eigenvalues

$$\lambda_{\,1n} = (1+\varepsilon)\,\pi^2 (n+0.5)^2 \tag{10}$$

and the eigenfunctions

$$\vartheta_{1n}(x) = \cos\, \pi(n + 0.5)x, \quad n \in \mathbb{N}. \tag{11}$$

In the case of odd eigenfunctions, equation (7) takes the form of

$$(1 - \varepsilon)\vartheta_2''(x) + \lambda_2 \vartheta_2(x) = 0. \tag{12}$$

Solving the differential equation (12) with conditions (8), we find the eigenvalues and the corresponding eigenfunctions of the following problem:

$$\lambda_{2n} = (1 - \varepsilon)\, \pi^2 n^2, \quad |\, \varepsilon \,| < 1, \tag{13}$$

$$\vartheta_{2n}(x) = \sin\, \pi n\, x, \quad n \in \mathbb{N}. \tag{14}$$

Note that the eigenfunctions $\vartheta_{in}(x)\,(i = 1, 2)$ determined by (11) and (14) form a complete system of orthonormal eigenfunctions in the space $L_2[-1, 1]$. Therefore, we seek nontrivial solutions to the inhomogeneous partial integro-differential equation (1) in the form $U(t, x) = U_1(t, x) + U_2(t, x)$, where

$$U_i(t, x) = \sum_{n=1}^{\infty} u_{in}(t)\, \vartheta_{in}(x), \quad i = 1, 2, \tag{15}$$

are the Fourier series, and $U_1(t, x)$, $U_2(t, x)$ satisfy the following integro-differential equation:

$$U_{i\,tt}(t, x) - \omega^2 \left[U_{i\,xx}(t, x) + \varepsilon\, U_{i\,xx}(t, -x)\right] = \alpha(t)\, U_i(t, x) + \nu \int_0^T K(t, s)\, U_i(s, x)\, ds,$$

$$u_{in}(t) = \int_{-1}^{1} U_i(t, x)\, \vartheta_{in}(x)\, dx, \quad i = 1, 2. \tag{16}$$

Substituting the Fourier series (15) into this integro-differential equation, we obtain a countable system of second order ordinary differential equations

$$u_{in}''(t) + \omega^2 \lambda_{in}\, u_{in}(t) = \nu \sum_{r=1}^{k} a_r(t)\, \tau_{inr} + \alpha(t)\, u_{in}(t), \quad i = 1, 2, \tag{17}$$

where λ_{in} are the eigenvalues determined by (10) and (13),

$$\tau_{inr} = \int_0^T b_r(s)\, u_{in}(s)\, ds. \tag{18}$$

Solving the countable system of inhomogeneous differential equations (17) by the method of variation of arbitrary constants, we obtain

$$u_{in}(t) = A_{1n} \cos \sqrt{\lambda_{in}}\, \omega\, t + A_{2n} \sin \sqrt{\lambda_{in}}\, \omega\, t+$$

$$+\frac{\nu}{\sqrt{\lambda_{in}}\, \omega} \sum_{r=1}^{k} \tau_{inr} \int_0^t \sin \sqrt{\lambda_{in}}\, \omega\, (t-s)\, a_r(s)\, ds+$$

$$+\frac{1}{\sqrt{\lambda_{in}}\, \omega} \int_0^t \sin \sqrt{\lambda_{in}}\, \omega\, (t-s)\, \alpha(s)\, u_{in}(s)\, ds, \qquad (19)$$

where A_{1n} and A_{2n} are arbitrary coefficients of integration, to be determined later. By differentiating (19) with respect to t, we obtain

$$u'_{in}(t) = -\sqrt{\lambda_{in}}\omega A_{1n} \sin \sqrt{\lambda_{in}}\, \omega\, t + \sqrt{\lambda_{in}}\, \omega A_{2n} \cos \sqrt{\lambda_{in}}\, \omega\, t+$$

$$+\nu \sum_{r=1}^{k} \tau_{inr} \int_0^t \cos \sqrt{\lambda_{in}}\, \omega\, (t-s)\, a_r(s)\, ds+$$

$$+\int_0^t \cos \sqrt{\lambda_{in}}\, \omega\, (t-s)\, \alpha(s)\, u_{in}(s)\, ds. \qquad (20)$$

For redefinition functions $\varphi_1(x)$ and $\varphi_2(x)$ we set $\varphi_1(x) = \varphi_{11}(x)+\varphi_{21}(x)$ and $\varphi_2(x) = \varphi_{12}(x) + \varphi_{22}(x)$. Now, supposing that the redefinition functions $\varphi_{i1}(x)$ and $\varphi_{i2}(x)$ are expanding into a Fourier series and using the Fourier coefficients (16), from conditions (3) we obtain

$$u_{in}(T) = \int_{-1}^{1} U_i(T,x)\, \vartheta_{in}(x)\, dx = \int_{-1}^{1} \varphi_{i1}(x)\, \vartheta_{in}(x)\, dx = \varphi_{i1n}, \qquad (21)$$

$$u'_{in}(T) = \int_{-1}^{1} U_{it}(T,x)\, \vartheta_{in}(x)\, dx = \int_{-1}^{1} \varphi_{i2}(x)\, \vartheta_{in}(x)\, dx = \varphi_{i2n}, \quad i = 1,2. \qquad (22)$$

To find the unknown (arbitrary) coefficients A_{1n} and A_{2n} in (19) and (20), we use the boundary conditions (21) and (22). Then we arrive at the system of algebraic equations (SAE)

$$\begin{cases} A_{1n} \cos \sqrt{\lambda_{in}}\, \omega\, T + A_{2n} \sin \sqrt{\lambda_{in}}\, \omega\, T = \gamma_{1n}, \\ -A_{1n} \sin \sqrt{\lambda_{in}}\, \omega\, T + A_{2n} \cos \sqrt{\lambda_{in}}\, \omega\, T = \gamma_{2n}, \end{cases} \qquad (23)$$

where

$$\gamma_{1\,n}=\varphi_{i\,1\,n}-\frac{\nu}{\sqrt{\lambda_{i\,n}}\,\omega}\sum_{r=1}^{k}\tau_{i\,n\,r}\int\limits_{0}^{T}\sin\,\sqrt{\lambda_{i\,n}}\,\omega\,(T-s)\,a_r(s)\,ds-$$

$$-\frac{1}{\sqrt{\lambda_{i\,n}}\,\omega}\int\limits_{0}^{T}\sin\,\sqrt{\lambda_{i\,n}}\,\omega\,(T-s)\,\alpha(s)\,u_{i\,n}(s)\,ds,$$

$$\gamma_{2\,n}=\varphi_{i\,2n}-\nu\sum_{r=1}^{k}\tau_{i\,n\,r}\int\limits_{0}^{T}\cos\,\sqrt{\lambda_{i\,n}}\,\omega\,(T-s)\,a_r(s)\,ds-$$

$$-\int\limits_{0}^{T}\cos\,\sqrt{\lambda_{i\,n}}\,\omega\,(T-s)\,\alpha(s)\,u_{i\,n}(s)\,ds.$$

For unique solvability of SAE (23), the condition

$$\delta_{0n}=\begin{vmatrix}\cos\,\sqrt{\lambda_{i\,n}}\,\omega\,T & \sin\,\sqrt{\lambda_{i\,n}}\,\omega\,T\\ -\sin\,\sqrt{\lambda_{i\,n}}\,\omega\,T & \cos\,\sqrt{\lambda_{i\,n}}\,\omega\,T\end{vmatrix}\neq 0$$

must be fulfilled. Since $\delta_{0n}=1$, this condition holds for all values of the parameter ω. Consequently, SAE (23) has a unique solution

$$A_{1n}=\begin{vmatrix}\gamma_{1n} & \sin\,\sqrt{\lambda_{i\,n}}\,\omega\,T\\ \gamma_{2n} & \cos\,\sqrt{\lambda_{i\,n}}\,\omega\,T\end{vmatrix}=\varphi_{i\,1n}\,\cos\,\sqrt{\lambda_{i\,n}}\,\omega\,T-\varphi_{i\,2n}\,\sin\,\sqrt{\lambda_{i\,n}}\,\omega\,T+$$

$$+\frac{\nu}{\sqrt{\lambda_{i\,n}}\,\omega}\sum_{r=1}^{k}\tau_{i\,n\,r}\int\limits_{0}^{T}\sin\,\sqrt{\lambda_{i\,n}}\,\omega\,s\,a_r(s)\,ds+$$

$$+\frac{1}{\sqrt{\lambda_{i\,n}}\,\omega}\int\limits_{0}^{T}\sin\,\sqrt{\lambda_{i\,n}}\,\omega\,s\,\alpha(s)\,u_{i\,n}(s)\,ds, \tag{24}$$

$$A_{2n}=\begin{vmatrix}\cos\,\sqrt{\lambda_{i\,n}}\,\omega\,T & \gamma_{1n}\\ -\sin\,\sqrt{\lambda_{i\,n}}\,\omega\,T & \gamma_{2n}\end{vmatrix}=\varphi_{\,i1n}\,\sin\,\sqrt{\lambda_{i\,n}}\,\omega\,T+\varphi_{\,i2n}\,\cos\,\sqrt{\lambda_{i\,n}}\,\omega\,T+$$

$$+\nu\sum_{r=1}^{k}\tau_{i\,n\,r}\int\limits_{0}^{T}\cos\,\sqrt{\lambda_{i\,n}}\,\omega\,s\,a_r(s)\,ds+\int\limits_{0}^{T}\cos\,\sqrt{\lambda_{i\,n}}\,\omega\,s\,\alpha(s)\,u_{i\,n}(s)\,ds. \tag{25}$$

Substituting (24) and (25) into (19), we obtain

$$u_{in}(t,\nu,\omega)=\varphi_{i1n}\,\chi_{i1n}(t,\omega)+\varphi_{i2n}\,\chi_{i2n}(t,\omega)+$$

$$+\frac{\nu}{\sqrt{\lambda_{in}}\,\omega}\sum_{r=1}^{k}\tau_{inr}\,\chi_{3nr}\,(t,\omega)+\frac{1}{\sqrt{\lambda_{in}}\,\omega}\int\limits_{0}^{T}H_{in}(t,s,\omega)\,\alpha(s)\,u_{in}(s,\nu,\omega)\,ds, \quad (26)$$

where

$$\chi_{i1n}(t,\omega)=\cos\,\sqrt{\lambda_{in}}\,\omega\,(T-t)-\sin\,\sqrt{\lambda_{in}}\,\omega\,(T-t),$$
$$\chi_{i2n}(t,\omega)=\cos\,\sqrt{\lambda_{in}}\,\omega\,(T+t)-\sin\,\sqrt{\lambda_{in}}\,\omega\,(T-t),$$
$$\chi_{i3rn}\,(t,\omega)=\int\limits_{0}^{T}H_{in}(t,s,\omega)\,a_r(s)\,ds,$$

$$H_{in}(t,s,\omega)=\begin{cases}\sin\,z\,(t+s), \quad z=\sqrt{\lambda_{in}}\,\omega, \quad i=1,2, \quad t<s\leq T,\\ \sin\,z\,(t-s)+\cos\,z\,t\,\sin\,z\,s+z\,\sin\,z\,t\,\sin\,z\,s, \;\; 0\leq s<t.\end{cases}$$

Although the functions (26) are the Fourier coefficients of the solution to the direct problem (1)–(3), they contain extra quantities τ_{inr} that are still unknown. To find these quantities, we substitute (26) into (18) and arrive at a new SAE:

$$\tau_{inr}-\frac{\nu}{\bar{\lambda}}\sum_{j=1}^{k}\tau_{ijnr}\,\sigma_{i3rjn}(t)=\varphi_{i1n}\,\sigma_{i1rn}+\varphi_{i2n}\sigma_{i2rn}+\sigma_{i4rn}(u_{in}), \qquad (27)$$

where

$$\sigma_{i1rn}=\int\limits_{0}^{T}b_r(s)\,\chi_{i1n}(s,\omega)\,ds, \quad \sigma_{i2rn}=\int\limits_{0}^{T}b_r(s)\chi_{i2n}(s,\omega)\,ds, \quad \bar{\lambda}=\sqrt{\lambda_{in}}\,\omega,$$

$$\sigma_{i3rjn}=\int\limits_{0}^{T}b_r(s)\int\limits_{0}^{T}H_{in}(s,\theta,\omega)\,a_j(\theta)\,d\theta\,ds,$$

$$\sigma_{i4rn}\,(u_{in})=\frac{1}{\bar{\lambda}}\int\limits_{0}^{T}b_r(s)\int\limits_{0}^{T}H_{in}(s,\theta,\omega)\,\alpha(\theta)\,u_{in}(\theta)\,d\theta\,ds.$$

To establish the unique solvability of SAE (27), we introduce the matrix

$$\Theta_{i0n}(\nu,\omega)=\begin{pmatrix}1-\frac{\nu}{\bar{\lambda}}\,\sigma_{i31\,1n} & \frac{\nu}{\bar{\lambda}}\,\sigma_{i31\,2n} & \ldots & \frac{\nu}{\bar{\lambda}}\,\sigma_{i31kn}\\ \frac{\nu}{\bar{\lambda}}\,\sigma_{i32\,1n} & 1-\frac{\nu}{\bar{\lambda}}\,\sigma_{i32\,2n} & \ldots & \frac{\nu}{\bar{\lambda}}\,\sigma_{i32kn}\\ \ldots & \ldots & \ldots & \ldots\\ \frac{\nu}{\bar{\lambda}}\,\sigma_{i3k\,1n} & \frac{\nu}{\bar{\lambda}}\,\sigma_{i3k\,2n} & \ldots & 1-\frac{\nu}{\bar{\lambda}}\,\sigma_{i3\,kkn}\end{pmatrix}$$

and consider the values of the parameter ν, for which the Fredholm determinant differs from zero:

$$\Delta_{i0n}(\nu,\omega) = \det\, \Theta_{i0n}\,(\nu,\omega) \neq 0. \tag{28}$$

Determinant $\Delta_{i0n}(\nu,\omega)$ in (28) is a polynomial with respect to $\frac{\nu}{\lambda}$ of degree no more than k. The countable system of algebraic equations $\Delta_{i0n}(\nu,\omega) = 0$ has no more than k different real roots for every value of n. We denote them by $\mu_l\,(l = \overline{1,p},\ 1 \leq p \leq k)$. Then $\nu_{in} = \nu_{iln} = \bar{\lambda}\,\mu_l = \sqrt{\lambda_{in}}\omega\,\mu_l$ are the characteristic (irregular) values of the kernel of the integro-differential equation (1). So, we introduce the following notations:

$$\Lambda_{i\,1} = \left\{(\nu_{in},\,\omega) :\, \nu_{in} = \sqrt{\lambda_{in}}\,\omega\,\mu_l,\ i = 1,2,\ \omega \in (0,\infty)\right\},$$

$$\Lambda_{i\,2} = \left\{(\nu_{in},\,\omega) :\, |\,\Delta_{\,i\,0n}\,(\nu_i,\omega)\,| > 0,\ \nu_{in} \neq \sqrt{\lambda_{in}}\,\omega\,\mu_l,\ i = 1,2,\ \omega \in (0,\infty)\right\}.$$

On the number set Λ_{i2}, we consider a matrix $\Theta_{irmn}(\nu,\omega) =$

$$= \begin{pmatrix} 1 - \frac{\nu_i}{\lambda}\sigma_{i31\,1n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i31\,(j-1)n} & \sigma_{i\,m\,1n} & \frac{\nu_i}{\lambda}\sigma_{i31\,(j+1)n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i31\,kn} \\ \frac{\nu_i}{\lambda}\sigma_{i32\,1n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i32\,(j-1)n} & \sigma_{i\,m\,2n} & \frac{\nu_i}{\lambda}\sigma_{i32\,(j+1)n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i32kn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\nu_i}{\lambda}\sigma_{i3k1n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i3k\,(j-1)n} & \sigma_{i\,m\,kn} & \frac{\nu_i}{\lambda}\sigma_{i\,3k\,(j+1)n} & \cdots & 1 - \frac{\nu_i}{\lambda}\sigma_{i3kkn} \end{pmatrix},$$

$m = 1,2,4$. Taking into account the known properties of the matrix $\Theta_{irmn}(\nu,\omega)$, we use the modified Cramer method on the set Λ_{i2} and obtain the solutions of SAE (27) in the form

$$\tau_{irn} = \varphi_{i1n}\frac{\Delta_{\,i1\,r\,n}(\nu,\omega)}{\Delta_{i0n}(\nu,\omega)} + \varphi_{i2n}\frac{\Delta_{\,i2\,r\,n}(\nu,\omega)}{\Delta_{i0n}\,(\nu,\omega)} + \frac{\Delta_{\,i4\,r\,n}(\nu,\omega,u_{in})}{\Delta_{i\,0n}(\nu,\omega)}, \tag{29}$$

where $i = 1,2,\ r = \overline{1,k},\ (\nu,\omega) \in \Lambda_{i2},\ \Delta_{\,irmn}(\nu,\omega) = \det\,\Theta_{irmn}(\nu,\omega),\ m = 1,2,4$. Substituting (29) into (26), we obtain

$$u_{in}(t,\,\nu,\omega) = \varphi_{\,i1n}\,h_{\,i1n}(t,\nu,\omega) + \varphi_{i2n}\,h_{\,i2n}(t,\nu,\omega) +$$

$$+ \frac{\nu_i}{\sqrt{\lambda_{in}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{\,i4\,r\,n}(\nu,\omega,u_{in})}{\Delta_{i0n}(\nu,\omega)}\,\chi_{\,i3\,r\,n}(t) +$$

$$+ \frac{1}{\sqrt{\lambda_{in}}\omega}\int_0^T H_{in}(t,s,\omega)\,u_{in}(s,\nu,\omega)\,ds, \quad (\nu,\omega) \in \Lambda_{i\,2}, \tag{30}$$

where

$$h_{ijn}(t,\nu,\omega)=\chi_{ijn}(t,\omega)+\frac{\nu_i}{\sqrt{\lambda_{in}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{ijrn}(\nu,\omega)}{\Delta_{i0n}(\nu,\omega)}\,\chi_{i3rn}(t,\omega),\ \ i,j=1,2,$$

$$\chi_{i1n}(t,\omega)=\cos\,\sqrt{\lambda_{in}}\,\omega\,(T-t)-\sin\,\sqrt{\lambda_{in}}\,\omega\,(T-t),$$
$$\chi_{i2n}(t,\omega)=\cos\,\sqrt{\lambda_{in}}\,\omega\,(T+t)-\sin\,\sqrt{\lambda_{in}}\,\omega\,(T-t),$$
$$\chi_{i3rn}(t,\omega)=\int_0^T H_{in}(t,s,\omega)\,a_r(s)\,ds,$$

$$H_{in}(t,s,\omega)=\begin{cases}\sin\,z\,(t+s),\ \ z=\sqrt{\lambda_{in}}\,\omega,\ \ t<s\le T,\\ \sin\,z\,(t-s)+\cos\,z\,t\,\sin\,z\,s+z\,\sin\,z\,t\,\sin\,z\,s,\ \ 0\le s<t.\end{cases}$$

The relation (30) is a countable system of functional-integral equations. Substituting (30) in the Fourier series (15), we obtain a formal solution of the direct problem (1)–(3) on the domain Ω :

$$U_i(t,x)=\sum_{n=1}^{\infty}\vartheta_{in}(x)\Big[\varphi_{i1n}\,h_{i1n}(t,\nu,\omega)+\varphi_{i2n}\,h_{i2n}(t,\nu,\omega)+$$
$$+\frac{\nu_i}{\sqrt{\lambda_{in}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{i4rn}(\nu,\omega,u_{in})}{\Delta_{i0n}(\nu,\omega)}\,\chi_{i3rn}(t)+$$
$$+\frac{1}{\sqrt{\lambda_{in}}\,\omega}\int_0^T H_{in}(t,s,\omega)\,u_{in}(s,\nu,\omega)\,ds\Big],\ \ (\nu,\omega)\in\Lambda_{i2}\ \ i=1,2. \tag{31}$$

However, there are two unknown quantities φ_{i1n} and φ_{i2n} in (31).

3. Formal solution of the inverse problem (1)-(4)

We will now formally define the redefinition functions $\varphi_{i1}(x)$ and $\varphi_{i2}(x)$. We subordinate function (30) to conditions (4). For this purpose, we differentiate (31) with respect to the time-variable t:

$$U_{it}(t,x)=\sum_{n=1}^{\infty}\vartheta_{in}(x)\left[\varphi_{i1n}\,h'_{i1n}(t,\nu,\omega)+\varphi_{i2n}\,h'_{i2n}(t,\nu,\omega)+\right.$$
$$+\frac{\nu_i}{\sqrt{\lambda_{in}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{i4rn}(\nu,\omega,u_{in})}{\Delta_{i0n}(\nu,\omega)}\,\chi'_{i3rn}(t)+$$

$$+\frac{1}{\sqrt{\lambda_{i\,n}}\,\omega}\int\limits_0^T H'_{i\,n}(t,s,\omega)\,u_{i\,n}(s,\nu,\omega)\,ds\Bigg],\quad (\nu,\omega)\in\Lambda_{i2},\ \ i=1,2. \tag{32}$$

Then, applying intermediate conditions (4) to functions (31) and (32), we arrive at the solution of the following SAE:

$$\begin{cases} \varphi_{i1n}\ [\chi_{\,i1n}(t_1,\omega)+\varepsilon_{i11n}]+\varphi_{i2n}\ [\chi_{\,i2n}(t_1,\omega)+\varepsilon_{i12n}]=\bar{\psi}_{1n}, \\ \varphi_{i1n}\ [\chi'_{\,i1n}(t_1,\omega)+\varepsilon_{i21n}]+\varphi_{i2n}\ [\chi'_{i2n}(t_1,\omega)+\varepsilon_{i22n}]=\bar{\psi}_{2n}, \end{cases} \tag{33}$$

where

$$\varepsilon_{\,i1j\,n}=\frac{\nu_i}{\sqrt{\lambda_{i\,n}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{i\,j\,r\,n}(\nu,\omega)}{\Delta_{i0n}(\nu,\omega)}\,\chi_{\,i\,3\,r\,n}(t_1,\omega),$$

$$\varepsilon_{\,i2j\,r\,n}=\frac{\nu_i}{\sqrt{\lambda_{i\,n}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{i\,j\,r\,n}(\nu,\omega)}{\Delta_{i0n}(\nu,\omega)}\,\chi'_{\,i3\,r\,n}(t_1,\omega),\ \ j=1,2,$$

$$\bar{\psi}_{1n}=\psi_{i1n}-\frac{\nu_i}{\sqrt{\lambda_{i\,n}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{\,i4\,r\,n}(\nu,\omega,u_{i\,n})}{\Delta_{i0n}(\nu,\omega)}\,\chi_{\,i3\,r\,n}(t_1,\omega)+$$

$$+\frac{1}{\sqrt{\lambda_{i\,n}}\,\omega}\int\limits_0^T H_{i\,n}(t_1,s,\omega)\,u_{i\,n}(s,\nu,\omega)\,ds, \tag{34}$$

$$\bar{\psi}_{2n}=\psi_{i2n}-\frac{\nu_i}{\sqrt{\lambda_{i\,n}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{\,i4\,r\,n}(\nu,\omega,u_{i\,n})}{\Delta_{i0n}(\nu,\omega)}\,\chi'_{\,i3\,r}(t_1,\omega)+$$

$$+\frac{1}{\sqrt{\lambda_{i\,n}}\,\omega}\int\limits_0^T H'_{i\,n}(t_1,s,\omega)\,u_{i\,n}(s,\nu,\omega)\,ds,\ \ i=1,2, \tag{35}$$

$$\psi_{i\,j\,n}=\int\limits_{-1}^{1}\psi_{\,i\,j}(x)\,\vartheta_{\,i\,n}(x)\,dx,\quad i,j=1,2.$$

The fulfillment of the following condition provides the unique solvability of SAE (33):

$$V_{i0n}(\omega)=\begin{vmatrix} \chi_{\,i1n}(t_1,\omega)+\varepsilon_{i11n} & \chi_{\,i2n}(t_1,\omega)+\varepsilon_{i12n} \\ \chi'_{\,i1n}(t_1,\omega)+\varepsilon_{i21n} & \chi'_{\,i2n}(t_1,\omega)+\varepsilon_{i22n} \end{vmatrix}=$$

$$=-z\,\sin\,2zT-z\,\cos\,2zT+2z\,\sin\,z(T-t_1)\,\cos\,z(T-t_1)-z\,\cos\,2z(T-t_1)-$$

$$-z\,\varepsilon_{i11n}[\sin\,z(T+t_1)+\cos\,z(T-t_1)]-z\varepsilon_{i12n}[\sin\,z(T-t_1)+\cos\,z(T-t_1)]-$$

$$-\varepsilon_{i21n}[\cos z(T+t_1) - z \sin z(T-t_1)] - \varepsilon_{i22n}[\sin z(T-t_1) - z \cos z(T-t_1)]+$$

$$+\varepsilon_{i11n}\varepsilon_{i22n} - \varepsilon_{i21n}\varepsilon_{i12n} \neq 0, \quad z = \sqrt{\lambda_{in}}\,\omega, \quad i = 1, 2. \tag{36}$$

Before proceeding to find a solution of SAE (33), we consider condition (36). Suppose the opposite:

$$V_{i0n}(\omega) = 0, \quad i = 1, 2. \tag{37}$$

Condition (37) is a transcendental equation. Let us denote the set of its solutions with respect to ω by $\Im$. So, on the set

$$\Lambda_{i3} = \left\{(\nu_{in}, \omega) : |\,\Delta_{i0n}(\nu, \omega)\,| > 0, \ \nu_{in} \neq \sqrt{\lambda_{in}}\omega\ \mu_l, \ i = 1, 2, \ \omega \in \Im\right\}$$

SAE (33) is not uniquely solvable. However, on the other set

$$\Lambda_{i4} = \left\{(\nu, \omega) : |\Delta_{i0n}(\nu, \omega)| > 0, \, |V_{i0n}(\omega)| > 0, \, \nu_{in} \neq \sqrt{\lambda_{in}}\omega\,\mu_l, \, \omega \in (0, \infty) \setminus \Im\right\}$$

SAE (33) is uniquely solvable. Now let us start solving SAE (33). Taking into account notations (34) and (35), we obtain

$$\varphi_{ijn} = \psi_{i1n}w_{ij1n}(\omega) + \psi_{i2n}w_{ij2n}(\omega) + \frac{\nu_i}{\sqrt{\lambda_{in}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{i4rn}(\nu, \omega, u_{in})}{\Delta_{i0n}(\nu, \omega)}\,w_{ij3rn}(\omega)+$$

$$+\frac{1}{\sqrt{\lambda_{in}}\,\omega}\int_0^T W_{ijn}(s, \omega)\,u_{in}(s, \nu, \omega)\,ds, \quad i, j = 1, 2, \quad (\nu, \omega) \in \Lambda_{i4}, \tag{38}$$

where

$$w_{i11n}(\omega) = V_{i0n}^{-1}\left(\chi'_{i2n}(t_1, \omega) + \varepsilon_{i22n}(\omega)\right), \quad w_{i12n}(\omega) = V_{i0n}^{-1}\left(-\chi_{i2n}(t_1, \omega) + \varepsilon_{i12n}(\omega)\right),$$

$$w_{i21n}(\omega) = V_{i0n}^{-1}\left(\chi'_{i1n}(t_1, \omega) + \varepsilon_{i21n}(\omega)\right), \quad w_{i22n}(\omega) = V_{i0n}^{-1}\left(\chi_{i1n}(t_1, \omega) + \varepsilon_{i11n}(\omega)\right),$$

$$w_{i13rn}(\omega) = -\left[\chi_{i3rn}(t_1, \omega)w_{i11n}(\omega) + \chi'_{i3rn}(t_1, \omega)\,w_{i12n}(\omega)\right],$$

$$w_{i23rn}(\omega) = -\left[\chi_{i3rn}(t_1, \omega)\,w_{i21n}(\omega) + \chi'_{i3rn}(t_1, \omega)\,w_{i22n}(\omega)\right],$$

$$W_{i1n}(s, \omega) = H_{in}(t_1, s)\,w_{i11n}(\omega) + H'_{in}(t_1, s)\,w_{i12n}(\omega),$$

$$W_{i2n}(s, \omega) = H_{in}(t_1, s)\,w_{i21n}(\omega) + H'_{in}(t_1, s)\,w_{i22n}(\omega), \quad i = 1, 2.$$

Since φ_{i1n} and φ_{i2n} are the Fourier coefficients, from (38) we obtain the Fourier series

$$\varphi_{ij}(x) = \sum_{n=1}^{\infty}\vartheta_{in}(x)\left[\psi_{i1n}w_{ij1n} + \psi_{i2n}w_{ij2n} + \frac{\nu_i}{\sqrt{\lambda_{in}}\omega}\sum_{r=1}^{k}\frac{\Delta_{i4rn}(\nu, \omega, u_{in})}{\Delta_{i0n}(\nu, \omega)}\,w_{ij3rn}+\right.$$

$$+\frac{1}{\sqrt{\lambda_{i\,n}}\,\omega}\int\limits_0^T W_{i\,j\,n}(s,\omega)\,u_{i\,n}(s,\nu,\omega)\,ds\Bigg],\ \ (\nu,\omega)\in\Lambda_{i4},\ \ i=1,2. \qquad (39)$$

The functions $u_{i\,n}(t,\nu,\omega)$ in (39) are the Fourier coefficients of the unknown function $U_i(t,x,\nu,\omega)$. Therefore, we need to uniquely define the Fourier coefficients $u_{i\,n}(t,\nu,\omega)$. Substituting (38) into (30), we finally obtain the following countable system of functional-integral equations:

$$u_{i\,n}(t,\nu,\omega)=S(t,\nu,\omega;u_{i\,n})\equiv\psi_{i1n}\,g_{i1n}(t,\nu,\omega)+\psi_{i2n}\,g_{i2n}(t,\nu,\omega)+$$

$$+\frac{\nu_i}{\sqrt{\lambda_{i\,n}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{i4\,r\,n}(\nu,\omega,u_{i\,n})}{\Delta_{i0n}(\nu,\omega)}\,g_{i\,3\,r\,n}(t,\omega)+$$

$$+\frac{1}{\sqrt{\lambda_{i\,n}}\,\omega}\int\limits_0^T G_{i\,n}(t,s,\nu,\omega)\,u_{i\,n}(s,\nu,\,\omega)\,ds,\ \ \ (\nu,\omega)\in\Lambda_{i\,4},\ \ i=1,2, \qquad (40)$$

where

$$g_{i1n}(t,\nu,\omega)=w_{i11n}(\omega)h_{\,i1n}(t,\nu,\omega)+w_{i21n}(\omega)h_{\,i2n}(t,\nu,\omega),$$

$$g_{i2n}(t,\nu,\omega)=w_{i12n}(\omega)\,h_{\,i1n}(t,\nu,\omega)+w_{i22n}(\omega)\,h_{\,i2n}(t,\nu,\omega),$$

$$g_{\,i\,3\,r\,n}(t,\omega)=g_{i\,1n}(t,\nu,\omega)\,\chi_{i3\,r\,n}(t_1,\omega)+g_{i2n}(t,\nu,\omega)\,\chi'_{i3\,r\,n}(t_1,\omega)+\chi_{i3\,r\,n}(t,\omega),$$

$$G_{i\,n}(t,s,\nu,\omega)=g_{i1n}(t,\nu,\omega)\,H_{i\,n}(t_1,s,\omega)+g_{i2n}(t,\nu,\omega)\,H'_{i\,n}(t_1,s,\omega)+H_{i\,n}(t,s,\omega).$$

Note that the functional-integral equations (40) make sense only for values of parameters ν, ω from the set Λ_{i4}. In addition, the unknown functions $u_{\,i\,n}(t,\nu,\omega)$ in (40) are under the determinant sign and under the of integral sign.

4. Solvability of the countable system of functional-integral equations

Let us investigate the system of functional-integral equations (40) for unique solvability. To this end, consider the following well-known Banach spaces, which we will use in our further actions:
The space B_2 of function sequences $\{\,u_n(t)\,\}_{n=1}^{\infty}$ on the segment $[0,T]$ with the norm

$$\|\,u(t)\,\|_{\,B_{\,2}}=\left\{\sum_{n=1}^{\infty}\left(\max_{t\in[0,T]}\,|\,u_{\,n}(t)\,|\right)^2\right\}^{0.5}<\infty.$$

The Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_n\}_{n=1}^{\infty}$ with the norm

$$\|\varphi\|_{\ell_2} = \left\{ \sum_{n=1}^{\infty} |\varphi_n|^2 \right\}^{0.5} < \infty.$$

The space $L_2[-1,1]$ of square-integrable functions on $[-1,1]$ with the norm

$$\|\vartheta(x)\|_{L_2[-1,1]} = \left\{ \int_{-1}^{1} |\vartheta(x)|^2 \, dx \right\}^{0.5} < \infty.$$

Smoothness conditions. Let the functions $\psi_i(x) \in C^3[-1,1]$, $i = 1,2$ have peace-wise continuous derivatives with respect to x up to fourth order on $[-1,1]$. Then, after integrating the integrand functions $\psi_{ijn} = \int_{-1}^{1} \psi_{ij}(x)\,\vartheta_{in}(x)dx$, $i,j = 1,2$ by parts four times with respect to x, we obtain the following relation:

$$|\psi_{ijn}| \le \left(\frac{1}{\pi}\right)^4 \frac{\left|\psi_{ijn}^{(4)}\right|}{n^4}, \; i, \, j = 1,2, \tag{41}$$

where $\psi_{ijn}^{(4)} = \int_{-1}^{1} \frac{\partial^4 \psi_{ij}(x)}{\partial x^4} \vartheta_{in}(x)\,dx$, $i,j = 1,2$. Here the Bessel inequality is valid:

$$\sum_{n=1}^{\infty} \left[\psi_{ijn}^{(4)}\right]^2 \le \int_{-1}^{1} \left[\frac{\partial^4 \psi_i(x)}{\partial x^4}\right]^2 dx, \quad i,j = 1,2. \tag{42}$$

Theorem 1. *Let the smoothness conditions and the following conditions be fulfilled:*

$$\max_{t\in[0,T]} [|\, g_{i1n}(t,\nu,\omega)\,|\, ; \, |\, g_{i2n}(t,\nu,\omega)\,|] \le \delta_{i1} < \infty, \tag{43}$$

$$\rho_i = |\,\nu_i\,|\,\delta_{i2} \left\| \sum_{r=1}^{k} \left| \frac{\bar{\Delta}_{i4r}(\nu,\omega)}{\Delta_{i0}(\nu,\omega)} \right| \delta_{i0r} \right\|_{\ell_2} + \delta_{i3} < 1, \quad i = 1,2, \tag{44}$$

where δ_{i0r}, δ_{i2} and δ_{i0r} will be defined by (48) *and* (49)*, while $\bar{\Delta}_{i4rn}(\nu,\omega)$ will be defined by* (51)*. Then the countable system of functional-integral equations* (40) *is uniquely solvable in the space B_2. In this case, the desired solution can be found by the following iterative process:*

$$\begin{cases} u_{in}^{0}(t,\nu,\omega) = \psi_{i1n}\, g_{i1n}(t,\nu,\omega) + \psi_{i2n}\, g_{i2n}(t,\nu,\omega), \\ u_{in}^{m+1}(t,\nu,\omega) = S(t,\nu,\omega; u_{in}^{m}), \quad i = 1,2, \quad m = 0,1,2,... \end{cases} \tag{45}$$

Proof. By virtue of smoothness condition (41) and estimate (43), applying the Cauchy–Schwartz inequality and Bessel inequality (42), from the approximations (45) we obtain the following estimate:

$$\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left|\,u_{i\,n}^{0}(t)\,\right|\leq\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left[\left|\,\psi_{i\,1\,n}\right|\cdot\left|\,g_{i\,1n}(t,\nu,\omega)\,\right|+\left|\,\psi_{i\,2\,n}\right|\cdot\left|\,g_{i\,2n}(t,\nu,\omega)\,\right|\right]\leq$$

$$\leq\delta_{i\,1}\left(\frac{1}{\pi}\right)^{4}\left[\sum_{n=1}^{\infty}\frac{\left|\psi_{i\,1,n}^{(4)}\right|}{n^{4}}+\sum_{n=1}^{\infty}\frac{\left|\psi_{i\,2,n}^{(4)}\right|}{n^{4}}\right]\leq\delta_{i\,1}\left(\frac{1}{\pi}\right)^{4}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^{8}}}\times$$

$$\times\left[\left\|\frac{\partial^{4}\psi_{i\,1}(x)}{\partial x^{4}}\right\|_{L_{2}[-1,1]}+\left\|\frac{\partial^{4}\psi_{i\,2}(x)}{\partial x^{4}}\right\|_{L_{2}[-1,1]}\right]=\delta_{i\,0}<\infty. \qquad (46)$$

Taking into account the estimate (46), applying the Cauchy-Schwartz inequality, for the first difference of approximations (45) we obtain:

$$\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left|\,u_{i\,n}^{1}(t)-u_{i\,n}^{0}(t)\,\right|\leq$$

$$\leq\left|\,\nu_{i}\,\right|\sum_{n=1}^{\infty}\frac{1}{\lambda_{i\,n}^{3/2}\,\omega}\sum_{r=1}^{k}\left|\frac{\Delta_{i\,4\,r\,n}\left(\nu,\omega,u_{i\,n}^{0}\right)}{\Delta_{i\,0n}(\nu,\omega)}\right|\max_{t\in[0,T]}\left|g_{i\,3\,r\,n}(t,\omega)\right|+$$

$$+\sum_{n=1}^{\infty}\frac{1}{\lambda_{i\,n}^{3/2}\omega}\max_{t\in[0,T]}\left|\int_{0}^{T}G_{i\,n}(t,s,\nu,\omega)\,u_{i\,n}^{0}(s,\nu,\omega)\,ds\right|\leq$$

$$\leq\left|\,\nu_{i}\,\right|\delta_{i\,2}\sqrt{\sum_{n=1}^{\infty}\left[\sum_{r=1}^{k}\left|\frac{\Delta_{i\,4\,r\,n}\left(\nu,\omega,u_{i\,n}^{0}\right)}{\Delta_{i\,0n}(\nu,\omega)}\right|\delta_{i\,0\,r}\right]^{2}}+\delta_{i\,3}\delta_{i\,0}<\infty, \qquad (47)$$

where

$$\delta_{i\,0}\geq\max_{t\in[0,T]}\left|g_{i\,3\,r\,n}(t,\omega)\right|,\quad\delta_{i\,2}=\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_{i\,n}^{3}\omega^{2}}}, \qquad (48)$$

$$\delta_{i\,3}=\sqrt{\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left[\frac{1}{\lambda_{i\,n}^{3/2}\omega}\int_{0}^{T}\left|\,G_{i\,n}(t,s,\nu,\omega)\,\right|\,ds\right]^{2}},\quad i=1,2. \qquad (49)$$

Continuing this process, similarly to the estimate (47) we obtain

$$\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left|\,u_{i\,n}^{m+1}(t)-u_{i\,n}^{m}(t)\,\right|\leq$$

$$\leq |\nu_i|\,\delta_{i2}\sqrt{\sum_{n=1}^{\infty}\left[\sum_{r=1}^{k}\left|\frac{\Delta_{i4rn}(\nu,\omega,u_{in}^{m})-\Delta_{i4rn}\left(\nu,\omega,u_{in}^{m-1}\right)}{\Delta_{i0n}(\nu,\omega)}\right|\delta_{i0r}\right]^2}+$$

$$+\delta_{i3}\sqrt{\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left|u_{in}^{m}(t,\nu,\omega)-u_{in}^{m-1}(t,\nu,\omega)\right|^2}\leq$$

$$\leq |\nu_i|\,\delta_{i2}\sqrt{\sum_{n=1}^{\infty}\left[\sum_{r=1}^{k}\left|\frac{\bar{\Delta}_{i4rn}(\nu,\omega)}{\Delta_{i0n}(\nu,\omega)}\right|\delta_{i0r}\right]^2}\left\|u_i^m(t,\nu,\omega)-u_i^{m-1}(t,\nu,\omega)\right\|_{B_2}+$$

$$+\delta_{i3}\left\|u_i^m(t,\nu,\omega)-u_i^{m-1}(t,\nu,\omega)\right\|_{B_2}\leq\rho_i\left\|u_i^m(t,\nu,\omega)-u_i^{m-1}(t,\nu,\omega)\right\|_{B_2},\quad(50)$$

where

$$\rho_i=|\nu_i|\,\delta_{i2}\left\|\sum_{r=1}^{k}\left|\frac{\bar{\Delta}_{i4rn}(\nu,\omega)}{\Delta_{i0n}(\nu,\omega)}\right|\delta_{i0r}\right\|_{\ell_2}+\delta_{i3},\quad i=1,2,\quad \bar{\Delta}_{i4rn}(\nu,\omega)=$$

$$\begin{vmatrix} 1-\frac{\nu_i}{\lambda}\sigma_{i311n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i31(j-1)n} & \bar{\sigma}_{i41n} & \frac{\nu_i}{\lambda}\sigma_{i31(j+1)n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i31kn}\\ \frac{\nu_i}{\lambda}\sigma_{i321n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i32(j-1)n} & \bar{\sigma}_{i42n} & \frac{\nu_i}{\lambda}\sigma_{i32(j+1)n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i32kn}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \frac{\nu_i}{\lambda}\sigma_{i3k1n} & \cdots & \frac{\nu_i}{\lambda}\sigma_{i3k(j-1)n} & \bar{\sigma}_{i4kn} & \frac{\nu_i}{\lambda}\sigma_{i3k(j+1)n} & \cdots & 1-\frac{\nu_i}{\lambda}\sigma_{i3kkn}\end{vmatrix},\quad(51)$$

$$\bar{\sigma}_{i4rn}=\frac{1}{\lambda}\int_0^T|b_r(s)|\int_0^T|H_{in}(s,\theta,\omega)\,\alpha(\theta)|\,d\theta\,ds.$$

According to the condition (44), $\rho_i<1$. Consequently, it follows from the estimate (50) that the operator on the right-hand side of the countable system of functional-integral equations (40) is contracting. Then the estimates (46), (47) and (50) imply that there is a unique fixed point, which is a solution to (40) in the space B_2. Theorem 1 is proved. ◄

5. Uniform convergence of series

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. Then the series in* (39) *are convergent in* $[-1,1]$.

Proof. According to the Theorem 1, $u_{in}(t,\nu,\omega)\in B_2$ is a solution of the system (40). As in the case of estimates (46) and (50), we obtain

$$|\varphi_{ij}(x)|\leq\left(\frac{1}{\pi}\right)^4\delta_{i1}\delta_{i2}\left[\left\|\frac{\partial^4\psi_{i1}(x)}{\partial x^4}\right\|_{L_2[-1,1]}+\left\|\frac{\partial^4\psi_{i2}(x)}{\partial x^4}\right\|_{L_2[-1,1]}\right]+$$

$$+\left|\nu_i\right|\delta_{i2}\sqrt{\sum_{n=1}^{\infty}\left[\sum_{r=1}^{k}\left|\frac{\Delta_{i4rn}(\nu,\omega,u_{in})}{\Delta_{i0n}(\nu,\omega)}\right|\delta_{i0r}\right]^2}+\delta_{i3}\left\|u_i(t,\nu,\omega)\right\|_{B_2}<\infty, \tag{52}$$

where $i,j=1,2$. The estimate (52) implies the absolute convergence of the series (39). Hence, it is obvious that $|\,\varphi_j(x)\,|\le|\,\varphi_{1j}(x)\,|+|\,\varphi_{2j}(x)\,|<\infty,\ \ j=1,2.$ ◀

Substituting the system (40) into the Fourier series (15), we obtain

$$U_i(t,x)=\sum_{n=1}^{\infty}\vartheta_{\,i\,n}(x)\left[\psi_{i\,1n}\,g_{\,i\,1n}(t,\nu,\omega)+\psi_{i\,2n}\,g_{\,i\,2n}(t,\nu,\omega)+\right.$$

$$+\frac{\nu_i}{\sqrt{\lambda_{\,i\,n}}\,\omega}\sum_{r=1}^{k}\frac{\Delta_{\,i\,4\,r\,n}(\nu,\omega,u_{\,i\,n})}{\Delta_{\,i\,0n}\,(\nu,\omega)}\,g_{\,i\,3\,r\,n}(t,\omega)+$$

$$\left.+\frac{1}{\sqrt{\lambda_{\,i\,n}}\,\omega}\int_0^T G_{i\,n}(t,s,\nu,\omega)\,u_{\,i\,n}(s,\nu,\omega)\,ds\right],\quad (\nu,\omega)\in\Lambda_{i\,4},\ \ i=1,2. \tag{53}$$

Theorem 3. *Let the conditions of Theorem 1 be fulfilled. Then the main unknown function $U(t,x)=U_1(t,x)+U_2(t,x)$ of the inverse problem* (1)–(4) *is defined by the Fourier series* (53), *and this series* (53) *converges absolutely in the domain Ω for all $(\nu,\omega)\in\Lambda_{i\,4}$. Moreover, the function* (53) *belongs to the class $C\,(\overline{\Omega})\cap C^{2,2}_{t,x}(\Omega)$.*

The proof of Theorem 3 is similar that of Theorem 2.

6. Conclusion

In the rectangular domain $\Omega=\{0<t<T,\ -1<x<1\}$, we consider an inhomogeneous hyperbolic type integro-differential equation (1) with degenerate kernel, two redefinition functions (3) given at the endpoint of the segment and involution. To find these redefinition functions, we use intermediate data (4). We also use Dirichlet boundary value conditions (2) with respect to spatial variable x. The Fourier method of separation of variables is applied. The countable system of functional-integral equations (40) is obtained. Theorem 1 on unique solvability of countable system of functional-integral equations (40) is proved. The method of successive approximations is used in combination with the method of contraction mappings. The triple of solutions of the inverse problem is obtained in the form of Fourier series (39) and (53). The absolute convergence of Fourier series is proved (Theorems 2 and 3).

Remark 1. *For values of parameters* (ν, ω) *from the set* Λ_{i3}*, the uniqueness of the solution to the inverse problem* (1)–(4) *is violated. Because condition* (36) *is not satisfied in this case.*

Remark 2. *For values of parameters* (ν, ω) *from the set* Λ_{i1}*, the inverse problem* (1)–(4) *does not make sense. Because condition* (28) *is not satisfied in this case. But, the direct problem* (1)–(3) *has an infinite set of solutions, if* $\varphi_1(x) = \varphi_2(x) = 0$ *for all* $x \in [-1, 1]$ *and* $\alpha(t) \equiv 0$ *for all* $t \in [0, T]$.

References

[1] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. Ferreira, *Existence and uniform decay for a nonlinear viscoelastic equation with strong damping*, Math. Methods in the Appl. Sciences, **24**, 2001, 1043–1053.

[2] Ya. V. Bykov, *On some problems in the theory of integro-differential equations*, Izdatelstvo Kirg. Gos. Univ-ta, Frunze, 1957, 327 p. (in Russian).

[3] D. S. Dzhumabaev, E. A. Bakirova, *Criteria for the well-posedness of a linear two-point boundary value problem for systems of integro-differential equations*, Differential Equations, **46(4)**, 2010, 553–567.

[4] D. S. Dzhumabaev, S. T. Mynbayeva, *New general solution to a nonlinear Fredholm integro-differential equation*, Eurasian Math. Journal, **10(4)**, 2019, 24–33.

[5] M. M. Vainberg, *Integro-differential equations*, Itogi Nauki. 1962. VINITI, Moscow, 1964, **5(37)** (in Russian).

[6] M. V. Falaleev, *Integro-differential equations with a Fredholm operator at the highest derivative in Banach spaces and their applications*, Izv. Irkutsk. Gos. Universiteta, Matematika, **5(2)**, 2012, 90–102 (in Russian).

[7] N. A. Sidorov, *Solution of the Cauchy problem for a class of integro-differential equations with analytic nonlinearities*, Differ. Uravneniya, **4(7)**, 1968, 1309–1316 (in Russian).

[8] E. I. Ushakov, *Static stability of electrical circuits*, Nauka, Novosibirsk, 1988, 273 p. (in Russian).

[9] T. K. Yuldashev, R. N. Odinaev, S. K. Zarifzoda, *On exact solutions of a class of singular partial integro-differential equations*, Lobachevskii Journal of Mathematics, **42(3)**, 2021, 676–684.

[10] T. K. Yuldashev, S. K. Zarifzoda, *On a new class of singular integro-differential equations*, Bulletin of the Karaganda University. Mathematics series, **101(1)**, 2021, 138–148.

[11] A. T. Abildayeva, R. M. Kaparova, A. T. Assanova, *To a unique solvability of a problem with integral condition for integro-differential equation*, Lobachevskii Journal of Mathematics, **42(12)**, 2021, 2697–2706.

[12] A. T. Asanova, D. S. Dzhumabaev, *Correct solvability of a nonlocal boundary value problem for systems of hyperbolic equations*, Doklady Mathematics, **68(1)**, 2003, 46–49.

[13] A. T. Assanova, *On the solvability of nonlocal problem for the system of Sobolev-type differential equations with integral condition*, Georgian Mathematical Journal, **28(1)**, 2021, 49–57.

[14] A. T. Asanova, D. S. Dzhumabaev, *Well-posedness of nonlocal boundary value problems with integral condition for the system of hyperbolic equations*, Journal of Mathematical Analysis and Applications, **402(1)**, 2013, 167–178.

[15] A. T. Assanova, A. E. Imanchiyev, Zh. M. Kadirbayeva, *A nonlocal problem for loaded partial differential equations of fourth order*, Bulletin of the Karaganda university-Mathematics, **97(1)**, 2020, 6–16.

[16] A. T. Assanova, Z. S. Tokmurzin, *A nonlocal multipoint problem for a system of fourth-order partial differential equations*, Eurasian Math. Journal, **11(3)**, 2020, 8–20.

[17] D. G. Gordeziani, G. A. Avalishvili, *Gordeziani solutions of nonlocal problems for one-dimensional vibrations of a medium*, Matematicheskoye Modelirovaniye, **12(1)**, 2000, 94–103 (in Russian).

[18] D. S. Dzhumabaev, *Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution*, Journal of Mathematical Analysis and Applications, **461(1)**, 2018, 1439–1462.

[19] N. Sh. Isgenderov, S. I. Allahverdiyeva, *On solvability of an inverse boundary value problem for the Boussinesq-Love equation with periodic and integral condition*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, **41(1)**, 2021, 118–132.

[20] N. I. Ivanchov, *Boundary value problems for a parabolic equation with integral conditions*, Differential Equations, **40(4)**, 2004, 591–609.

[21] G. S. Mammedzadeh, *On a boundary value problem with spectral parameter quadratically contained in the boundary condition*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, **42(1)**, 2022, 141–150.

[22] N. K. Ochilova, T. K. Yuldashev, *On a nonlocal boundary value problem for a degenerate parabolichyperbolic equation with fractional derivative*, Lobachevskii Journal of Mathematics, **43(1)**, 2022, 229–236.

[23] I. V. Tikhonov, *Theorems on uniqueness in linear nonlocal problems for abstract differential equations*, Izvestiya: Mathematics, **67(2)**, 2003, 333–363.

[24] V. A. Yurko, *Inverse problems for first-order integro-differential operators*, Math. Notes, **100(6)**, 2016, 876–882.

[25] T. K. Yuldashev, *Determination of the coefficient and boundary regime in boundary value problem for integro-differential equation with degenerate kernel*, Lobachevskii Journal of Mathematics, **38(3)**, 2017, 547–553.

[26] T. K. Yuldashev, F. D. Rakhmonov, *Nonlocal problem for a nonlinear fractional mixed type integro-differential equation with spectral parameters*, AIP Conference Proceedings, **2365(060003)**, 2021, 1–20.

[27] S. K. Zaripov, *Construction of an analog of the Fredholm theorem for a class of model first order integrodifferential equations with a singular point in the kernel*, Vestnik Tomsk. gos. universiteta. Matematika i mekhanika, **46**, 2017, 24–35.

[28] S. K. Zaripov, *A construction of analog of Fredgolm theorems for one class of first order model integrodifferential equation with logarithmic singularity in the kernel*, Vestnik Samar. gos. tekh. univ. Fiz.-mat. nauki, **21(2)**, 2017, 236–248.

[29] S. K. Zaripov, *On a new method of solving of one class of model first-order integro-differential equations with singularity in the kernel*, Matematicheskaya fizika i compyuternoe modelirovanie, **20(4)**, 2017, 68–75.

[30] T. K. Yuldashev, *On Fredholm partial integro-differential equation of the third order*, Russian Mathematics, **59(9)**, 2015, 62–66.

[31] T. K. Yuldashev, *On a boundary-value problem for a fourth-order partial integro-differential equation with degenerate kernel*, Journal of Mathematical Sciences, **245(4)**, 2020, 508–523.

[32] T. K. Yuldashev, *Determining of coefficients and the classical solvability of a nonlocal boundary-value problem for the Benney-Luke integro-differential equation with degenerate kernel*, Journal of Mathematical Sciences, **254(6)**, 2021, 793–807.

[33] T. K. Yuldashev, *Inverse boundary-value problem for an integro-differential Boussinesq-type equation with degenerate kernel*, Journal of Mathematical Sciences, **250(5)**, 2020, 847–858.

[34] T. K. Yuldashev, Yu. P. Apakov, A. Kh. Zhuraev, *Boundary value problem for third order partial integro-differential equation with a degenerate kernel*, Lobachevskii Journal of Mathematics, **42(6)**, 2021, 1317–1327.

[35] T. K. Yuldashev, *On a boundary value problem for a fifth order partial integro-differential equation*, Azerbaijan Journal of Mathematics, **12(2)**, 2022, 154–172.

[36] T. K. Yuldashev, *On a nonlocal boundary value problem for a partial integro-differential equations with degenerate kernel*, Vladikavkaz Mathematical Journal, **24(2)**, 2022, 130–141.

Tursun K. Yuldashev
Tashkent State University of Economics, Karimov str., 49, Tashkent, 100066 Uzbekistan
E-mail: t.yuldashev@tsue.uz

Oybek Sh. Kilichev
Tashkent State University of Economics, Karimov str., 49, Tashkent, 100066 Uzbekistan
E-mail: oybe2402@mail.ru

Received 30 September 2022
Accepted 01 April 2023