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On the Fourier Transform of the Convolution of a Distribution and a Function Belonging to the Space $S_0(\mathbb{R})$

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Abstract. In this paper we consider the Fourier transform of the convolution of a distribution and a function which is an element of the space $S_0(\mathbb{R})$. Also, we give an application of the obtained result to the sequences that converge in the same space, and we give their analytic representation.

Key Words and Phrases: Fourier transform, convolution, distribution, space $S_0(\mathbb{R})$.

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1. Introduction

We will use general notations found in [2,4,5]. We denote with $S(\mathbb{R})$ the space of all functions of rapid decrease $\varphi \in C^\infty(\mathbb{R})$ for which

$$\rho^1_{k,n}(\varphi) = \sup_{x\in\mathbb{R}} \left| x^k \varphi^{(n)}(x) \right| < \infty, \ \ \forall k,\, n \in \mathbb{N}_0.$$

The dual space of $S(\mathbb{R})$ is the space of tempered distributions denoted by $S'(\mathbb{R})$.

L. Schwarts has considered the Fourier transform F of distributions in S'. The space S' has the important property that the Fourier transform of distribution in S' is also distribution in S'.

If $\varphi \in S$, then the Fourier transform of the function φ is defined as

$$F(\varphi, z) = \int_{\mathbb{R}} \varphi(t) e^{itz} dt$$

and it is an element of S.

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Also, for $\psi \in S$, the inverse of Fourier transform is defined as

$$F^{-1}(\psi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t) e^{-itz} dt$$

and it is an elements of the space S.

For $T \in S'$, the Fourier transform and the inverse Fourier transform are defined by
$\langle F(T), \varphi \rangle = \langle T_t, F(\varphi, t) \rangle$ *and* $\left\langle F^{-1}(T), \varphi \right\rangle = \left\langle T_t, F^{-1}(\varphi, t) \right\rangle$, $\varphi \in S$, respectively ([3,8]).

The function $\varphi \in L^2(\mathbb{R})$ is called a progressive (regressive) function if and only if $\operatorname{supp}\hat{\varphi} \subseteq (0, \infty]$ ($\operatorname{supp}\hat{\varphi} \subseteq [-\infty, 0)$), where $\hat{\varphi}(z) = F(\varphi, -2\pi z)$.

Lemma 1 ([6])**.** *Let $\varphi \in L^2(\mathbb{R})$ be a progressive function. Then the following conditions are equivalent:*

1. $\sup\limits_{x \in \mathbb{R}} (1 + |x|^2)^{p/2} |\varphi(x)| + \sup\limits_{w \geq 0} \frac{(1+w)^{2p+1}}{w^p} |\widehat{\varphi}(w)| < \infty, \ \forall p > 0;$

2. $\sup\limits_{x \in \mathbb{R}} (1 + |x|^2)^{p/2} |\varphi(x)| + \sup\limits_{w \geq 0} (1 + w^2)^{p/2} |\widehat{\varphi}(w)| < \infty, \ \forall p > 0.$

Definition 1. *i) Let $\varphi \in L^2(\mathbb{R})$ be a progressive function. Then $\varphi \in S_+(\mathbb{R})$ if and only if condition 1) or condition 2) from Lemma 1 is true.*
ii) $\varphi \in S_-(\mathbb{R}) \Leftrightarrow \varphi(-x) \in S_+(\mathbb{R})$.
iii) $S_0(\mathbb{R}) = S_+(\mathbb{R}) \otimes S_-(\mathbb{R})$.

The space $S_0(\mathbb{R})$ may be defined as a space of all functions of $S(\mathbb{R})$ with all its moments zero, i.e. $\varphi \in S_0(\mathbb{R})$ if and only if $\int_{\mathbb{R}} x^m \varphi(x) dx = 0$, $\forall m \in \mathbb{N}_0$, or $\widehat{\varphi}^{(n)}(0) = 0, \ \forall n \in \mathbb{N}_0$.

It is true that $S_0(\mathbb{R}) \subset S(\mathbb{R})$ is dense and $S_0'(\mathbb{R}) \simeq S'(\mathbb{R})/P(\mathbb{R})$, where $P(\mathbb{R})$ is a space of polynomials and $S_0'(\mathbb{R})$ is space of Lizorkin distributions.

For $\alpha \in \mathbb{Z}^+ \cup \{0\}$, the functions

$$x_+^\alpha = \begin{cases} x^\alpha, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad \text{and} \quad x_-^\alpha = \begin{cases} (-x)^\alpha, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

define Lizorkin distributions

$$x_+^\alpha : \varphi \to \int_0^\infty x^\alpha \varphi(x) dx,$$

and

$$x_-^\alpha : \varphi \to \int_{-\infty}^0 (-x)^\alpha \varphi(x) dx, \ \varphi(x) \in S(\mathbb{R}),$$

i.e. $\left\langle x_+^\alpha, \varphi \right\rangle = \int_0^\infty x^\alpha \varphi(x) dx$ and $\left\langle x_-^\alpha, \varphi \right\rangle = \int_{-\infty}^0 (-x)^\alpha \varphi(x) dx$, $\varphi(x) \in S(\mathbb{R})$.

Theorem 1 ([1, 7])**.** *Let $f \in S$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}/\{0\}$. Then*

1) $F(f^{(n)}, \omega) = (-i\omega)^n F(f(\omega))$;

2) $F(f(t-a), \omega) = e^{awi} F(f(\omega))$;

3) $F(f(at), \omega) = \frac{1}{|a|} F(f(\frac{\omega}{a}))$.

Theorem 2 ([1])**.** *Let $T \in S'$. Then*

1) $F(T^{(n)}) = (-it)^n F(T)$,

2) $F(T) = S, \quad S^{(n)} = F((i\omega)^n T)$.

Theorem 3 ([3])**.** *If $T \in D'$ is an arbitrary distribution, then $T = \sum_{j=1}^{\infty} T_j$, where each T_j has compact support and the following two conditions hold:*

a) Any compact subset of the real line intersects with supports of only finitely many supports of T_j.

b) $\lim\limits_{N\to\infty} \sum_{j=1}^{N} \langle T_j\, , \phi\rangle = \langle T, \phi\rangle$ for all $\phi \in D$.

2. Main results

Theorem 4. *Let $T \in \mathcal{D}'$ be with compact support and let $\varphi \in S_0(\mathbb{R})$. Then the Fourier transform of the convolution of the distribution T and the function φ is a function of the space $S_0(\mathbb{R})$ and equals to the product of their Fourier transforms, i.e.*

$$F(T * \varphi, \omega) = F(T, \omega) \cdot F(\varphi, \omega).$$

Proof. Since T has compact support, T is a tempered distribution and the convolution $T * \varphi$, for $\varphi \in S_0(\mathbb{R})$, is a function of the space $S(\mathbb{R})$. We will prove that it belongs to the space $S_0(\mathbb{R})$.
Since

$$\begin{aligned} &F^{(n)}\left((T*\varphi), \omega\right) = F\left((i\omega)^n \left(T*\varphi\right), \omega\right) = F\left((i\omega)^n \left\langle T_t, \varphi(x-t)\right\rangle, \omega\right) = \\ &= F\left(\left\langle T_t, (i\omega)^n \varphi(x-t)\right\rangle, \omega\right) = \left\langle T_t, F\left((i\omega)^n \varphi(x-t), \omega\right)\right\rangle = \\ &= \left\langle T_t, (i\omega)^n F(\varphi(x-t), \omega)\right\rangle = (i\omega)^n \left\langle T_t, F(\varphi(x-t), \omega)\right\rangle, \end{aligned}$$

we have $F^{(n)}\left((T*\varphi), 0\right) = 0$, so $\widehat{T*\varphi}^{(n)}(0) = 0$ which proves that $T*\varphi$ belongs to the space $S_0(\mathbb{R})$.

Next, we will prove that for a function $\varphi \in S_0(\mathbb{R})$, the Fourier transform of the convolution of the distribution T and the function φ belongs to $S_0(R)$, i.e. $F(T*\varphi, \omega) \in S_0(\mathbb{R})$. Firstly, we will prove that if $\varphi \in S_0(\mathbb{R})$, then $F(\varphi, \omega) \in S_0(\mathbb{R})$. Indeed, since $F^{(n)}\left(F(\varphi, \omega), \omega\right) = F^{(n+1)}\left(\varphi, \omega\right)$, for $\varphi \in S_0(\mathbb{R})$ we have

$F^{(n)}\left(F(\varphi,\omega),0\right)=0$, so $\widehat{F(\varphi,\omega)}^{(n)}(0)=0$, which proves that $F(\varphi,\omega)$ belongs to the space $S_0(\mathbb{R})$. Now, since we already proved that $T*\varphi\in S_0(\mathbb{R})$, for $\varphi\in S_0(\mathbb{R})$ we conclude that $F(T*\varphi,\omega)\in S_0(\mathbb{R})$.

We have

$$F(T*\varphi,\omega)=\int_{\mathbb{R}}(T*\ \varphi)(x)e^{i\omega x}dx=\int_{\mathbb{R}}\langle T_t,\varphi(x-t)\rangle\, e^{i\omega x}dx. \tag{1}$$

Since the integral on the right-hand side of (1) is a Riemann integral, we may rewrite it in the following form:

$$\int_{\mathbb{R}}\langle T_t,\varphi(x-t)\rangle\, e^{i\omega x}dx=\lim_{N\to\infty}\int_{-N}^{N}\langle T_t,\varphi(x-t)\rangle\, e^{i\omega x}dx,$$

for $N=1,2,3,...$

$$\begin{aligned}&\textstyle\int_{\mathbb{R}} l^m F(T*\varphi)dl=\int_{\mathbb{R}} l^m\int_{\mathbb{R}}\langle T_t,\varphi(x-t)\rangle\, e^{i\omega x}dxdl\\ &=\textstyle\int_{\mathbb{R}} l^m\lim\limits_{N\to\infty}\int_{-N}^{N}\langle T_t,\varphi(x-t)\rangle\, e^{i\omega x}dxdl=\lim\limits_{N\to\infty}\int_{\mathbb{R}} l^m\int_{-N}^{N}\langle T_t,\varphi(x-t)\rangle\, e^{i\omega x}dxdl.\end{aligned}$$

The function $f(x)=\langle T_t,\varphi(x-t)\rangle\, e^{i\omega x}$ is continuous and, by the first mean value theorem for integrals, it follows that that there exists a point $x_N\in[-N,N]$ such that

$$\int_{\mathbb{R}} l^m\int_{-N}^{N}\langle T_t,\varphi(x-t)\rangle\, e^{i\omega x}dxdl=2N\int_{\mathbb{R}} l^m\left\langle T_t,\varphi(x_N-t)e^{i\omega x_N}\right\rangle dl.$$

Now, we consider the sequences of functions $(f_N(t))$, where

$$f_N(t)=2N\varphi(x_N-t)e^{i\omega x_N}=\int_{-N}^{N}\varphi(x-t)e^{i\omega x}dx.$$

We will show that the sequence $(f_N(t))$ is uniformly bounded and equicontinuous. Since

$$|f_N(t)|=\left|\int_{-N}^{N}\varphi(x-t)e^{i\omega x}dx\right|\le\int_{-N}^{N}|\varphi(x-t)|\,dx\le\|\varphi\|_1\,,$$

$(f_N(t))$ is a uniformly bounded sequence.

Now, let $\varepsilon>0$ be a given number and $t',t''\in[-N,N]$ be points such that $|t'-t''|<\delta$ for some $\delta>0$.

Then

$$\left|f_N(t'')-f_N(t')\right|=\left|\int_{-N}^{N}\left[\varphi(x-t'')-\varphi(x-t')\right]e^{i\omega x}dx\right|$$

$$\leq \int_{-N}^{N} \left|\varphi(x-t'') - \varphi(x-t')\right| dx.$$

For a given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t', t'' \in [-N, N]$ with $|t'' - t'| < \delta$, we have

$$\int_{-N}^{N} \left|\varphi(x-t'') - \varphi(x-t')\right| dx < \varepsilon.$$

Thus, the sequence $(f_N(t))$ is equicontinuous.
Since

$$\lim_{N\to\infty} f_N(t) = \int_{-\infty}^{\infty} \varphi(x-t)e^{i\omega x}dx,$$

the Arzela-Ascoli theorem asserts that the sequence $(f_N(t))$ converges uniformly on every compact subset of $\mathbb{R}$ to the function

$$\int_{\mathbb{R}} \varphi(x-t)e^{i\omega x}dx.$$

The same is true for every sequence $({f_N}^{(k)}(t))$. Thus, we have shown that the sequence $(f_N(t))$ converges to the function

$$\int_{-\infty}^{\infty} \varphi(x-t)e^{i\omega x}dx$$

in E.

Since T is a continuous linear functional in the space E, the sequence

$$\int_{\mathbb{R}} l^m \left\langle T_t, \varphi(x_N - t)e^{ilx_N} \right\rangle dl,$$

converges to the function

$$\int_{\mathbb{R}} l^m \left\langle T_t, \int_{-\infty}^{\infty} \varphi(x-t)e^{ilx}dx \right\rangle dl.$$

If we set $u = x - t$, then

$$\begin{aligned} &\textstyle\int_{\mathbb{R}} l^m \left\langle T_t, \int_{-\infty}^{\infty} \varphi(x-t)e^{ilx}dx \right\rangle dl = \int_{\mathbb{R}} l^m \lim\limits_{N\to\infty} \left\langle T_t, f_N\,(t)\right\rangle dl \\ &= \textstyle\int_{\mathbb{R}} l^m \left\langle T_t, e^{ilt} \int_{\mathbb{R}} \varphi(u)e^{ilu}du \right\rangle dl \\ &= \textstyle\int_{\mathbb{R}} l^m \left\langle T_t, e^{ilt} \right\rangle \cdot \int_{\mathbb{R}} \varphi(u)e^{ilu}du\, dl = \int_{\mathbb{R}} l^m F(T,l) \cdot F(\varphi, l)dl. \end{aligned}$$

We conclude that $\int_{\mathbb{R}} l^m F(T,l) \cdot F(\varphi, l)dl = 0$. This implies that $F(T) \cdot F(\varphi) \in S_0$ and $F(T * \varphi, \omega) = F(T, \omega) \cdot F(\varphi, \omega)$, which completes the proof. ◄

Theorem 5. *Let $T \in D'$ have compact support and let (φ_k) be a sequence in $S_0(\mathbb{R})$ such that $\varphi_k \to \varphi$ in S_0. Then the sequence $F(T * \varphi_k, \omega)$ converges and it is element of S_0.*

Proof. The proof is similar to that of Theorem 4. Since $T * \varphi_k$ belongs to the space S_0, for every $k = 1, 2, 3, \dots$, it has a Fourier transform.

Thus,

$$F(T * \varphi_k, \omega) = \int_{\mathbb{R}} (T * \varphi_k)(x) e^{i\omega x} dx \quad for \quad k = 1, 2, 3, \dots .$$

We will show that

$$\lim_{k\to\infty} F(T * \varphi_k, \omega) = F(T, \omega) \cdot \lim_{k\to\infty} F(\varphi_k, \omega) =$$

$$= F(T, \omega) \cdot F(\varphi, \omega).$$

We have

$$\lim_{k\to\infty} \int_{\mathbb{R}} l^m \left\langle T_t, \int_{-\infty}^{\infty} \varphi_k(x-t) e^{ilx} dx \right\rangle dl$$

$$= \lim_{k\to\infty} \lim_{N\to\infty} \int_{\mathbb{R}} l^m \int_{-N}^{N} \left\langle T_t, \varphi_k(x-t) e^{ilx} dx \right\rangle dl.$$

Now we consider the sequence $(f_{N,k}(t))$, where

$$f_{N,k}(t) = 2N \varphi_k(x_N - t) e^{i\omega x_N}$$

$$= \int_{-N}^{N} \varphi_k(x-t) e^{i\omega x} dx.$$

The sequence $(f_{N,k})$ is uniformly bounded and equicontinuous.

Thus, the sequence

$$f_{N,k}(t) = 2N \varphi_k(x_N - t) e^{i\omega x_N}$$

converges to the function

$$\int_{-\infty}^{\infty} \varphi_k(x-t) e^{i\omega x} dx$$

in E.

Finally, if we take a limit as $k \to \infty$, we get

$$\lim_{k\to\infty} \int_{\mathbb{R}} l^m F(T * \varphi_k, l)\, dl = \lim_{k\to\infty} \int_{\mathbb{R}} l^m \left\langle T_t, \int_{-\infty}^{\infty} \varphi_k(x-t) e^{ilx} dx \right\rangle dl$$
$$= \lim_{k\to\infty} \lim_{N\to\infty} \int_{\mathbb{R}} l^m \int_{-N}^{N} \left\langle T_t, \varphi_k(x_N - t) e^{ilx_N} dx \right\rangle dl$$
$$= \lim_{k\to\infty} \int_{\mathbb{R}} l^m \int_{-\infty}^{\infty} \left\langle T_t, \varphi_k(x-t) e^{ilx} dx \right\rangle dl = \lim_{k\to\infty} \int_{\mathbb{R}} l^m F(T, l) \cdot F(\varphi_k, l) dl = 0.$$

So we have $\lim\limits_{k\to\infty} F\left(T * \varphi_k, \omega\right) = \lim\limits_{k\to\infty} F\left(T, \omega\right) \cdot F\left(\varphi_k, \omega\right) = F\left(T, \omega\right) \cdot F\left(\varphi, \omega\right).$ The proof is complete. ◀

Theorem 6. *Let $T_k \in D'$ be distributions with compact support, $\varphi \in S_0(\mathbb{R})$ and $\lim\limits_{k\to\infty} F\left(T_k * \varphi; \omega\right)$ and $\lim\limits_{k\to\infty} F(T_k, \omega) \cdot F(\varphi, \omega)$ exist. Then*

$$\lim_{k\to\infty} F\left(T_k * \varphi; \omega\right) = \lim_{k\to\infty} F(T_k, \omega) \cdot F(\varphi, \omega).$$

Proof. Since every T_k has a compact support, for every $\varphi \in S_0(\mathbb{R})$ the convolution $T_k * \varphi$ belongs to $S_0(\mathbb{R})$ and hence it has the Fourier transform $F\left(T_k * \varphi; \omega\right)$, which also belongs to the space $S_0(\mathbb{R})$. Thus, from the above lemma, we have

$$F\left(T_k * \varphi; \omega\right) = F(T_k, \omega) \cdot F(\varphi, \omega).$$

Since the sequence of the Fourier transforms of $S_0(\mathbb{R})$ converges uniformly on $\mathbb{R}$ to the Fourier transform of $S_0(\mathbb{R})$, by taking limits on both sides we get

$$\lim_{k\to\infty} F\left(T_k * \varphi; \omega\right) = \lim_{k\to\infty} F(T_k, \omega) \cdot F(\varphi, \omega).$$

◀

Theorem 7. *Let $f, g \in S_0$. Then $\int_{-\infty}^{\infty} f(t) F(g, t) dt = \int_{-\infty}^{\infty} F(f, w) g(w) dw$.*

Proof. Since $f, g \in S_0$, we have $F(f, w)$, $F(g, t) \in S_0$. From $S_0 \subset S \subset L^P$, we have $f, g \in L^1$. On the other hand, the product of two S_0 functions is in L^1. Hence the integral $\int_{-R}^{R} \int_{-A}^{A} \left| f(t) g(w) e^{iwt} \right| dwdt$ exists and, by Fubini's theorem, we get

$$\int_{-R}^{R} f(t) [\int_{-A}^{A} e^{iwt} g(w)\, dw] dt = \int_{-A}^{A} g(w) [\int_{-R}^{R} e^{iwt} f(t)\, dt] dw.$$

So,

$$\int_{-\infty}^{\infty} f(t) [\int_{-A}^{A} e^{iwt} g(w)\, dw] dt = \lim_{R\to\infty} \int_{-A}^{A} g(w) [\int_{-R}^{R} e^{iwt} f(t)\, dt] dw.$$

By Schwartz's inequality and Plancherel transform, we have

$$\left| \int_{-A}^{A} g(w) [\int_{|t|>R} e^{iwt} f(t)\, dt] dw \right|^2 \le \int_{-A}^{A} |g(w)|^2\, dw\, 2\pi \int_{|t|>R} |f(t)|^2\, dt,$$

and the right-hand side tends to zero. Hence

$$\lim_{R\to\infty} \int_{-A}^{A} g(w) [\int_{-R}^{R} e^{iwt} f(t)\, dt] dw = \int_{-A}^{A} g(w) F(f, w) dw,$$

which proves the theorem. ◀

Theorem 8. *Let $\varphi \in S_0$. Let $g(w) = F(\varphi, w)$ be the Fourier transform of φ. Let $\hat{g}(z)$ be the Cauchy representation of g, $z = x + iy$. Then*

$$\hat{g}(z) = \begin{cases} \int_0^\infty \varphi(t) e^{itz} dt, & y > 0, \\ -\int_{-\infty}^0 \varphi(t) e^{itz} dt, & y < 0. \end{cases}$$

Proof. If $\varphi \in S_0$, while $S_0 \subset L_1$, then $\varphi, g \in L_1$, which is defined as

$$\hat{g}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(w)}{w - z} dw$$

exists for $y \neq 0$. We have

$$\frac{1}{2\pi i(w - z)} = \begin{cases} F^{-1}(H(t)e^{itz}, w), & y > 0, \\ -F^{-1}(H(-t)e^{itz}, w), & y < 0. \end{cases}$$

Using Parseval's formula and the fact that $\varphi \in L_1$ and $\mathrm{H(t)e}^{itz} \in L_1$, we get

$$\hat{g}(z) = \begin{cases} \int_0^\infty \varphi(t) e^{itz} dt, & y > 0, \\ -\int_{-\infty}^0 \varphi(t) e^{itz} dt, & y < 0\,. \end{cases}$$

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