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Integration of the Loaded Sine-Gordon Equation by the Inverse Scattering Problem Method

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Abstract. In this paper, we consider the Cauchy-Goursat problem for a loaded sine-Gordon equation. The main results of the work are the theorem on the uniqueness of the solution of the problem under consideration and the theorem on the evolution of the scattering data of the Dirac operator whose potential is related to the solution of the loaded sine-Gordon equation. The equalities obtained in the scattering data evolution theorem make it possible to apply the method of inverse scattering problem to solve the considered problem.

Key Words and Phrases: loaded sine-Gordon equation, inverse scattering method, Cauchy-Goursat problem, Dirac operator, evolution of scattering data.

2010 Mathematics Subject Classifications: 34A55, 35A01, 35A25, 35A24

1. Introduction

Nonlinear evolution equations represent important physical applications. Therefore, it is always interesting for researchers to find soliton solutions of these equations [1, 2, 3] using direct and inverse methods. The sine-Gordon Equation

$$u_{xx} + u_{tt} = \sin u, \quad u = u(x,t), \;\; x \in R, \;\; t \geq 0,$$

is a non-linear partial differential equation that appears in differential geometry as an embedding equation for the Lobachevsky plane in three-dimensional Euclidean space. It also has applications in the study of superconductivity and Josephson effects [4].

Ablowitz, Kaup, Newell, and Segur [5] showed that the Cauchy problem for the sine-Gordon equation in light-cone coordinates

$$u_{xt} = \sin u, \quad u = u(x,t), \;\; x \in R, \;\; t \geq 0,$$

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is integrable and can be solved by the inverse scattering method. The integrability of the sine-Gordon equation was also studied by V.E. Zakharov, L.A. Tahtadjan, L.D. Faddeev [6]. The work [7] considers the dynamics of a sine-Gordon breather under the action of defects in condensed media, which is described by the perturbed sine-Gordon equation.

In this paper, we consider the Cauchy problem for the following loaded sine-Gordon equation:

$$\frac{\partial^2 u}{\partial x \partial t} = \sin u + \gamma(t) u_x(0,t) u_{xx}, \quad u = u(x,t), \ x \in R, \ t \geq 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in R, \tag{2}$$

where $\gamma(t)$ is a given bounded, continuous function, and the initial function $u_0(x)$ $(-\infty < x < \infty)$ has the following properties:

1) $u_0(x) \equiv 0 (mod\, 2\pi)$ as $|x| \to \infty$; $\int_{-\infty}^{\infty} ((1+|x|)\,|u_0'(x)| + |u_0''(x)|)\,dx < \infty$;

2) The operator $L(0) = i \begin{pmatrix} \frac{d}{dx} & \frac{u_0'}{2} \\ \frac{u_0'}{2} & -\frac{d}{dx} \end{pmatrix}$ has exactly N simple eigenvalues $\xi_1(0), \xi_2(0),\ \ldots,\ \xi_N(0)$ lying in the upper half-plane of the complex plane without spectral singularities.

Equation (1) belongs to the class of so-called loaded equations [8, 9, 10].

The solution $u(x,t)$ to problem (1)-(2) is sought in the class of functions that have sufficient smoothness and rather quickly tend to their limits as $x \to \pm\infty$:

$$u(x,t) \equiv 0 (mod\, 2\pi) \text{ as } |x| \to \infty; \int_{-\infty}^{\infty} ((1+|x|)\,|u_x(x,t)| + |u_{xx}(x,t)|)\,dx < \infty. \tag{3}$$

2. Uniqueness of the solution

In this section, we will use the method of [11].

Theorem 1. *If problem* (1)-(3) *has a solution, then it is unique.*

Proof. Let $u(x,t)$, $v(x,t)$ be different solutions of (1)-(3). Denoting $w(x,t) = u_{xx} - v_{xx}$, we have

$$w_t = \tfrac{1}{2}\left[(\cos\, u - \cos\, v)\,(u+v)_x + (\cos\, u + \cos\, v)\,(u-v)_x\right] +$$

$$+\frac{1}{2}\gamma(t)\left[(u_x(0,t) - v_x(0,t))\,(u+v)_{xxx} + (u_x(0,t) + v_x(0,t))\,w_x\right].$$

Multiplying this equality by $w(x,t)$ and integrating over x on the interval $(-\infty,\infty)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2 dx = -\int_{-\infty}^{\infty} \sin\,\frac{u+v}{2}\,\sin\,\frac{u-v}{2}\,(u+v)_x\,wdx+$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}(\cos\,u+\cos\,v)\,(u-v)_x\,wdx+$$

$$+\frac{1}{2}\gamma(t)\int_{-\infty}^{\infty}(u_x(0,t)-v_x(0,t))\,(u+v)_{xxx}\,wdx+$$

$$+\frac{1}{4}\gamma(t)\,(u_x(0,t)+v_x(0,t))\int_{-\infty}^{\infty}(w^2)_x dx.$$

Denoting $\max|(u+v)_x|$ by k , $\max|\cos\;u+\cos\;v|$ by l, $\max|(u+v)_{xxx}|$ by m and using the decreasing of $w(x,t)$ as $x\to\pm\infty$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2 dx \le \frac{k}{2}\int_{-\infty}^{\infty}|u-v|\,wdx+\frac{l}{2}\int_{-\infty}^{\infty}(u-\;v)_x\,wdx+$$

$$+\frac{\gamma(t)}{2}m\;\max|(u-v)_x|\int_{-\infty}^{\infty}wdx.$$

Using the Cauchy-Schwartz inequality, we get:

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2 dx \le k\sqrt{\int_{-\infty}^{\infty}(u-v)^2dx}\sqrt{\int_{-\infty}^{\infty}w^2dx}+$$

$$+l\sqrt{\int_{-\infty}^{\infty}((u-v)_x)^2dx}\sqrt{\int_{-\infty}^{\infty}w^2dx}+$$

$$+m\;\gamma(t)\;\max|(u-v)_x|\sqrt{\int_{-\infty}^{\infty}w^2dx}.$$

According to [12], there are constants $k_1,\;l_1,\;m_1>0$ such that

$$\sqrt{\int_{-\infty}^{\infty}(u-v)^2dx}\le k_1\sqrt{\int_{-\infty}^{\infty}((u-v)_x)^2dx}\,,$$

$$\sqrt{\int_{-\infty}^{\infty}((u-v)_x)^2dx}\le l_1\sqrt{\int_{-\infty}^{\infty}((u-v)_{xx})^2dx},$$

$$\max\ |(u-v)_x| \le m_1 \sqrt{\int_{-\infty}^{\infty} ((u-v)_{xx})^2 dx}.$$

So,

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2 dx \le k\,k_1\,l_1 \int_{-\infty}^{\infty} w^2 dx + l\,l_1 \int_{-\infty}^{\infty} w^2 dx + m\,m_1\,\gamma(t)\int_{-\infty}^{\infty} w^2 dx.$$

Denoting $\int_{-\infty}^{\infty} w^2 dx$ by $E(t)$ and setting $k\,k_1 l_1 + l\,l_1 + m\,m_1\,\gamma(t) = c(t)$, we obtain the inequality

$$\frac{dE(t)}{dt} \le c(t)E(t).$$

Consequently,

$$E(t) \le\ E(0)\exp\left(\int_0^t c(s)\,ds\right).$$

Since $E(0) = 0$, from the last inequality it follows $E(t) = 0$, i.e. $w(x,t) = u_{xx} - v_{xx} = 0$. Using conditions (2) and (3) we have $u_x = v_x$, therefore $u(x,t) = v(x,t)$.

Theorem 1 is proved. ◄

3. Scattering problem

In this section, the function $u(x,t)$ does not depend on t. Consider the system of Dirac equations in the form

$$\begin{cases} \nu_{1x} + i\xi\nu_{1x} = \frac{u'(x)}{2}\nu_2, \\ \nu_{2x} - i\xi\nu_2 = \frac{u'(x)}{2}\nu_1 \end{cases} \tag{4}$$

on the entire axis $-\infty < x < \infty$, with a potential $u(x)$ that satisfies the condition

$$\begin{array}{l} u(x) \equiv 0(mod\,2\pi)\ \ at\ \ |x| \to \infty; \\ \int_{-\infty}^{\infty}((1+|x|)\,|u'(x)|)dx < \infty. \end{array} \tag{5}$$

Let us state some important facts concerning the direct and inverse scattering problems for (4)-(5).

Under condition (5), the system of equations (4) has a Jost solutions with the following asymptotics:

$$\left.\begin{array}{l} \varphi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \\ \overline{\varphi} \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\xi x} \end{array}\right\} atx \to -\infty \tag{6}$$

$$\left.\begin{array}{l}\psi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} \\ \overline{\psi} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x}\end{array}\right\} at x \to \infty \tag{7}$$

(Note that $\bar{\varphi}$ is not a complex conjugate of φ).

For real ξ, the pairs of vector functions $\{\varphi(x,\xi),\ \bar{\varphi}(x,\xi)\}$ and $\{\psi(x,\xi),\ \bar{\psi}(x,\xi)\}$ are the pairs of linearly independent solutions for the system of equations (4). Therefore,

$$\left\{\begin{array}{l}\varphi = a(\xi)\bar{\psi} + b(\xi)\psi, \\ \bar{\varphi} = -\bar{a}(\xi)\psi + \bar{b}(\xi)\bar{\psi}.\end{array}\right. \tag{8}$$

It is easy to see that the following equality is true:

$$a(\xi) = W\{\varphi,\psi\} \equiv \varphi_1\psi_2 - \varphi_2\psi_1,$$

Moreover, for real ξ,

$$a(\xi)\bar{a}(\xi) + b(\xi)\bar{b}(\xi) = 1. \tag{9}$$

The function $a(\xi)$ admits an analytic continuation to the upper half-plane $Im\ \xi > 0$. For $|\xi| \to \infty$, $Im\xi \geq 0$, the function $a(\xi)$ has an asymptotics $a(\xi) = 1 + O\left(\frac{1}{|\xi|}\right)$. A function $a(\xi)$ can have only a finite number of zeros $\xi_k,\ k = 1,\ 2,\ ...,\ N$ in the half-plane $Im\ \xi > 0$. Zeros ξ_k of the function $a(\xi)$ correspond to the eigenvalues of the operator $L = i\begin{pmatrix} \frac{d}{dx} & \frac{u'}{2} \\ \frac{u'}{2} & -\frac{d}{dx} \end{pmatrix}$ in the upper half plane.

Note that the operator L can have multiple eigenvalues. We assume that the operator L has no spectral singularities and all its eigenvalues are simple, so that

$$\varphi(x,\xi_k) = C_k\psi(x,\xi_k)\ ,\ k = 1, 2, ..., N.$$

The requirement that the operator L has no spectral singularities means that the functions $a(\xi)$ and $\bar{a}(\xi)$ have no real zeros. The class of such operators is not empty. In particular, it contains operators with reflectionless potentials, i.e., $b(\xi) = 0$, because, in this case, according to (9), $a(\xi)\bar{a}(\xi) = 1$.

The following integral representation holds for the vector function ψ:

$$\psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} + \int_x^{\infty} \mathbf{K}\ (x,s)\, e^{i\xi s} ds, \tag{10}$$

where

$$\mathbf{K}\ (x,s) = \begin{pmatrix} K_1(x,s) \\ K_2(x,s) \end{pmatrix}.$$

In (10), the kernel $\mathbf{K}\,(x,s)$ does not depend on ξ and is related to $u(x)$ by means of the equality

$$u'(x) = 4K_1\,(x,x)\,. \tag{11}$$

The component of this kernel $K_1(x,y)$ in representation (11), as $y > x$ is a solution of the Gelfand-Levitan-Marchenko integral equation

$$K_1(x,y) - F(x+y) + \int_x^\infty \int_x^\infty K_1(x,z)F(z+s)F(s+y)dsdz = 0,$$

where

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(\xi)}{a(\xi)} e^{i\xi x} d\xi - i\sum_{j=1}^{N} C_j e^{i\xi_j x}.$$

Now the potential $u(x)$ is determined from the equality (11).

The set $\left\{ r^+\,(\xi) \equiv \frac{b(\xi)}{a(\xi)},\;\; \xi_k,\;\; C_k,\;\; k = 1,2,...,N \right\}$ is called scattering data for the system of equations (4).

Note that the vector functions

$$h_n\,(x) = \frac{\left.\frac{d}{d\xi}\,(\varphi - C_n\psi)\right|_{\xi = \xi_n}}{\dot{a}\,(\xi_n)}, \quad n = 1,2,...,N \tag{12}$$

are the solutions to the equation $Lh_n = \xi_n h_n$. According to the equality $a(\xi) = W\,\{\varphi,\psi\}$, we obtain the following asymptotics:

$$\psi \sim a(\xi)\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} \quad \text{as} \quad x \to -\infty,$$

$$\varphi \sim a(\xi)\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \quad \text{as} \quad x \to \infty,$$

which are valid for $Im\;\xi > 0$. It follows from these estimates and the equality (12) that

$$\begin{aligned} h_n &\sim -C_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi_n x} \quad \text{as} \quad x \to -\infty, \\ h_n &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi_n x} \quad \text{as} \quad x \to \infty. \end{aligned} \tag{13}$$

In particular,

$$W\,\{\varphi_n, h_n\} \equiv \varphi_{n1}h_{n2} - \varphi_{n2}h_{n1} = -C_n,$$

where

$$\varphi_n \equiv \varphi\,(x,\xi_n)\,, \quad n = 1,2,...,N.$$

4. Evolution of scattering data

In this section, we consider the equation

$$u_{xt} = \sin u + G, \tag{14}$$

where $G = G(x,t)$ is a sufficiently smooth function that, for any nonnegative value of t, satisfies the condition

$$G(x,t) = o(1) \quad as \quad x \to \pm\infty.$$

Equation (14) is considered under the initial condition (2).

The following theorem is true [13].

Theorem 2. *If the potential $u\,(x,t)$ is a solution to equation* (14) *in the class of functions* (3)*, then the scattering data of the system of equations* (4) *with the potential $u\,(x,t)$ depend on t as follows:*

$$\frac{\partial r^+}{\partial t} = -\frac{i}{2\xi}r^+ + \frac{1}{2a^2}\int_{-\infty}^{\infty}\left(G\varphi_2^2 + G\varphi_1^2\right)dx, \quad (Im\xi = 0)\,,$$

$$\frac{dC_n}{dt} = \left(-\frac{i}{2\xi_n} + \int_{-\infty}^{\infty}\frac{G}{2}\left(h_{n2}\psi_{n2} + h_{n1}\psi_{n1}\right)dx\right)C_n,$$

$$\frac{d\xi_n}{dt} = \frac{i\int_{-\infty}^{\infty}\left(G\varphi_{n2}^2 + G\varphi_{n1}^2\right)dx}{4\int_{-\infty}^{\infty}\varphi_{n1}\varphi_{n2}dx}, \quad n = 1,2,...,N.$$

Let us apply the result of Theorem 2 to equation (1) assuming

$$G(x,t) = \gamma(t)u_x(0,t)u_{xx},$$

$$\frac{\partial r^+}{\partial t} = -\frac{i}{2\xi}r^+ + \frac{\gamma(t)u_x(0,t)}{2a^2}\int_{-\infty}^{\infty}\left(u_{xx}\varphi_2^2 + u_{xx}\varphi_1^2\right)dx, \quad (Im\xi = 0)\,.$$

Using the system of equations (4), expansion formulas (8), and asymptotic formulas (6), we obtain

$$\int_{-\infty}^{\infty}\left(u_{xx}\varphi_2^2 + u_{xx}\varphi_1^2\right)dx =$$

$$= \lim_{R\to\infty}\left[\left(u_x\varphi_2^2 + u_x\varphi_1^2\right)\Big|_{-R}^{R} - \int_{-R}^{R}\left(u_x\varphi_2\varphi_{2x} + u_x\varphi_1\varphi_{1x}\right)dx\right] =$$

$$= \lim_{R\to\infty}\int_{-R}^{R}\left[(-2\varphi_{1x} - 2i\xi\varphi_1)\varphi_{2x} + (2\varphi_{2x} - 2i\xi\varphi_2)\varphi_{1x}\right]dx =$$

$$= 4i\xi \lim_{R\to\infty} (\varphi_1\varphi_2) \big|_{-R}^{R} = 4i\xi a(\xi) b(\xi).$$

Consequently,

$$\frac{\partial r^+}{\partial t} = -\frac{i}{2\xi} r^+ + 2i\xi\gamma(t) u_x(0,t) r^+, \quad (Im\xi = 0)\,. \tag{15}$$

From equalities (4), (6) it follows

$$\int_{-\infty}^{\infty} u_{xx} \left(\varphi_{n2}^2 + \varphi_{n1}^2\right) dx =$$

$$= \lim_{R\to\infty} \left[u_x \left(\varphi_{n2}^2 + \varphi_{n1}^2\right) \big|_{-R}^{R} - 2 \int_{-R}^{R} (u_x \varphi_{n2} (\varphi_{n2})_x + u_x \varphi_{n1} (\varphi_{n1})_x) dx \right] =$$

$$= -4 \lim_{R\to\infty} \int_{-R}^{R} \left[(-(\varphi_{n1})_x - i\xi_n \varphi_{n1})(\varphi_{n2})_x + ((\varphi_{n2})_x - i\xi_n\varphi_{n2})(\varphi_{n1})_x)\right] dx =$$

$$= 4i\xi_n \lim_{R\to\infty} \int_{-R}^{R} (\varphi_{n1}\varphi_{n2})_x \, dx = 0,$$

and consequently,

$$\frac{d\xi_n}{dt} = 0, \qquad n = 1, 2, ..., N. \tag{16}$$

So,

$$\frac{dC_n}{dt} = \left(-\frac{i}{2\xi_n} + \gamma(t) u_x(0,t) \int_{-\infty}^{\infty} \frac{u_{xx}}{2} \left(h_{n2}\psi_{n2} + h_{n1}\psi_{n1}\right) dx \right) C_n.$$

Hence, from the system of equations (4), asymptotic formulas (13) and (6) we obtain

$$\int_{-\infty}^{\infty} \frac{u_{xx}}{2} \left(h_{n2}\psi_{n2} + h_{n1}\psi_{n1}\right) dx =$$

$$= \lim_{R\to\infty} \left[\frac{u_x}{2} \left(h_{n2}\psi_{n2} + h_{n1}\psi_{n1}\right) \big|_{-R}^{R} - \int_{-R}^{R} \frac{u_x}{2} (h_{n2}\psi_{n2} + h_{n1}\psi_{n1})_x dx \right] =$$

$$= i\xi_n \lim_{R\to\infty} \left[(h_{n2}\psi_{n1}) \big|_{-R}^{R} + (h_{n1}\psi_{n2}) \big|_{-R}^{R} \right] = 2i\xi_n,$$

i.e.

$$\frac{dC_n}{dt} = \left(-\frac{i}{2\xi_n} + 2i\xi_n \gamma(t) u_x(0,t) \right) C_n, \qquad n = 1, 2, ..., N. \tag{17}$$

We combine (15), (16) and (17) into the following theorem.

Theorem 3. *If the potential $u(x,t)$ is a solution to problem* (1)-(2) *in the class of functions* (3)*, then the scattering data of the system of equations* (4) *with the potential $u(x,t)$ depend on t as follows:*

$$\frac{\partial r^{+}}{\partial t} = -\frac{i}{2\xi_n} r^{+} + 2i\xi_n \gamma(t) u_x(0,t) r^{+}, \ (Im\xi = 0),$$

$$\tfrac{dC_n}{dt} = \left(-\tfrac{i}{2\xi_n} + 2i\xi_n\gamma(t)u_x(0,t)\right) C_n,$$

$$\frac{d\xi_n}{dt} = 0, \qquad n = 1, 2, ..., N.$$

The obtained equalities completely determine the evolution of the scattering data, which makes it possible to apply the method of inverse scattering problem to solve the problem (1)–(3).

Example 1. *Consider the following problem*

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial t} = \sin u + \gamma(t) u_x(0,t) u_{xx}, \\ u(x,0) = 4arctg\left(e^{2x}\right), \end{cases}$$

where $\gamma(t) = \frac{\sqrt{1+\frac{t^2}{4}}\left(ch\,\frac{t}{2}-1\right)}{16}$. *For the operator* $L(0) = i\begin{pmatrix} \frac{d}{dx} & \frac{u_0'}{2} \\ \frac{u_0'}{2} & -\frac{d}{dx}\end{pmatrix}$, *the scattering data have the form*

$$N = 1, \ r^{+}(\xi,0) = 0, \quad \xi_1(0) = i, \quad C_1(0) = 2i.$$

According to Theorem 2, we have

$$r^{+}(\xi,t) = 0, \quad \xi_1(t) = i, \quad C_1(0) = 2i\,\exp\left(\delta(t)\right),$$

where $\delta(t) = -\frac{1}{2}t - 2\int_0^t \gamma(\tau)u_x(0,\tau)d\tau$. *Applying the inverse problem method, we obtain*

$$u(x,t) = 4arctg\left(\exp(2x - \frac{1}{2}t - 2\int_0^t \gamma(\tau)u_x(0,\tau)d\tau)\right).$$

Differentiating the last equality, we have

$$u_x(x,t) = 4arctg\left(\exp(2x - \frac{1}{2}t - 2\int_0^t \gamma(\tau)u_x(0,\tau)d\tau)\right).$$

Assuming $x = 0$, for the function $f(t) = 2\int_0^t \gamma(\tau)u_x(0,\tau)d\tau$ we obtain the following Cauchy problem:

$$\begin{cases} \frac{f'(t)}{\gamma(t)} = \frac{8}{ch\left(\frac{t}{2}+f(t)\right)}, \\ f(0) = 0. \end{cases}$$

and solving this problem we find $f(t) = sh\, t - \frac{t}{2}$. *Then,*

$$u(x,t) = 4arctg\left(\exp(2x + sh\,\frac{t}{2})\right). \tag{18}$$

As it is known, the solution of the sine-Gordon equation for a given initial condition has the form

$$u(x,t) = 4arctg\left(\exp(2x + \frac{t}{2})\right). \tag{19}$$

The difference between the solution of the sine-Gordon equation (19) and the one of the loaded sine-Gordon equation (18) for a given initial condition is shown in the following figure:

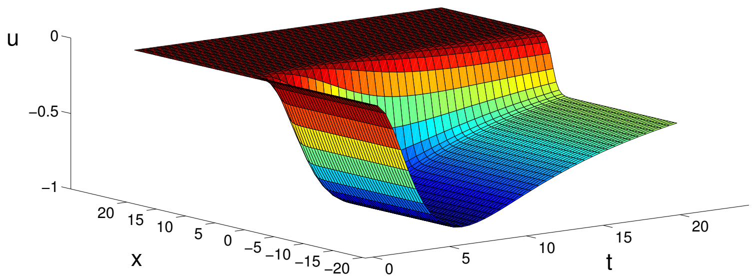

Figure 1: The difference between the solution of the sine-Gordon and loaded sine-Gordon equation.

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