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The Three -Variables Differentiable Form of Hilbert's Inequality

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Abstract. Through this research, an extension for a differential form of Hilbert's integral inequality for three variables is provided, and the form of the reverse of the main conclusion will also be given.

Key Words and Phrases: inequality of Hilbert's integral, Hölder's inequality, differential form, best possible constant.

1. Introduction

In the middle of the 20^{th} Century, Hardy presented the famous Hardy - Hilbert inequality, for $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and for the $+ve$ functions f and g given by

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}, \tag{1}$$

where the constant $C = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible. The inequality (1) has been the starting point for many extensions of inequalities related to it, see for example [1, 3-8].

For the form of differential in Hilbert's integral inequality of two variables, Nizar Kh. [2] gave an extension as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy$$

$$< C\left(\int_0^\infty x^{p(n+1)-p\gamma-\lambda-1}\left(f^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(n+1)+q\gamma-\lambda-1}\left(g^{(n)}(y)\right)^q dy\right)^{\frac{1}{q}}$$

where $C=\frac{\Gamma(\frac{\lambda}{p}+\gamma-n)\Gamma(\frac{\lambda}{q}-\gamma-n)}{\Gamma(\lambda)}$.

2. Preliminaries and lemmas

In this research we will need the following well-known functions:

$$\Gamma(\varpi)=\int_0^\infty t^{\varpi-1}e^{-t}dt, \varpi>0,$$

$$B\left(s,r\right)=\int_0^\infty \frac{t^{s-1}}{\left(t+1\right)^{s+r}}dt, \quad s,r>0,$$

and the following representations of the above special functions:

$$\frac{1}{x^\varpi}=\frac{1}{\Gamma\left(\varpi\right)}\int_0^\infty t^{\varpi-1}e^{-xt}dt, \tag{2}$$

$$\Gamma(x+1)=x\Gamma(x), \quad x>0,$$

$$B(s,r)=\frac{\Gamma(s)\Gamma(r)}{\Gamma(s+r)}. \tag{3}$$

Next, we introduce the following two lemmas.

Lemma 1. *Suppose that* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $t>0$, $l(z)>0$, *the derivatives* $l', l'', \cdots, l^{(n)}$ *exist and are positive,* $l^{(n)}\in L(0,\infty), (n=0,1,2,\cdots), (l^{(0)}:=l)$, L^P *is the space of all functions that are Lebesgue integrable. Furthermore, let* $l(0)=l'(0)=l''(0)=\cdots=l^{(n-1)}(0)=0$, *for* $t>0$ *and for* $\eta q+1>0$. *Then*

$$\int_0^\infty e^{-zt}l(z)dz \le t^{-n-\frac{1}{p}-\eta}\left(\Gamma(p\eta+1)\right)^{\frac{1}{p}}\left(\int_0^\infty z^{-q\eta}e^{-zt}\left(l^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}}. \tag{4}$$

Proof. Integrating by parts with respect to z, we obtain

$$\int_0^\infty e^{-zt} l(z) dz = \frac{1}{t} \int_0^\infty e^{-zt} l'(z) dz.$$

Repeating integration by parts n times with respect to the variable z, we get

$$\int_0^\infty e^{-zt} l(z) dz = \frac{1}{t^n} \int_0^\infty e^{-zt} l^{(n)}(z) dz.$$

Applying Hölder's inequality, we have

$$\begin{aligned}
\int_0^\infty e^{-zt} l(z) dz &= \frac{1}{t^n} \int_0^\infty \left(z^\eta e^{-\frac{z}{p}t} \right) \left(z^{-\eta} e^{-\frac{z}{q}t} l^{(n)}(z) \right) dz \\
&\leq \frac{1}{t^n} \left(\int_0^\infty z^{p\eta} e^{-zt} dz \right)^{\frac{1}{p}} \left(\int_0^\infty z^{-q\eta} e^{-zt} \left(l^{(n)}(z) \right)^q dz \right)^{\frac{1}{q}} \\
&= \frac{1}{t^n} \left((t^{-p\eta-1})^{\frac{1}{p}} \Gamma(1+p\eta) \right)^{\frac{1}{p}} \left(\int_0^\infty z^{-q\eta} e^{-zt} \left(l^{(n)}(z) \right)^q dz \right)^{\frac{1}{q}} \\
&= t^{-\eta-n-\frac{1}{p}} \left(\Gamma(1+p\eta) \right)^{\frac{1}{p}} \left(\int_0^\infty z^{-q\eta} e^{-zt} \left(l^{(n)}(z) \right)^q dz \right)^{\frac{1}{q}}. \quad \blacktriangleleft
\end{aligned}$$

Lemma 2. *For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t > 0$, and $h(x,y) > 0$, suppose that the derivatives $\frac{\partial h(x,y)}{\partial x}$, $\frac{\partial h(x,y)}{\partial y}$, $\frac{\partial^2 h(x,y)}{\partial y \partial x}$, $\frac{\partial^3 h(x,y)}{\partial y \partial x^2}$, $\frac{\partial^4 h(x,y)}{\partial y^2 \partial x^2}, \cdots, \frac{\partial^{2n} h(x,y)}{\partial y^n \partial x^n}$ with respect to the variable x and the variable y exist and are positive, $\frac{\partial^{2n} h(x,y)}{\partial y^n \partial x^n} \in L(0,\infty)$, $(n = 0, 1, 2, \cdots)$, L^P is the space of all functions that are Lebesgue integrable. Further, assume that $h(x,y) = \frac{\partial^2 h(x,y)}{\partial y \partial x} = \cdots = \frac{\partial^{2n-2} h(x,y)}{\partial^{n-1} y \partial^{n-1} x} = 0$, for $x = 0$ or $y = 0$. Then for $2 - \mu q > 0$ we have*

$$\int_0^\infty \int_0^\infty h(x,y) e^{-(x+y)t} dx dy$$

$$\leq t^{\mu-2n-\frac{2}{q}}\Gamma(2-\mu q)^{\frac{1}{q}}\left(\int_0^\infty\int_0^\infty (x+y)^{\mu p}e^{-(x+y)t}\left(\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}\right)^p dxdy\right)^{\frac{1}{p}}. \qquad (5)$$

Proof. Integrating by parts with respect to the variable x and let $u=h(x,y)$ and $dv=e^{-xt}dx$, we obtain

$$\begin{aligned} E &= \int_0^\infty\int_0^\infty h(x,y)e^{-(x+y)t}dxdy = \int_0^\infty e^{-yt}\left(\int_0^\infty h(x,y)e^{-xt}dx\right)dy \\ &= \int_0^\infty e^{-yt}\left(\frac{1}{t}\int_0^\infty \frac{\partial h(x,y)}{\partial x}e^{-xt}dx\right)dy. \end{aligned}$$

Integrating by parts with respect to the variable y and denoting $u=\frac{\partial h}{\partial x}(x,y)$ and $dv=e^{-yt}dy$, we obtain

$$\begin{aligned} E &= \frac{1}{t}\int_0^\infty e^{-xt}\left(\int_0^\infty e^{-yt}\frac{\partial h(x,y)}{\partial x}dy\right)dx \\ &= \frac{1}{t}\int_0^\infty e^{-xt}\left(\frac{1}{t}\int_0^\infty e^{-yt}\frac{\partial^2 h(x,y)}{\partial y\partial x}dy\right)dx \\ &= \frac{1}{t^2}\int_0^\infty\int_0^\infty e^{-(x+y)t}\frac{\partial^2 h(x,y)}{\partial y\partial x}dxdy. \end{aligned}$$

Repeating integration by parts n times, we get

$$E=\frac{1}{t^{2n}}\int_0^\infty\int_0^\infty e^{-(x+y)t}\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}dxdy\ .$$

Applying Hölder's inequality, we have

$$\begin{aligned} E &= \int_0^\infty\int_0^\infty h(x,y)e^{-(x+y)t}dxdy \\ &= \frac{1}{t^{2n}}\int_0^\infty\int_0^\infty e^{-(x+y)t}\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}dxdy \end{aligned}$$

$$= \frac{1}{t^{2n}}\int_0^\infty\int_0^\infty \left\{(x+y)^{-\mu}e^{-\frac{(x+y)}{q}t}\right\}\left\{(x+y)^{\mu}e^{-\frac{(x+y)}{p}t}\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}\right\}dxdy$$

$$\le \frac{1}{t^{2n}}\left(\int_0^\infty\int_0^\infty (x+y)^{-\mu q}e^{-(x+y)t}dxdy\right)^{\frac{1}{q}}$$

$$\times\left(\int_0^\infty\int_0^\infty (x+y)^{\mu p}e^{-(x+y)t}\left(\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}\right)^p dxdy\right)^{\frac{1}{p}}. \qquad (6)$$

It is easy to calculate the first integral on the right-hand side of (6). Using the substitutions $y=\omega x$ and $x=\frac{\tau}{t(\omega+1)}$, respectively, we have

$$\int_0^\infty\int_0^\infty (x+y)^{-\mu q}e^{-(x+y)t}dxdy$$

$$= \int_0^\infty (1+\omega)^{-\mu q}\left(\int_0^\infty x^{1-\mu q}e^{-x(1+\omega)t}dx\right)d\omega$$

$$= \int_0^\infty (1+\omega)^{-\mu q}\left(\int_0^\infty (\frac{\tau}{t(\omega+1)})^{1-\mu q}e^{-\frac{\tau}{(\omega+1)t}(1+\omega)t}\left(\frac{d\tau}{t(\omega+1)}\right)\right)d\omega$$

$$= t^{\mu q-2}\int_0^\infty (1+\omega)^{-2}d\omega\int_0^\infty \tau^{1-\mu q}e^{-\tau}d\tau = t^{\mu q-2}\Gamma(2-\mu q).$$

Finally, substituting the last equation into (6), we complete the proof of Lemma 2. ◄

Remark 1. *For* $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, *and using the reverse form of Hölder's inequality, it is easy to prove the following inequalities:*

$$\int_0^\infty e^{-zt}l(z)dz \ge t^{-n-\frac{1}{p}-\eta}\left(\Gamma(p\eta+1)\right)^{\frac{1}{p}}\left(\int_0^\infty z^{-q\eta}e^{-zt}\left(l^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}} \qquad (7)$$

and

$$\int_0^\infty\int_0^\infty h(x,y)e^{-(x+y)t}dxdy \geq$$

$$\geq t^{\mu-2n-\frac{2}{q}}\Gamma(2-\mu q)^{\frac{1}{q}}\left(\int_0^\infty\int_0^\infty (x+y)^{\mu p}e^{-(x+y)t}\left(\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}\right)^p dxdy\right)^{\frac{1}{p}}. \quad (8)$$

Inequalities (7) *and* (8) *are the reverse form of* (4) *and* (5).

3. Main result

Theorem 1. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\tau>0$, $2n-\frac{\tau}{p}<\xi<\frac{\tau}{q}-n$. *And let the functiont* $h(x,y)>0$, *is continuous on* $(0,\infty)\times(0,\infty)$, *and* $l(z)$ *is a postive function continuous on* $(0,\infty)$. *If* $\int_0^\infty\int_0^\infty (x+y)^{-\tau-\xi p+2np+\frac{2p}{q}}\left(\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}\right)^p dxdy<\infty$ *and* $\int_0^\infty z^{-\tau+\xi q+nq+\frac{q}{p}}\left(l^{(n)}(z)\right)^q dz<\infty$, *then*

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{h(x,y)l(z)}{(x+y+z)^\tau}dxdydz$$

$$\leq C\left(\int_0^\infty\int_0^\infty (x+y)^{-\tau-\xi p+2np+\frac{2p}{q}}\left(\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}\right)^p dxdy\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^\infty z^{-\tau+\xi q+nq+\frac{q}{p}}\left(l^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}}, \quad (9)$$

where $C=\frac{\Gamma(\frac{\tau}{p}+\xi-2n)\Gamma(\frac{\tau}{q}-\xi-n)}{\Gamma(\tau)}$.

Proof. Let

$$I=\int_0^\infty\int_0^\infty\int_0^\infty \frac{h(x,y)l(z)}{(x+y+z)^\tau}dxdydz.$$

Using Hölder's inequality, we get

$$\begin{aligned} I &= \frac{1}{\Gamma(\tau)}\int_0^\infty\int_0^\infty\int_0^\infty h(x,y)l(z)\left(\int_0^\infty t^{\tau-1}e^{-(x+y+z)t}dt\right)dxdydz \\ &= \frac{1}{\Gamma(\tau)}\int_0^\infty\left(t^{\frac{\tau-1}{p}+\xi}\int_0^\infty\int_0^\infty h(x,y)e^{-(x+y)t}dxdy\right)\left(t^{\frac{\tau-1}{q}-\xi}\int_0^\infty e^{-zt}l(z)dz\right)dt \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\tau)}\left(\int_0^\infty t^{\tau-1+\xi p}\left(\int_0^\infty\int_0^\infty h(x,y)e^{-(x+y)}dxdy\right)^p dt\right)^{\frac{1}{p}} \\ &\times\left(\int_0^\infty t^{\tau-1-\xi q}\left(\int_0^\infty e^{-zt}l(z)dz\right)^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{10}$$

Substituting (4) and (5) into (10), we obtain:

$$I \leq \frac{1}{\Gamma(\tau)}\left(\int_0^\infty t^{\tau-1+\xi p}\left[t^{\mu-2n-\frac{2}{q}}\Gamma(2-\mu q)^{\frac{1}{q}}\right.\right.$$

$$\left.\left.\times\left(\int_0^\infty\int_0^\infty (x+y)^{\mu p}e^{-(x+y)t}\left(\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}\right)^p dxdy\right)^{\frac{1}{p}}\right]^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^\infty t^{\tau-1-\xi q}\left[t^{-\eta-n-\frac{1}{p}}\left(\Gamma(1+p\eta)\right)^{\frac{1}{p}}\left(\int_0^\infty z^{-q\eta}e^{-zt}\left(l^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}}\right]^q dt\right)^{\frac{1}{q}},$$

$$I = \frac{\Gamma(2-\mu q)^{\frac{1}{q}}\Gamma(1+p\eta)^{\frac{1}{p}}}{\Gamma(\tau)}$$

$$\times\left(\int_0^\infty\int_0^\infty (x+y)^{\mu p}\left(\frac{\partial^{2n}h(x,y)}{\partial y^n\partial x^n}\right)^p\left(\int_0^\infty t^{\tau-1+\xi p+\mu p-2np-\frac{2p}{q}}e^{-(x+y)t}dt\right)dxdy\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^\infty z^{-q\eta}\left(l^{(n)}(z)\right)^q\int_0^\infty t^{\tau-1-\xi q-\eta q-nq-\frac{q}{p}}\quad e^{-zt}dtdz\right)^{\frac{1}{q}}$$

$$= \frac{\Gamma(2-\mu q)^{\frac{1}{q}}\Gamma(1+p\eta)^{\frac{1}{p}}\Gamma(\tau+\xi p+\mu p-2np-\frac{2p}{q})^{\frac{1}{p}}\Gamma(\tau-\xi q-\eta q-nq-\frac{q}{p})^{\frac{1}{q}}}{\Gamma(\tau)}$$

$$\times \left(\int_0^\infty\int_0^\infty (x+y)^{-\tau-\xi p+2np+\frac{2p}{q}} \left(\frac{\partial^{2n}h(x,y)}{\partial y^n \partial x^n}\right)^p dxdy\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty z^{-\tau+\xi q+nq+\frac{q}{p}} \left(l^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}}$$

$$= C_{\mu,\eta} \left(\int_0^\infty\int_0^\infty (x+y)^{-\tau-\xi p+2np+\frac{2p}{q}} \left(\frac{\partial^{2n}h(x,y)}{\partial y^n \partial x^n}\right)^p dxdy\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty z^{-\tau+\xi q+nq+\frac{q}{p}} \left(l^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}}.$$

Setting $\mu = \frac{2p+2np-\tau-\xi p}{pq}$ and $\eta = \frac{\tau-q\xi-qn-q}{pq}$, we complete the proof of our main result with $C_{\frac{2p+2np-\tau-\xi p}{pq},\frac{\tau-q\xi-qn-q}{pq}} = C_{\mu,\eta} = C$.

To show that C which appears in (9) is the best constant, let us define the following two functions:

$$h_\epsilon(x,y) = \begin{cases} 0, & 0<x<1, 0<y<1 \\ \frac{\Gamma(\frac{\tau}{p}+\xi-\frac{\epsilon}{p}-2n)}{\Gamma(\frac{\tau}{p}+\xi-\frac{\epsilon}{p})}(x+y)^{\frac{\tau+\xi p-\frac{2p}{q}-\epsilon-2}{p}}, & x\geq 1, y\geq 1 \end{cases}$$

and

$$l_\epsilon(z) = \begin{cases} 0, & 0<z<1 \\ \frac{\Gamma(\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-n)}{\Gamma(\frac{\tau}{q}-\xi-\frac{\epsilon}{q})} z^{\frac{\tau-\xi q-\frac{q}{p}-\epsilon-1}{q}}, & z\geq 1 \end{cases}$$

where $0<\epsilon<\min\{\tau+\xi p-\frac{2p}{q}-2nq, \tau-\xi q-\frac{q}{p}-nq\}$. It is easy to find $\frac{\partial^{2n}h_\epsilon(x,y)}{\partial y^n\partial x^n} = (x+y)^{\frac{\tau+\xi p-\frac{2p}{q}-\epsilon-2}{p}-2n}$, $x>1, y>1$, and $l_\epsilon^{(n)}(z) = z^{\frac{\tau-\xi q-\frac{q}{p}-\epsilon-1}{q}-n}$, $z>1$.

Now suppose that our C is not the best constant. Then $\exists\ 0<Q<C$, we get

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{h_\epsilon(x,y)gl_\epsilon(z)}{(x+y+z)^\tau}dxdydz < Q\left(\int_0^\infty\int_0^\infty (x+y)^{-\tau-\xi p+2np+\frac{2p}{q}}\left(\frac{\partial^{2n}h_\epsilon}{\partial y^n\partial x^n}(x,y)\right)^p dxdy\right)^{\frac{1}{p}}$$

$$\begin{aligned}
&\times\left(\int_0^\infty z^{-\tau+\xi q+nq+\frac{q}{p}}\left(l_\epsilon^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}} \\
<\ & Q\left(\int_1^\infty\int_1^\infty (x+y)^{-\epsilon-2}dxdy\right)^{\frac{1}{p}}\left(\int_1^\infty z^{-\epsilon-1}dz\right)^{\frac{1}{q}} \\
=\ & \frac{Q}{\epsilon(1+\epsilon)2^\epsilon}. \qquad (11)
\end{aligned}$$

To estimate the left integral in our main inequality (9), we use the substitution $z = u(x+y)$ and let $S = \frac{\Gamma(\frac{\tau}{p}+\xi-\frac{\epsilon}{p}-2n)\Gamma(\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-n)}{\Gamma(\frac{\tau}{p}+\xi-\frac{\epsilon}{p})\Gamma(\frac{\tau}{q}-\xi-\frac{\epsilon}{q})}$. Then we find

$$\begin{aligned}
I &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{h_\epsilon(x,y)l_\epsilon(z)}{(x+y+z)^\tau}dxdydz \\
&= S\int_1^\infty\int_1^\infty\int_1^\infty \frac{(x+y)^{\frac{\tau+\xi p-\frac{2p}{q}-\epsilon-2}{p}} z^{\frac{\tau-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(x+y+z)^\tau}dxdydz
\end{aligned}$$

$$\begin{aligned}
&= S\int_1^\infty\int_1^\infty (x+y)^{\frac{\tau+\xi p-\frac{2p}{q}-\epsilon-1}{p}}\left(\int_{\frac{1}{x+y}}^\infty \frac{\left[[u(x+y)]^{\frac{\tau-\xi q-\frac{q}{p}-\epsilon-2}{q}}\right]}{(x+y)^\tau(1+u)^\tau}[(x+y)du]\right)dxdy \\
&= S\int_1^\infty\int_1^\infty (x+y)^{-2-\epsilon}\left(\int_{\frac{1}{x+y}}^\infty \frac{u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}}{(1+u)^\tau}du\right)dxdy \\
&= S\int_1^\infty\int_1^\infty (x+y)^{-2-\epsilon}\left(\int_0^\infty \frac{u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}}{(1+u)^\tau}du - \int_0^{\frac{1}{x+y}} \frac{u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}}{(1+u)^\tau}du\right)dxdy
\end{aligned}$$

$$= S\int_1^\infty\int_1^\infty (x+y)^{-2-\epsilon}\left(B(\frac{\tau}{q}-\xi-\frac{\epsilon}{q},\frac{\tau}{p}+\xi+\frac{\epsilon}{q})-\int_0^{\frac{1}{x+y}}\frac{u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}}{(1+u)^\tau}du\right)dxdy$$

$$= \frac{S}{2^\epsilon\epsilon(1+\epsilon)}\left(B(\frac{\tau}{q}-\xi-\frac{\epsilon}{q},\frac{\tau}{p}+\xi+\frac{\epsilon}{q})-\int_0^{\frac{1}{x+y}}\frac{u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}}{(1+u)^\tau}du\right)$$

$$> \frac{S}{2^\epsilon\epsilon(1+\epsilon)}\left(B(\frac{\tau}{q}-\xi-\frac{\epsilon}{q},\frac{\tau}{p}+\xi+\frac{\epsilon}{q})-\int_0^{\frac{1}{x+y}}u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}du\right)$$

$$= S\frac{B(\frac{\tau}{q}-\xi-\frac{\epsilon}{q},\frac{\tau}{p}+\xi+\frac{\epsilon}{q})}{2^\epsilon\epsilon(1+\epsilon)}-O(1), \tag{12}$$

It is clear that if we let $\epsilon\to 0^+$, then from (11) and (12) the contradiction is clear. the contradiction will be obtained. By this, we finish the proof of the first theorem. ◄

Theorem 2. *Let* $0<p<1, (q<0)$, $\frac{1}{p}+\frac{1}{q}=1, \tau>0$. *Let* $h(x,y)>0$ *on* $(0,\infty)\times(0,\infty)$, *and* $l(z)$ *be a positive function continuous on* $(0,\infty)$. *If*

$$\int_0^\infty\int_0^\infty (x+y)^{-\tau-\xi p+2np+\frac{2p}{q}}\left(\frac{\partial^{2n}h(x,y)}{\partial^n y\partial^n x}\right)^p dxdy<\infty,$$

and

$$\int_0^\infty z^{-\tau+\xi q+nq+\frac{q}{p}}\left(l^{(n)}(z)\right)^q dz<\infty,$$

then

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{h(x,y)l(z)}{(x+y+z)^\tau}dxdydz \geq C\left(\int_0^\infty\int_0^\infty (x+y)^{-\tau-\xi p+2np+\frac{2p}{q}}\left(\frac{\partial^{2n}h_\epsilon(x,y)}{\partial y^n\partial x^n}\right)^p dxdy\right)^{\frac{1}{p}} \times\left(\int_0^\infty z^{-\tau+\xi q+nq+\frac{q}{p}}\left(l^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}}, \tag{13}$$

where $C=\frac{\Gamma(\frac{\tau}{p}+\xi-2n)\Gamma(\frac{\tau}{q}-\xi-n)}{\Gamma(\tau)}$ *as in* (9).

Proof. Using the reverse Hölder's inequality and (2), we have

$$
\begin{aligned}
I &= \frac{1}{\Gamma(\tau)}\int_0^\infty \left(t^{\frac{\tau-1}{p}+\xi}\int_0^\infty\int_0^\infty h(x,y)e^{-(x+y)t}dxdy\right)\left(t^{\frac{\tau-1}{q}-\xi}\int_0^\infty e^{-zt}l(z)dz\right)dt \\
&\geq \frac{1}{\Gamma(\tau)}\left(\int_0^\infty t^{\tau-1+\xi p}\left(\int_0^\infty\int_0^\infty h(x,y)e^{-(x+y)}dxdy\right)^p dt\right)^{\frac{1}{p}} \\
&\quad \times\left(\int_0^\infty t^{\tau-1-\xi q}\left(\int_0^\infty e^{-zt}l(z)dz\right)^q dt\right)^{\frac{1}{q}}. \qquad (14)
\end{aligned}
$$

Substituting (7), (8) into (14), we obtain:

$$
\begin{aligned}
I \geq \frac{1}{\Gamma(\tau)}&\left(\int_0^\infty t^{\tau-1+\xi p}\left(\int_0^\infty\int_0^\infty h(x,y)e^{-(x+y)}dxdy\right)^p dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^\infty t^{\tau-1-\xi q}\left(\int_0^\infty e^{-zt}l(z)dz\right)^q dt\right)^{\frac{1}{q}} \\
&\geq \frac{1}{\Gamma(\tau)}\left(\int_0^\infty t^{\tau-1+\xi p}\left(t^{\mu-2n-\frac{2}{q}}\Gamma(2-\mu q)^{\frac{1}{q}}\right.\right. \\
&\times\left.\left.\left(\int_0^\infty\int_0^\infty (x+y)^{\mu p}e^{-(x+y)t}\left(\frac{\partial^{2n}h_\epsilon(x,y)}{\partial y^n\partial x^n}\right)^p dxdy\right)^{\frac{1}{p}}\right)^p dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^\infty t^{\tau-1-\xi q}\left(t^{-n-\frac{1}{p}-\eta}\left(\Gamma(p\eta+1)\right)^{\frac{1}{p}}\left(\int_0^\infty z^{-q\eta}e^{-zt}\left(l^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}}\right)^q dt\right)^{\frac{1}{q}} \\
= C&\left(\int_0^\infty\int_0^\infty (x+y)^{-\tau-\xi p+2np+\frac{2p}{q}}\left(\frac{\partial^{2n}h_\epsilon(x,y)}{\partial y^n\partial x^n}\right)^p dxdy\right)^{\frac{1}{p}}
\end{aligned}
$$

$$\times\left(\int_0^\infty z^{-\tau+\xi q+nq+\frac{q}{p}}\left(l^{(n)}(z)\right)^q dz\right)^{\frac{1}{q}}, \tag{15}$$

where $C = C_{\mu,\eta}$ is the same as in (9). Further, as in (9) we can prove that C is also the best possible constant, by defining the same functions $h_\epsilon(x, y)$ and $l_\epsilon(z)$ as in the proof of (9). Suppose that we can find a constant $R : R > C$ such that (15) holds when we replace C by R. Then

$$\begin{aligned}\int_0^\infty\int_0^\infty\int_0^\infty \frac{h_\epsilon(x,y)l_\epsilon(z)}{(x+y+z)^\tau}dxdydz &= \int_1^\infty\int_1^\infty\int_1^\infty \frac{h_\epsilon(x,y)l_\epsilon(z)}{(x+y+z)^\tau}dxdydz \\ &> \frac{R}{\epsilon(1+\epsilon)2^\epsilon}.\end{aligned} \tag{16}$$

On the other hand,

$$\begin{aligned} I &= \int_1^\infty\int_1^\infty\int_1^\infty \frac{h_\epsilon(x,y)l_\epsilon(z)}{(x+y+z)^\tau}dxdydz \\ &= S\int_1^\infty\int_1^\infty\int_1^\infty \frac{(x+y)^{\frac{\tau+\xi p-\frac{2p}{q}-\epsilon-2}{p}} z^{\frac{\tau-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(x+y+z)^\tau}dxdydz \\ &= S\int_1^\infty\int_1^\infty (x+y)^{-2-\epsilon}\int_{\frac{1}{x+y}}^\infty \frac{u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}}{(1+u)^\tau}dudxdy \\ &= S\int_1^\infty\int_1^\infty (x+y)^{-2-\epsilon}\left(\int_0^\infty \frac{u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}}{(1+u)^\tau}du - \int_0^{\frac{1}{x+y}} \frac{u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}}{(1+u)^\tau}du\right)dxdy \\ &= \frac{S}{2^\epsilon\epsilon(1+\epsilon)}\left(B(\frac{\tau}{q}-\xi-\frac{\epsilon}{q},\frac{\tau}{p}+\xi+\frac{\epsilon}{q}) - \int_0^{\frac{1}{x+y}} u^{\frac{\tau}{q}-\xi-\frac{\epsilon}{q}-1}du\right) \\ &< S\frac{B(\frac{\tau}{q}-\xi-\frac{\epsilon}{q},\frac{\tau}{p}+\xi+\frac{\epsilon}{q})}{2^\epsilon\epsilon(1+\epsilon)}. \end{aligned} \tag{17}$$

If we let $\epsilon \to 0^+$ in (16) and (17), the contradiction will be obtained. By this, we finish the proof of the theorem 2. ◄

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