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$*$-Operator Frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$

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Abstract. In this work, we introduce the concept of $*$-operator frame, which is a generalization of $*$-frames in Hilbert pro-C^*-modules, and we establish some results. We also study the tensor product of $*$-operator frame for Hilbert pro-C^*-modules.

Key Words and Phrases: $*-$frame, $*-$operator frame, pro-C^*-algebra, Hilbert pro-C^*-modules, tensor product.

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1. Introduction

In 1952, Duffin and Schaeffer [3] introduced the notion of frame in nonharmonic Fourier analysis. In 1986 the work of Duffin and Schaeffer was continued by Grossman and Meyer [8]. After their works, the theory of frame was developed and has been popular.

The notion of frame on Hilbert space has been successfully extended to frames in Hilbert pro-C^*-modules. In 2008, Joita [10] proposed the concept of frames of multipliers in pro-C^*-Hilbert modules and demonstrated that many properties of frames in Hilbert C^*-modules are preserved in these frames of multipliers.

The concept of $*$-frames was introduced by Alijani and Dehghan [1], providing a significant advancement in the theory of frames in Hilbert spaces. Building upon this, the notion of $*$-operator frames was developed as a generalization of $*$-frames, extending the framework to more complex structures within the realm of operator theory.

The first purpose of this paper is to give the definition of $*$-operator frame in pro-C^*-modules and some properties.

The second purpose is to investigate the tensor product of Hilbert pro-C^*-modules, and to show that tensor product of $*$-operator frames for Hilbert pro-C^*-modules $\mathcal{X}$ and $\mathcal{Y}$, present $*$-operator frame for $\mathcal{X} \otimes \mathcal{Y}$.

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In the next section, we give some definitions and basic properties of Hilbert C^*-modules.

2. Preliminaries

The basic information about pro-C^*-algebras can be found in the works [5, 6, 7, 9, 12, 13, 14].

C^*-algebra whose topology is induced by a family of continuous C^*-seminorms instead of a C^*-norm is called pro-C^*-algebra. Hilbert pro-C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a pro-C^*-algebra rather than in the field of complex numbers.

Pro-C^*-algebra is defined as a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_\alpha\}$ converges to 0 if and only if $p(a_\alpha)$ converges to 0 for all continuous C^*-seminorms p on $\mathcal{A}$ (see [4, 9, 11, 14]), and

1) $p(ab) \leq p(a)p(b)$,

2) $p(a^*a) = p(a)^2$,

for all $a, b \in \mathcal{A}$.

If the topology of pro-C^*-algebra is determined by only countably many C^*-seminorms, then it is called a σ-C^*-algebra.

We denote by $sp(a)$ the spectrum of a such that $sp(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a$ is not invertible$\}$ for all $a \in \mathcal{A}$, where $\mathcal{A}$ is a unital pro-C^*-algebra with an identity $1_{\mathcal{A}}$.

The set of all continuous C^*-seminorms on $\mathcal{A}$ is denoted by $S(\mathcal{A})$. $\mathcal{A}^+$ denotes the set of all positive elements of $\mathcal{A}$.

Example 1. *Every C^*-algebra is a pro-C^*-algebra.*

Proposition 1. *[9] Let $\mathcal{A}$ be a unital pro-C^*-algebra with an identity $1_{\mathcal{A}}$. Then for any $p \in S(\mathcal{A})$, we have:*

(1) $p(a) = p(a^*)$ for all $a \in A$,

(2) $p(1_{\mathcal{A}}) = 1$,

(3) If $a, b \in \mathcal{A}^+$ and $a \leq b$, then $p(a) \leq p(b)$,

(4) If $1_{\mathcal{A}} \leq b$, then b is invertible and $b^{-1} \leq 1_{\mathcal{A}}$,

(5) If $a, b \in \mathcal{A}^+$ are invertible and $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$,

(6) If $a, b, c \in \mathcal{A}$ and $a \leq b$, then $c^*ac \leq c^*bc$,

(7) If $a, b \in \mathcal{A}^+$ and $a^2 \leq b^2$, then $0 \leq a \leq b$.

Definition 1. *[14] A pre-Hilbert module over pro-C^*-algebra $\mathcal{A}$, is a complex vector space E, which is also a left $\mathcal{A}$-module compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle$ $E \times E \to \mathcal{A}$, which is $\mathbb{C}$-and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:*

1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$,

2) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$,

3) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$,

for all $\xi, \eta \in E$. We say E is a Hilbert $\mathcal{A}$-module (or Hilbert pro-C^*-module over $\mathcal{A}$) if it is complete with respect to the topology determined by the family of seminorms

$$\bar{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)} \quad \xi \in E, p \in S(\mathcal{A}).$$

Let $\mathcal{A}$ be a pro-C^*-algebra and let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and assume that I and J are countable index sets. A bounded $\mathcal{A}$-module map from $\mathcal{X}$ to $\mathcal{Y}$ is called an operator from $\mathcal{X}$ to $\mathcal{Y}$. We denote the set of all operators from $\mathcal{X}$ to $\mathcal{Y}$ by $Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$.

Definition 2. *[2] An $\mathcal{A}$-module map $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is adjointable if there is a map $T^* : \mathcal{Y} \longrightarrow \mathcal{X}$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for all $\xi \in \mathcal{X}, \eta \in \mathcal{Y}$, and is called bounded if for all $p \in S(\mathcal{A})$ there is $M_p > 0$ such that $\bar{p}_{\mathcal{Y}}(T\xi) \leq M_p \bar{p}_{\mathcal{X}}(\xi)$ for all $\xi \in \mathcal{X}$.*

We denote by $Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ the set of all adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$, and $Hom^*_{\mathcal{A}}(\mathcal{X}) = Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{X})$.

Definition 3. *[2] Let $\mathcal{A}$ be a pro-C^*-algebra and $\mathcal{X}, \mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules. The operator $T : \mathcal{X} \to \mathcal{Y}$ is called uniformly bounded below, if there exists $C > 0$ such that for each $p \in S(\mathcal{A})$,*

$$\bar{p}_{\mathcal{Y}}(T\xi) \leqslant C \bar{p}_{\mathcal{X}}(\xi), \quad \text{for all } \xi \in \mathcal{X},$$

and is called uniformly bounded above if there exists $C' > 0$ such that for each $p \in S(\mathcal{A})$,

$$\bar{p}_{\mathcal{Y}}(T\xi) \geqslant C' \bar{p}_{\mathcal{X}}(\xi), \quad \text{for all } \xi \in \mathcal{X},$$

$$\|T\|_\infty = \inf\{M : M \text{ is an upper bound for } T\},$$

$$\hat{p}_{\mathcal{Y}}(T) = \sup\left\{\bar{p}_{\mathcal{Y}}(T(x)) : \xi \in \mathcal{X}, \quad \bar{p}_{\mathcal{X}}(\xi) \leqslant 1\right\}.$$

It's clear that $\hat{p}(T) \leqslant \|T\|_\infty$ for all $p \in S(\mathcal{A})$.

Proposition 2. *[2]. Let $\mathcal{X}$ be a Hilbert module over pro-C^*-algebra $\mathcal{A}$ and T be an invertible element in $Hom^*_{\mathcal{A}}(\mathcal{X})$ such that both are uniformly bounded. Then for each $\xi \in \mathcal{X}$,*

$$\left\|T^{-1}\right\|_{\infty}^{-2} \langle \xi, \xi \rangle \leq \langle T\xi, T\xi \rangle \leq \|T\|_{\infty}^{2} \langle \xi, \xi \rangle.$$

Similar to C^*-algebra, the $*$-homomorphism between two pro-C^*-algebras is increasing.

Lemma 1. *If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an $*$-homomorphism between pro-$\mathcal{C}^*$-algebras, then φ is increasing, that is, if $a \leq b$, then $\varphi(a) \leq \varphi(b)$.*

3. $*$-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$

Definition 4. *A family of adjointable operators $\{T_i\}_{i\in J}$ on a Hilbert $\mathcal{A}$-module $\mathcal{X}$ over a unital pro-C^*-algebra is said to be an operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$, if there exist positive constants $A, B > 0$ such that*

$$A\langle \xi, \xi \rangle \leq \sum_{i\in J} \langle T_i\xi, T_i\xi \rangle \leq B\langle \xi, \xi \rangle, \forall \xi \in \mathcal{X}. \tag{1}$$

The numbers A and B are called lower and upper bounds of the operator frame, respectively. If $A = B = \lambda$, the operator frame is λ-tight. If $A = B = 1$, it is called a normalized tight operator frame or a Parseval operator frame. If only upper inequality of (1) holds, then $\{T_i\}_{i\in J}$ is called an operator Bessel sequence for $Hom^*_{\mathcal{A}}(\mathcal{X})$.

Definition 5. *A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{X}$ over a pro-C^*-algebra is said to be an $*$-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$, if there exist two strictly nonzero elements A and B in $\mathcal{A}$ such that*

$$A\langle \xi, \xi \rangle A^* \leq \sum_{i\in I} \langle T_i\xi, T_i\xi \rangle \leq B\langle \xi, \xi \rangle B^*, \forall \xi \in \mathcal{X}. \tag{2}$$

The elements A and B are called lower and upper bounds of the $*$-operator frame, respectively. If $A = B = \lambda$, the $*$-operator frame is λ-tight. If $A = B = 1_{\mathcal{A}}$, it is called a normalized tight $*$-operator frame or a Parseval $*$-operator frame. If only upper inequality of (2) holds, then $\{T_i\}_{i\in i}$ is called an $*$-operator Bessel sequence for $Hom^*_{\mathcal{A}}(\mathcal{X})$.

We mentioned that the set of all operator frames for $Hom^*_{\mathcal{A}}(\mathcal{X})$ can be considered as a subset of $*$-operator frame. To illustrate this, let $\{T_j\}_{i\in I}$ be an operator frame for Hilbert $\mathcal{A}$-module $\mathcal{X}$ with operator frame real bounds A and B. Note that for $\xi \in \mathcal{X}$,

$$(\sqrt{A})1_{\mathcal{A}}\langle \xi, \xi\rangle(\sqrt{A})1_{\mathcal{A}} \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq (\sqrt{B})1_{\mathcal{A}}\langle \xi, \xi\rangle(\sqrt{B})1_{\mathcal{A}}.$$

Therefore, every operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$ with real bounds A and B is an $*$-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$ with $\mathcal{A}$-valued $*$-operator frame bounds $(\sqrt{A})1_{\mathcal{A}}$ and $(\sqrt{B})1_{\mathcal{B}}$.

Example 2. *Let $\mathcal{A}$ be a Hilbert pro-C^*-module over itself with the inner product $\langle a, b\rangle = ab^*$. Let $\{\xi_i\}_{i\in I}$ be an $*$-frame for $\mathcal{A}$ with bounds A and B, respectively. For each $i \in I$, we define $T_i : \mathcal{A} \to \mathcal{A}$ by $T_i\xi = \langle \xi, \xi_i\rangle$, $\forall \xi \in \mathcal{A}$. T_i is adjointable and $T_i^*a = a\xi_i$ for each $a \in \mathcal{A}$. And we have*

$$A\langle \xi, \xi\rangle A^* \leq \sum_{i\in I}\langle \xi, \xi_i\rangle\langle \xi_i, \xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{A}.$$

Then

$$A\langle \xi, \xi\rangle A^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{A}.$$

So $\{T_i\}_{i\in I}$ is an $*$-operator frame in $\mathcal{A}$ with bounds A and B, respectively.

Similar to $*$-frames, we introduce the $*$-operator frame transform and $*$-frame operator and establish some properties.

Theorem 1. *Let $\{T_i\}_{i\in I} \subset Hom^*_{\mathcal{A}}(\mathcal{X})$ be an $*$-operator frame with lower and upper bounds A and B, respectively. The $*$-operator frame transform $R : \mathcal{X} \to l^2(\mathcal{X})$ defined by $R\xi = \{T_i\xi\}_{i\in I}$ is injective and closed range adjointable $\mathcal{A}$-module map and $\bar{p}_{\mathcal{X}}(R) \leq \bar{p}_{\mathcal{X}}(B)$. The adjoint operator R^* is surjective and it is given by $R^*(\{\xi_i\}_{i\in I}) = \sum_{i\in I} T_i^*\xi_i$ for all $\{\xi_i\}_{i\in I}$ in $l^2(\mathcal{X})$.*

Proof. By the definition of norm in $l^2(\mathcal{X})$,

$$\bar{p}_{\mathcal{X}}(R\xi)^2 = p(\sum_{i\in I}\langle T_i\xi, T_i\xi\rangle) \leq \bar{p}_{\mathcal{X}}(B)^2 p(\langle \xi, \xi\rangle), \forall \xi \in \mathcal{X}. \quad (3)$$

This inequality implies that R is well defined and $\bar{p}_{\mathcal{X}}(R) \leq \bar{p}_{\mathcal{X}}(B)$. Clearly, R is a linear $\mathcal{A}$-module map. We now show that the range of R is closed. Let $\{R\xi_n\}_{n\in\mathbb{N}}$ be a sequence in the range of R such that $\lim_{n\to\infty} R\xi_n = \eta$. For $n, m \in \mathbb{N}$, we have

$$p(A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*) \leq p(\langle R(\xi_n - \xi_m), R(\xi_n - \xi_m)\rangle) = \bar{p}_{\mathcal{X}}(R(\xi_n - \xi_m))^2.$$

Since $\{R\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$, we have
$p(A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*) \to 0$, as $n, m \to \infty$.
Note that for $n, m \in \mathbb{N}$,

$$\begin{aligned} p(\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle) &= p(A^{-1}A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*(A^*)^{-1}) \\ &\leq p(A^{-1})^2 p(A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*). \end{aligned}$$

Therefore the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is Cauchy and hence there exists $\xi \in \mathcal{X}$ such that $\xi_n \to \xi$ as $n \to \infty$. Again by (3), we have

$$\bar{p}_{\mathcal{X}}(R(\xi_n - \xi_m))^2 \leq \bar{p}_{\mathcal{X}}(B)^2 p(\langle \xi_n - \xi, \xi_n - \xi\rangle).$$

Thus $p(R\xi_n - R\xi) \to 0$ as $n \to \infty$ implies that $R\xi = \eta$. It follows that the range of R is closed. Next we show that R is injective. Suppose that $\xi \in \mathcal{X}$ and $R\xi = 0$. Note that $A\langle \xi, \xi\rangle A^* \leq \langle R\xi, R\xi\rangle$. Then $\langle \xi, \xi\rangle = 0$, so $\xi = 0$, i.e. R is injective.

For $\xi \in \mathcal{X}$ and $\{\xi_i\}_{i\in I} \in l^2(\mathcal{X})$, we have

$$\langle R\xi, \{\xi_i\}_{i\in I}\rangle = \langle \{T_i\xi\}_{i\in I}, \{\xi_i\}_{i\in I}\rangle = \sum_{i\in I}\langle T_i\xi, \xi_i\rangle = \sum_{i\in I}\langle \xi, T_i^*\xi_i\rangle = \langle \xi, \sum_{i\in I} T_i^*\xi_i\rangle.$$

Then $R^*(\{\xi_i\}_{i\in I}) = \sum_{i\in I} T_i^*\xi_i$. By injectivity of R, the operator R^* has closed range and $\mathcal{X} = range(R^*)$, which completes the proof. ◄

Now we define $*$-frame operator and we study some of its properties.

Definition 6. *Let $\{T_i\}_{i\in I} \subset Hom^*_{\mathcal{A}}(\mathcal{X})$ be an $*$-operator frame with $*$-operator frame transform R and lower and upper bounds A and B, respectively. The $*$-frame operator $S : \mathcal{X} \to \mathcal{X}$ is defined by $S\xi = R^*R\xi = \sum_{i\in I} T_i^*T_i\xi,\ \ \forall \xi \in \mathcal{X}$.*

The following lemma is used to prove the next results.

Lemma 2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules and $T \in Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$.*

(i) If T is injective and T has a closed range, then the adjointable map T^*T is invertible and

$$\bar{p}_{\mathcal{X}}(T^*T^{-1})^{-1} I_{\mathcal{X}} \leq T^*T \leq \bar{p}_{\mathcal{X}}(T)^2 I_{\mathcal{X}}.$$

(ii) If T is surjective, then the adjointable map TT^* is invertible and

$$\bar{p}_{\mathcal{X}}((TT^*)^{-1})^{-1} I_{\mathcal{Y}} \leq TT^* \leq \bar{p}_{\mathcal{X}}(T)^2 I_{\mathcal{Y}}.$$

Proof.

1. Since the adjointable map T^* is surjective, it follows that for any $\xi \in \mathcal{X}$ there exists $\eta \in \mathcal{Y}$ such that $T^*\eta = \xi$. Since $\mathcal{Y} = kerT^* \oplus \operatorname{Im} T$, it follows that $\eta = \eta_1 + Th$ for some $\eta_1 \in \ker T^*$ and some $h \in \mathcal{X}$. Thus, $\xi = T^*(\eta_1 + Th) = T^*Th$, and hence T^*T is surjective. If $T^*T\xi = 0$, then $T\xi \in \ker T^* \cap \operatorname{Im} T = \{0\}$, which implies that $\xi = 0$. Therefore, T^*T is an injective positive map. Hence, T^*T is an invertible element of the set of all bounded $\mathcal{A}$-module maps, $0 \leq (T^*T)^{-1} \leq \bar{p}_{\mathcal{X}}((T^*T)^{-1})$ and $0 \leq (T^*T) \leq \bar{p}_{\mathcal{X}}((T^*T))$. Therefore, $\bar{p}_{\mathcal{X}}((T^*T)^{-1})^{-1} \leq T^*T \leq \bar{p}_{\mathcal{X}}(T)^2$

2. Let T be surjective. Then T^* is injective and has a closed range. By substituting T^* for T in (1), we see that TT^* is invertible and $\bar{p}_{\mathcal{X}}((TT^*)^{-1})^{-1} \leq TT^* \leq \bar{p}_{\mathcal{X}}(T)^2$.

◄

Theorem 2. *The $*$-operator frame S is bounded, positive, self-adjoint, invertible and* $\bar{p}_{\mathcal{X}}(A^{-1})^{-2} \leq \bar{p}_{\mathcal{X}}(S) \leq \bar{p}_{\mathcal{X}}(B)^2$.

Proof.
By definition we have, $\forall \xi, \eta \in \mathcal{X}$:

$$\begin{aligned}
\langle S\xi, \eta\rangle &= \left\langle \sum_{i\in I} T_i^* T_i \xi, \eta \right\rangle \\
&= \sum_{i\in I} \langle T_i^* T_i \xi, \eta\rangle \\
&= \sum_{i\in I} \langle \xi, T_i^* T_i \eta\rangle \\
&= \left\langle \xi, \sum_{i\in I} T_i^* T_i \eta \right\rangle \\
&= \langle \xi, S\eta\rangle.
\end{aligned}$$

Then S is selfadjoint.

By Lemma 2 and Theorem 1, S is invertible. Clearly S is positive.

By definition of an $*$-operator frame, we have

$$A\langle \xi, \xi\rangle A^* \leq \sum_{i\in I} \langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*.$$

So

$$A\langle \xi, \xi\rangle A^* \leq \langle S\xi, \xi\rangle \leq B\langle \xi, \xi\rangle B^*.$$

Then

$$\bar{p}_{\mathcal{X}}(A^{-1})^{-2}\bar{p}_{\mathcal{X}}(\xi)^2 \leq \bar{p}_{\mathcal{X}}(\langle S\xi, \xi\rangle) \leq \bar{p}_{\mathcal{X}}(B)^2\bar{p}_{\mathcal{X}}(\xi)^2, \forall \xi \in \mathcal{X}.$$

If we take supremum on all $\xi \in \mathcal{X}$, where $\bar{p}_{\mathcal{X}}(\xi) \leq 1$, then $\bar{p}_{\mathcal{X}}(A^{-1})^{-2} \leq \bar{p}_{\mathcal{X}}(S) \leq \bar{p}_{\mathcal{X}}(B)^2$. ◀

Corollary 1. *Let $\{T_i\}_{i\in I} \subset Hom^*_{\mathcal{A}}(\mathcal{X})$ be an $*$-operator frame with $*$-operator frame transform R and lower and upper bounds A and B, respectively. Then $\{T_i\}_{i\in I}$ is an operator frame for $\mathcal{X}$ with lower and upper bounds $\bar{p}_{\mathcal{X}}((R^*R)^{-1})^{-1}$ and $\bar{p}_{\mathcal{X}}(R)^2$, respectively.*

Proof. By Theorem 1, R is injective and has a closed range, and by Lemma 2

$$\bar{p}_{\mathcal{X}}((R^*R)^{-1})^{-1}I_{\mathcal{X}} \leq R^*R \leq \bar{p}_{\mathcal{X}}(R)^2 I_{\mathcal{X}}.$$

So

$$\bar{p}_{\mathcal{X}}((R^*R)^{-1})^{-1}\langle \xi, \xi\rangle \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq \bar{p}_{\mathcal{X}}(R)^2\langle \xi, \xi\rangle, \ \ \forall x \in \mathcal{X}.$$

Then $\{T_i\}_{i\in I}$ is an operator frame for $\mathcal{X}$ with lower and upper bounds $\bar{p}_{\mathcal{X}}((R^*R)^{-1})^{-1}$ and $\bar{p}_{\mathcal{X}}(R)^2$, respectively. ◀

Theorem 3. *Let $\{T_i\}_{i\in I} \subset Hom^*_{\mathcal{A}}(\mathcal{X})$ be an $*$-operator frame for $\mathcal{X}$, with lower and upper bounds A and B, respectively, and with $*$-frame operator S. Let $\theta \in Hom^*_{\mathcal{A}}(\mathcal{X})$ be injective and have a closed range. Then $\{T_i\theta\}_{i\in I}$ is an $*$-operator frame for $\mathcal{X}$ with $*$-frame operator $\theta^*S\theta$ with bounds $\bar{p}_{\mathcal{X}}((\theta^*\theta)^{-1})^{-\frac{1}{2}}A$, $\bar{p}_{\mathcal{X}}(\theta)B$.*

Proof. We have

$$A\langle \theta\xi, \theta\xi\rangle A^* \leq \sum_{i\in I}\langle T_i\theta\xi, T_i\theta\xi\rangle \leq B\langle \theta\xi, \theta\xi\rangle B^*, \forall \xi \in \mathcal{X}. \tag{4}$$

Using Lemma 2, we have $\bar{p}_{\mathcal{X}}((\theta^*\theta)^{-1})^{-1}\langle \xi, \xi\rangle \leq \langle \theta\xi, \theta\xi\rangle$, $\forall \xi \in \mathcal{X}$. This implies

$$\bar{p}_{\mathcal{X}}((\theta^*\theta)^{-1})^{-\frac{1}{2}}A\langle \xi, \xi\rangle(\bar{p}_{\mathcal{X}}((\theta^*\theta)^{-1})^{-\frac{1}{2}}A)^* \leq A\langle \theta\xi, \theta\xi\rangle A^*, \forall \xi \in \mathcal{X}. \tag{5}$$

And we know that $\langle \theta\xi, \theta\xi\rangle \leq \bar{p}_{\mathcal{X}}(\theta)^2\langle \xi, \xi\rangle$, $\forall \xi \in \mathcal{X}$. This implies that

$$B\langle \theta\xi, \theta\xi\rangle B^* \leq \bar{p}_{\mathcal{X}}(\theta)B\langle \xi, \xi\rangle(\bar{p}_{\mathcal{X}}(\theta)B)^*, \forall \xi \in \mathcal{X}. \tag{6}$$

Using (4), (5), (6), we have

$$\bar{p}_{\mathcal{X}}((\theta^*\theta)^{-1})^{-\frac{1}{2}}A\langle \xi, \xi\rangle(\bar{p}_{\mathcal{X}}((\theta^*\theta)^{-1})^{-\frac{1}{2}}A)^* \leq \sum_{i\in I}\langle T_i\theta x, T_i\theta\xi\rangle$$

$$\leq \bar{p}_{\mathcal{X}}(\theta)B\langle \xi,\xi\rangle(\bar{p}_{\mathcal{X}}(\theta)B)^*, \forall \xi \in \mathcal{X}.$$

So $\{T_i\theta\}_{i\in I}$ is an $*$-operator frame for $\mathcal{X}$.

Moreover, for every $\xi \in \mathcal{X}$, we have

$$\theta^* S\theta\xi = \theta^* \sum_{i\in I} T_i^* T_i \theta\xi = \sum_{i\in I} \theta^* T_i^* T_i \theta\xi = \sum_{i\in I} (T_i\theta)^*(T_i\theta)\xi.$$

This completes the proof. ◄

Corollary 2. *Let $\{T_i\}_{i\in I} \subset Hom_{\mathcal{A}}^{*}(\mathcal{X})$ be an $*$-operator frame for $\mathcal{X}$, with $*$-frame operator S. Then $\{T_iS^{-1}\}_{i\in I}$ is an $*$-operator frame for $\mathcal{X}$.*

Proof. The proof follows from Theorem 3 by taking $\theta = S^{-1}$. ◄

Corollary 3. *Let $\{T_i\}_{i\in I} \subset Hom_{\mathcal{A}}^{*}(\mathcal{X})$ be an $*$-operator frame for $\mathcal{X}$, with $*$-frame operator S. Then $\{T_iS^{-\frac{1}{2}}\}_{i\in I}$ is a Parseval $*$-operator frame for $\mathcal{X}$.*

Proof. The proof follows from Theorem 3 by taking $\theta = S^{-\frac{1}{2}}$. ◄

Theorem 4. *Let $\{T_i\}_{i\in I} \subset Hom_{\mathcal{A}}^{*}(\mathcal{X})$ be an $*$-operator frame for $\mathcal{X}$, with lower and upper bounds A and B, respectively. Let $\theta \in Hom_{\mathcal{A}}^{*}(\mathcal{X})$ be surjective. Then $\{\theta T_i\}_{i\in I}$ is an $*$-operator frame for $\mathcal{X}$ with bounds $A\bar{p}_{\mathcal{X}}((\theta\theta^*)^{-1})^{-\frac{1}{2}}$, $B\bar{p}_{\mathcal{X}}(\theta)$.*

Proof. By the definition of $*$-operator frame, we have

$$A\langle \xi,\xi\rangle A^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi,\xi\rangle B^*, \forall \xi \in \mathcal{X}. \tag{7}$$

Using Lemma 2, we have

$$\bar{p}_{\mathcal{X}}((\theta\theta^*)^{-1})^{-1}\langle T_i\xi, T_i\xi\rangle \leq \langle \theta T_i x, \theta T_i\xi\rangle \leq \bar{p}_{\mathcal{X}}^2\langle T_i\xi, T_i\xi\rangle, \forall \xi \in \mathcal{X}. \tag{8}$$

Using (7), (8), we have

$$\bar{p}_{\mathcal{X}}((\theta\theta^*)^{-1})^{-\frac{1}{2}}A\langle \xi,\xi\rangle(\bar{p}_{\mathcal{X}}((\theta\theta^*)^{-1})^{-\frac{1}{2}}A)^* \leq \sum_{i\in I}\langle \theta T_i\xi, \theta T_i\xi\rangle$$
$$\leq B\bar{p}_{\mathcal{X}}(\theta)\langle \xi,\xi\rangle(B\bar{p}_{\mathcal{X}}(\theta))^*, \forall \xi \in \mathcal{X}.$$

So $\{\theta T_i\}_{i\in I}$ is an $*$-operator frame for $\mathcal{X}$. ◄

Theorem 5. *Let $(\mathcal{X}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ and $(\mathcal{X}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ be two Hilbert pro-$\mathcal{C}^*$-modules and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be an $*$-homomorphism and θ be a map on $\mathcal{X}$ such that $\langle \theta\xi, \theta\eta\rangle_{\mathcal{B}} = \varphi(\langle \xi, \eta\rangle_{\mathcal{A}})$ for all $\xi, \eta \in \mathcal{X}$. Also, suppose that $\{T_i\}_{i\in I} \subset Hom^*_{\mathcal{A}}(\mathcal{X})$ is an $*$-operator frame for $(\mathcal{X}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ with $*$-frame operator $S_{\mathcal{A}}$ and lower and upper $*$-operator frame bounds A, B, respectively. If θ is surjective and $\theta T_i = T_i\theta$ for each i in I, then $\{T_i\}_{i\in I}$ is an $*$-operator frame for $(\mathcal{X}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ with $*$-frame operator $S_{\mathcal{B}}$ and lower and upper $*$-operator frame bounds $\varphi(A)$, $\varphi(B)$, respectively, and $\langle S_{\mathcal{B}}\theta\xi, \theta\eta\rangle_{\mathcal{B}} = \varphi(\langle S_{\mathcal{A}}\xi, \eta\rangle_{\mathcal{A}})$.*

Proof. Let $\eta \in \mathcal{X}$. Then there exists $\xi \in \mathcal{X}$ such that $\theta\xi = \eta$ (θ is surjective). By the definition of $*$-operator frames we have

$$A\langle \xi, \xi\rangle_{\mathcal{A}} A^* \leq \sum_{i\in I} \langle T_i\xi, T_i\xi\rangle_{\mathcal{A}} \leq B\langle \xi, \xi\rangle_{\mathcal{A}} B^*.$$

By Lemma 1 we have

$$\varphi(A\langle \xi, \xi\rangle_{\mathcal{A}} A^*) \leq \varphi(\sum_{i\in I} \langle T_i\xi, T_i\xi\rangle_{\mathcal{A}}) \leq \varphi(B\langle \xi, \xi\rangle_{\mathcal{A}} B^*).$$

By the definition of $*$-homomorphism, we have

$$\varphi(A)\varphi(\langle \xi, \xi\rangle_{\mathcal{A}})\varphi(A^*) \leq \sum_{i\in I} \varphi(\langle T_i\xi, T_i\xi\rangle_{\mathcal{A}}) \leq \varphi(B)\varphi(\langle \xi, \xi\rangle_{\mathcal{A}})\varphi(B^*).$$

From the relationship between θ and φ we get

$$\varphi(A)\langle \theta\xi, \theta\xi\rangle_{\mathcal{B}}\varphi(A)^* \leq \sum_{i\in I} \langle \theta T_i\xi, \theta T_i\xi\rangle_{\mathcal{B}} \leq \varphi(B)\langle \theta\xi, \theta\xi\rangle_{\mathcal{B}}\varphi(B)^*.$$

From the relationship between θ and T_i we have

$$\varphi(A)\langle \theta\xi, \theta\xi\rangle_{\mathcal{B}}\varphi(A)^* \leq \sum_{i\in I} \langle T_i\theta\xi, T_i\theta\xi\rangle_{\mathcal{B}} \leq \varphi(B)\langle \theta\xi, \theta\xi\rangle_{\mathcal{B}}\varphi(B)^*.$$

Then

$$\varphi(A)\langle \eta, \eta\rangle_{\mathcal{B}}(\varphi(A))^* \leq \sum_{i\in I} \langle T_i\eta, T_i\eta\rangle_{\mathcal{B}} \leq \varphi(B)\langle \eta, \eta\rangle_{\mathcal{B}}(\varphi(B))^*, \forall \eta \in \mathcal{X}.$$

On the other hand, we have

$$\begin{aligned}\varphi(\langle S_{\mathcal{A}}\xi,\eta\rangle_{\mathcal{A}}) &= \varphi(\langle\sum_{i\in I}T_i^*T_i\xi,\eta\rangle_{\mathcal{A}})\\ &= \sum_{i\in I}\varphi(\langle T_i\xi,T_i\eta\rangle_{\mathcal{A}})\\ &= \sum_{i\in I}\langle\theta T_i\xi,\theta T_i\eta\rangle_{\mathcal{B}}\\ &= \sum_{i\in I}\langle T_i\theta\xi,T_i\theta\eta\rangle_{\mathcal{B}}\\ &= \langle\sum_{i\in I}T_i^*T_i\theta\xi,\theta\eta\rangle_{\mathcal{B}}\\ &= \langle S_{\mathcal{B}}\theta\xi,\theta\eta\rangle_{\mathcal{B}},\end{aligned}$$

which completes the proof. ◄

4. Tensor product

The minimal or injective tensor product of the pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A}\otimes\mathcal{B}$, is the completion of the algebraic tensor product $\mathcal{A}\otimes_{\text{alg}}\mathcal{B}$ with respect to the topology determined by a family of C^*-seminorms. Suppose that $\mathcal{X}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{A}$ and $\mathcal{Y}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{B}$. The algebraic tensor product $\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ is a pre-Hilbert $\mathcal{A}\otimes\mathcal{B}$-module with the action of $\mathcal{A}\otimes\mathcal{B}$ on $\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}$ defined by

$$(\xi\otimes\eta)(a\otimes b)=\xi a\otimes\eta b \text{ for all } \xi\in\mathcal{X},\eta\in\mathcal{Y},a\in\mathcal{A} \text{ and } b\in\mathcal{B}$$

and the inner product

$$\langle\cdot,\cdot\rangle:(\mathcal{X}\otimes_{\text{alg}}\mathcal{Y})\times(\mathcal{X}\otimes_{\text{alg}}\mathcal{Y})\to\mathcal{A}\otimes_{\text{alg}}\mathcal{B}. \text{ defined by}$$

$$\langle\xi_1\otimes\eta_1,\xi_2\otimes\eta_2\rangle=\langle\xi_1,\xi_2\rangle\otimes\langle\eta_1,\eta_2\rangle$$

We also know that for $z=\sum_{i=1}^n\xi_i\otimes\eta_i$ in $\mathcal{X}\otimes_{alg}\mathcal{Y}$ we have $\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}}=\sum_{i,j}\langle\xi_i,\xi_j\rangle_{\mathcal{A}}\otimes\langle\eta_i,\eta_j\rangle_{\mathcal{B}}\geq 0$ and $\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}}=0$ iff $z=0$.
The external tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is the Hilbert module $\mathcal{X}\otimes\mathcal{Y}$ over $\mathcal{A}\otimes\mathcal{B}$ obtained by the completion of the pre-Hilbert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{X}\otimes_{\text{alg}}\mathcal{Y}$.

If $P\in M(\mathcal{X})$ and $Q\in M(\mathcal{Y})$, then there is a unique adjointable module morphism $P\otimes Q:\mathcal{A}\otimes\mathcal{B}\to\mathcal{X}\otimes\mathcal{Y}$ such that $(P\otimes Q)(a\otimes b)=P(a)\otimes Q(b)$ and $(P\otimes Q)^*(a\otimes b)=P^*(a)\otimes Q^*(b)$ for all $a\in A$ and for all $b\in B$ (see, for example, [10]). Let I and J be countable index sets.

Theorem 6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert pro-C^*-modules over pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\{T_i\}_{i\in I} \subset Hom^*_{\mathcal{A}}(\mathcal{X})$ and $\{L_j\}_{j\in J} \subset Hom^*_{\mathcal{B}}(\mathcal{Y})$ be two $*$-operator frames for $\mathcal{X}$ and $\mathcal{Y}$ with $*$-frame operators S_T and S_L and $*$-operator frame bounds (A, B) and (C, D), respectively. Then $\{T_i \otimes L_j\}_{i\in I, j\in J}$ is an $*$-operator frame for Hilbert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{X}\otimes\mathcal{Y}$ with $*$-frame operator $S_T \otimes S_L$ and lower and upper $*$-operator frame bounds $A \otimes C$ and $B \otimes D$, respectively.*

Proof. By the definition of $*$-operator frames $\{T_i\}_{i\in I}$ and $\{L_j\}_{j\in J}$, we have

$$A\langle\xi,\xi\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle_{\mathcal{A}} \leq B\langle\xi,\xi\rangle_{\mathcal{A}}B^*, \forall\xi\in\mathcal{X},$$

and

$$C\langle\eta,\eta\rangle_{\mathcal{B}}C^* \leq \sum_{j\in J}\langle L_j\eta, L_j\eta\rangle_{\mathcal{B}} \leq D\langle\eta,\eta\rangle_{\mathcal{B}}D^*, \forall\eta\in\mathcal{Y}.$$

Therefore,

$$\begin{aligned}
&(A\langle\xi,\xi\rangle_{\mathcal{A}}A^*)\otimes(C\langle\eta,\eta\rangle_{\mathcal{B}}C^*)\\
&\leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle_{\mathcal{A}}\otimes\sum_{j\in J}\langle L_j\eta, L_j\eta\rangle_{\mathcal{B}}\\
&\leq (B\langle\xi,\xi\rangle_{\mathcal{A}}B^*)\otimes(D\langle\eta,\eta\rangle_{\mathcal{B}}D^*), \forall\xi\in\mathcal{X}, \forall\eta\in\mathcal{Y}.
\end{aligned}$$

Then

$$\begin{aligned}
&(A\otimes C)(\langle\xi,\xi\rangle_{\mathcal{A}}\otimes\langle\eta,\eta\rangle_{\mathcal{B}})(A^*\otimes C^*)\\
&\leq \sum_{i\in I, j\in J}\langle T_i\xi, T_i\xi\rangle_{\mathcal{A}}\otimes\langle L_j\eta, L_j\eta\rangle_{\mathcal{B}}\\
&\leq (B\otimes D)(\langle\xi,\xi\rangle_{\mathcal{A}}\otimes\langle\eta,\eta\rangle_{\mathcal{B}})(B^*\otimes D^*), \forall\xi\in\mathcal{X}, \forall\eta\in\mathcal{Y}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&(A\otimes C)\langle\xi\otimes\eta, \xi\otimes\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}(A\otimes C)^*\\
&\leq \sum_{i\in I, j\in J}\langle T_i\xi\otimes L_j\eta, T_i\xi\otimes L_j\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}\\
&\leq (B\otimes D)\langle\xi\otimes\eta, \xi\otimes\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}(B\otimes D)^*, \forall\xi\in\mathcal{X}, \forall\eta\in\mathcal{Y}.
\end{aligned}$$

Then for all $\xi\otimes\eta\in\mathcal{X}\otimes\mathcal{Y}$ we have

$$\begin{aligned}
&(A\otimes C)\langle\xi\otimes\eta, \xi\otimes\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}(A\otimes C)^*\\
&\leq \sum_{i\in I, j\in J}\langle(T_i\otimes L_j)(\xi\otimes\eta), (T_i\otimes L_j)(\xi\otimes\eta)\rangle_{\mathcal{A}\otimes\mathcal{B}}\\
&\leq (B\otimes D)\langle\xi\otimes\eta, \xi\otimes\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}(B\otimes D)^*.
\end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ and then it's satisfied for all $z \in \mathcal{X} \otimes \mathcal{Y}$. It shows that $\{T_i \otimes L_j\}_{i\in I, j\in J}$ is an $*$-operator frame for Hilbert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{X} \otimes \mathcal{Y}$ with lower and upper $*$-operator frame bounds $A \otimes C$ and $B \otimes D$, respectively.

By the definition of $*$-frame operators S_T and S_L, we have:

$$S_T\xi = \sum_{i\in I} T_i^* T_i \xi, \forall \xi \in \mathcal{X},$$

and

$$S_L\eta = \sum_{j\in J} L_j^* L_j \eta, \forall \eta \in \mathcal{Y}.$$

Therefore,

$$\begin{aligned}
(S_T \otimes S_L)(\xi \otimes \eta) &= S_T\xi \otimes S_L\eta \\
&= \sum_{i\in I} T_i^* T_i \xi \otimes \sum_{j\in J} L_j^* L_j \eta \\
&= \sum_{i\in I, j\in J} T_i^* T_i \xi \otimes L_j^* L_j \eta \\
&= \sum_{i\in I, j\in J} (T_i^* \otimes L_j^*)(T_i\xi \otimes L_j\eta) \\
&= \sum_{i\in I, j\in J} (T_i^* \otimes L_j^*)(T_i \otimes L_j)(\xi \otimes \eta) \\
&= \sum_{i\in I, j\in J} (T_i \otimes L_j)^*)(L_i \otimes L_j)(\xi \otimes \eta).
\end{aligned}$$

Now by the uniqueness of $*$-frame operator, the last expression is equal to $S_{T\otimes L}(\xi \otimes \eta)$. Consequently we have $(S_T \otimes S_L)(\xi \otimes \eta) = S_{T\otimes L}(\xi \otimes \eta)$. The last equality is satisfied for every finite sum of elements in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ and then it's satisfied for all $z \in \mathcal{X} \otimes \mathcal{Y}$. It shows that $(S_T \otimes S_L)(z) = S_{T\otimes L}(z)$. So $S_{T\otimes L} = S_T \otimes S_L$. ◄

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