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Nonlinear Parabolic Problems in Anisotropic Sobolev Spaces

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Abstract. In this paper, our goal is to establish results on existence of renormalized solutions for a class of Stefan problems of the form $\beta(u)_t - div(a(x, Du) + F(u)) \ni f$, posed in an open bounded Ω, where data belongs to $L^1 - data$, β is a maximal monotone graph and $div(a(x, Du))$ is a Leary-Lions operator with anisotropic growth conditions.

Key Words and Phrases: anisotropic Sobolev spaces, maximal monotone graph, Stefan type problems.

2010 Mathematics Subject Classifications: 35K55, 47B44, 35D30

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1. Introduction

We investigate the existence and uniqueness of solutions for the following nonlinear parabolic probelm:

$$(P, f, b_0) \begin{cases} \beta(u)_t - \operatorname{div}(a(x, Du) + F(u)) \ni f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ \beta(u(0, .)) \ni b_0 & \text{in } \Omega, \end{cases}$$

where Ω is a domain in the space $\mathbb{R}^N, N \geqslant 1$, Q_T is cylindrical domain $Q_T = (0, T) \times \Omega$, a right-hand side $f \in L^1(Q_T)$. Furthermore, $F : \mathbb{R} \to \mathbb{R}^N$ is locally Lipschitz continuous and $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued, maximal monotone mapping such that $0 \in \beta(0)$. Moreover, we assume that $\beta^0(l) \in L^1(\Omega)$ for each $l \in \mathbb{R}$, where β^0 denotes the minimal selection of the graph of β. Namely $\beta_0(l)$ is the minimal in the norm element of $\beta(l)$, $\beta_0(l) = \inf\{|r|/r \in \mathbb{R} \text{ and } r \in \beta(l)\}$.

The Stefan problem is a classical mathematical problem in heat transfer that deals with the melting or solidification of a material. It is named after the Austrian mathematician Josef Stefan, who first formulated the problem in 1891. It

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is concerned with the temperature distribution in a material that is undergoing a phase change, such as the melting of an ice cube or the solidification of molten metal. The problem involves solving a set of partial differential equations that describe the heat transfer and mass transfer processes that occur during the phase change.

A large number of papers and researches have then been dedicated to this model and its extensions. Our problem appears in multiphase Stefan problem [11] and Hele-shaw problem [10], also in a various phenomena with changes of phases [16], [14], [9].

Our problem can be viewed as ganeralistion of the following problem with homogeneous Direchlet boundary conditions: [5]

$$DP_\gamma\,(f,u_0)\begin{cases} u_t - \operatorname{div} a(x,Du) = f \quad \text{ in } Q_T,\\ u=0 \text{ on } S_T,\\ u(0)=u_0 \quad \text{ in } \Omega.\end{cases}$$

In the case where $\beta_t = b_t$, with b a maximal monotone graph on $\mathbb{R}$ such that $b^{-1} \in C^0(\mathbb{R})$, D. Blanchard and A.Porretta deal in [7] the existence of renormalized solutions.

$$\begin{cases} b(u)_t - \operatorname{div}(a(x,u,\nabla u)) + \operatorname{div}(\Phi(u)) = f & \text{in } \Omega\times(0,T),\\ b(u)(t=0) = b_0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega\times(0,T).\end{cases}$$

The concept of these solutions was introduced by R.J. DiPerna and P.L. Lions [12] for the study of Boltzmann equations. This notion was then extended to the study of various problems of partial differential equations of parabolic, elliptic-parabolic and hyperbolic type, for more details see [3],[21].

In this work, we deal with the existence and uniqueness of renormalized solutions of (P, f, b_0) under a local Lipschitz condition on $a(x,.)$. We use mainly truncation techniques and a generalized Minty method in anisotropic Sobolev spaces. We recall that in the case with variable exponent Sobolev spaces this problem was treated by Wittbold and al. [23]. Note that other work in this direction can be found in [15].

In the elliptic case, we showed in [1] that there exists a renormalized solution to the elliptic problem (E, F)

$$\beta(u) - div(a(x, D(u) + F(u)) \ni f \quad in \quad \Omega.$$

These results will serve to deduce that there exists a mild solution of the abstract problem correspending to (P, f, b_0) in the sense of non linear semigroup theory. We detail that in next section.

This article is organized as follows. In Section 2, we recall some basic notations and properties of anisotropic Sobolev spaces, we make assumptions on the problem (P, f, b_0), and we give the definition of mild solutions of the abstract Cauchy problem. In Section 3, we present notions of renormalized solutions associated to our problem. In Section 4, we give the main results in this paper. In Section 5 and 6, we prove the existence of weak and renormalized solutions.

2. Preliminaries

2.1. Function spaces and basic assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$, $(N \geq 2)$ and let $1 \leq p_1, ..., p_N < \infty$ be N real numbers, $p^+ = max(p_1, ..., p_N)$, $p^- = min(p_1, ..., p_N)$ and $\overrightarrow{p} = (p_1, ..., p_N)$. The anisotropic spaces (see [22])

$$W^{1,\overrightarrow{p}}(\Omega) = \{u \in W^{1,1}(\Omega) : \partial_{x_i} u \in L^{p_i}(\Omega), i = 1, ..., N\}$$

is a Banach space with respect to norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}.$$

The space $W_0^{1,\overrightarrow{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm.

The dual space of anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ is equivalent to $W^{-1,\overrightarrow{p'}}(\Omega)$, where $\overrightarrow{p'} = (p'_1, ..., p'_N)$ and $p'_i = \dfrac{p_i}{p_i - 1}$ for all $i = 1, ..., , N$.

We recall now a Poincaré-type inequality:

Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, then for every $q \geq 1$ there exists a constant C_p (depending on q and p_i (see [13]) such that

$$\|u\|_{L^q(\Omega)} \leq C_p \|\partial_{x_i} u\|_{L^{p_i}(\Omega)} \text{ for } i = 1, ..., N. \tag{1}$$

Moreover, a Sobolev-type inequality holds. Let us denote by $\overline{p}$ the harmonic mean of these numbers, i.e. $\dfrac{1}{\overline{p}} = \dfrac{1}{N} \displaystyle\sum_{i=1}^{N} \dfrac{1}{p_i}$. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$. It follows from [22] that there exists a constant C_s such that

$$\|u\|_{L^q(\Omega)} \leq C_s \prod_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \tag{2}$$

where $q = \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$ if $\overline{p} < N$ or $q \in [1, +\infty[$ if $\overline{p} \geq N$. On the right-hand side of (2) it is possible to replace the geometric mean by the arithmetic mean: let

$a_1, ..., a_N$ be positive numbers. Then

$$\prod_{i=1}^{N} a_i^{\frac{1}{N}} \leq \frac{1}{N}\sum_{i=1}^{N} a_i,$$

which implies by (2) that

$$\|u\|_{L^q(\Omega)} \leq \frac{C_s}{N}\sum_{i=1}^{N}\|\partial_{x_i}u\|_{L^{p_i}(\Omega)}. \tag{3}$$

Note that when the following inequality holds

$$\overline{p} < N \tag{4}$$

holds, then the inequality (3) implies the continuous embedding of the space $W_0^{1,\overrightarrow{p}}(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \overline{p}^*]$. On the other hand, the continuity of the embeding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^+}(\Omega)$ with $p^+ := max\{p_1, ..., p_N\}$ relies on inequality (1).

It may happen that $\overline{p}^* < p^+$ if the exponents p_i are closed enough, then $p_\infty := max\{\overline{p}^*, p^+\}$ turns out to be the critical exponent in the anisotrpic Sobolev embedding (see [22]).

Proposition 1. *If the condition* (4) *holds, then for* $q \in [1, p_\infty]$ *there is a continuous embedding* $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$*. For* $q < p_\infty$ *the embedding is compact.*

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega). \tag{5}$$

We can introduce the subspace $X = W_0^{1,\overrightarrow{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{W^{1,\overrightarrow{p}}(\Omega)}$.

We consider the parabolic anisotropic Sobolev space (see [18])

$$L^{\overrightarrow{p}}(0,T; W_0^{1,\overrightarrow{p}}(\Omega)) = \{u : [0,T] \to W_0^{1,\overrightarrow{p}}(\Omega)\text{mesurable}/\partial_i u \in L^{p_i}(Q_T), i = 1, ..., N\}$$

where $Q_T = \Omega \times [0,T]$.

The norm on this space is defined as follows:

$$\|u\|_X = \sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i}(Q_T)}.$$

Now we give our assumptions on the problem (P, f, b_0). Let the function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions:

(A1): $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.
(A2): Coerciveness: there exists a positive constant λ such that

$$\sum_{i=1}^{N} a_i(x,\xi) \cdot \xi_i \geq \lambda \sum_{i=1}^{N} |\xi_i|^{p_i}$$

holds for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.
(A3): Growth restriction:

$$|a_i(x,\xi)| \leq \gamma(d_i(x) + |\xi_i^{p_i-1}|)$$

for almost every $x \in \Omega$, γ is a positive constant for $i = 1, ..., N$, d_i is a positive function in $L^{p_i'}(\Omega)$ and every $\xi \in \mathbb{R}^N$.
(A4): Monotonicity in $\xi \in \mathbb{R}^N$:

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) \geq 0$$

for almost every $x \in \Omega$ and every $\xi, \eta \in \mathbb{R}^N$.

2.2. Mild solution

We outline some of the main points of the theory of nonlinear semigroups and evolution equations governed by accretive operators, called 'mild' solution for abstract Cauchy problems of the form

$$\frac{du}{dt} + Au \ni f,$$

where V is a real Banach space with norm denoted by $\| \cdot \|$, $f : (0,T) \to V$ and $A : D(A) \to 2^V$ is a (multivalued) operator. The use of multivalued nonlinear operators permits to obtain a coherent theory but also it is quite useful in applications. We refer to [6].

It follows from Theorem 4.1 (see [1]) that for all $f \in L^1(\Omega)$, there exists a renormalized solution (u,b) to (E,f). For $f, \tilde{f} \in L^1(\Omega)$ let (u,b), and $(\tilde{u},\tilde{b})$ be renormalized solutions of $(E,f),(E,\tilde{f})$ respectively. Writing $|b-\tilde{b}| = (b-\tilde{b})^+ + (\tilde{b}-b)^+$ and applying the comparison principle from Lemma 6.6 (see [1]), we find that

$$\|b - \tilde{b}\|_{L^1(\Omega)} \leq \|f - \tilde{f}\|_{L^1(\Omega).} \tag{6}$$

In terms of nonlinear operators the preceeding results read as follows: if A_β is the nonlinear operator defined in $L^1(\Omega)$ by

$A_\beta := \{(b,w) \in L^1(\Omega) \times L^1(\Omega) : \exists u : \Omega \to \mathbb{R}$ measurable, $u \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ $b \in \beta(u)$ a.e in Ω, u is a renormalized solution of $-\operatorname{div}(a(x,Du)+F(u)) = w\}$. Then A_β is $m-$accrtive in $L^1(\Omega)$.

Here, we provide the definition of $m-$accretive operator A.

Definition 1. *An operator A in V is accretive if*

$$\|x-\hat{x}\| \leq \|x-\hat{x}+\lambda(y-\hat{y})\| \text{ whenever } \lambda>0 \text{ and } (x,y),(\hat{x},\hat{y}) \in A.$$

Note that A is accretive if and only if, for $\lambda>0$,

$$\left\|(I+\lambda A)^{-1}z-(I+\lambda A)^{-1}\hat{z}\right\| \leq \|z-\hat{z}\|,$$

that is, A is accretive if and only if $J_\lambda^A := (I+\lambda A)^{-1}$ (called the resolvent of A) is a single-valued nonexpansive map for $\lambda>0$. An operator in a Hilbert space is accretive if it is monotone, that is,

$$(x-\hat{x} \mid y-\hat{y}) \geq 0 \quad \text{for all } (x,y),(\hat{x},\hat{y}) \in A.$$

But apart from accretivity one should expect a range condition to get the existence of solution as well.

One could ask for $\mathrm{R}(I+\lambda A)=V$ for all $\lambda>0$:

An operator A is said to be m-accretive in V if A is accretive and $\mathrm{R}(I+\lambda A)=V$ for all $\lambda>0$; if and only if there exists one $\lambda>0$ such that $\mathrm{R}(I+\lambda A)=V$.

So, if we take $A=A_\beta$ and $V=L^1(\Omega)$, we have the following corollary:

Corollary 1. *Let A_β be the operator defined as follows: $A_\beta := \{(b,w) \in L^1(\Omega)\times L^1(\Omega) : \exists u:\Omega\to\mathbb{R}$ measurable, $u \in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$ $b \in \beta(u)$ a.e in Ω, u is a renormalized solution of $-\operatorname{div}(a(x,Du)+F(u))=w\}$ verifies the following properties:*

i) A_β is m -accretive in $L^1(\Omega)$,

ii) $R(I+\lambda A_\beta)=L^1(\Omega), \lambda>0$,

iii) $\overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} = \left\{b \in L^1(\Omega) : b \in \overline{R(\beta)} \text{ a.e in } \Omega\right\}$.

Proof. A_β is m-accretive in $L^1(\Omega)$, i.e., the resolvent mapping $f \in L^1(\Omega) \to (I+\lambda A_\beta)^{-1} f := J_{A_\beta}^\lambda(f)$ is a contraction in the L^1 -norm (because of 6) and the range condition

$$R(I+\lambda A_\beta)=L^1(\Omega) \tag{7}$$

holds. Indeed, for any $f \in L^1(\Omega), \lambda>0$ there exists $(b,w) \in A_\beta$ such that

$$b+\lambda w=f \tag{8}$$

almost everywhere in Ω. If (u,b) is the renormalized solution to (E,f), then we have $b \in \beta(u)$ almost everywhere in Ω and u is the renormalized solution to

$$-\lambda \operatorname{div}(a(x,Du)+F(u))=f-b.$$

Therefore, $\left(b,\frac{f-b}{\lambda}\right) \in A_\beta$ and (8) holds with $w=\frac{f-b}{\lambda}$. For iii) see Proposition 4.1.1 (see [23]). ◄

Remark 1. *By the general theory of nanlinear semigroups (see [4], [6]) we conclude that the abstract Cauchy problem corresponding to* (P, f, b_0)

$$(ACP)\,(f, b_0) \begin{cases} \dfrac{db}{dt} + A_\beta b \ni f & in \quad (0,T), \\ b(0) = b_0, \end{cases}$$

admits a unique mild solution $b \in C\left([0,T); L^1(\Omega)\right)$ *for any initial datum* $b_o \in \overline{D\left(A_\beta\right)}^{\|\cdot\|_{L^1(\Omega)}}, 0 < \varepsilon \leq 1, N(\varepsilon) \in \mathbb{N}$ *and any right-hand side* $f \in L^1\left(0,T; L^1(\Omega)\right) \simeq L^1\left(Q_T\right)$. *Roughly speaking, a mild solution is a contimuous abstract function* $b \in C(0,T]; L^1(\Omega))$ *which is the uniform limit of piecewise constant functions.*

Lemma 1. *For* $f \in L^1\left(Q_T\right), b_0 \in \overline{D\left(A_\beta\right)}^{\|\cdot\|_{L^1(\Omega)}}, 0 < \varepsilon \leq 1, N(\varepsilon) \in \mathbb{N}$ *and*

$$(D_\varepsilon) \begin{cases} t_0^\varepsilon = 0 < t_1^\varepsilon < \ldots < t_{N(\varepsilon)}^\varepsilon = T \\ t_j^\varepsilon - t_{j-1}^\varepsilon = \varepsilon, T - t_{N(\varepsilon)}^\varepsilon \leq \varepsilon \forall j = 1, \ldots.., N(\varepsilon) \\ f_j^\varepsilon \in L^\infty(\Omega), j = 1, \ldots\ldots, N(\varepsilon) : \displaystyle\sum_{j=1}^{N(\varepsilon)} \int_{t_{j-1}^\varepsilon}^{t_j^\varepsilon} \left\|f(t) - f_j^\varepsilon\right\|_{L^1(\Omega)} dt \leq \varepsilon \\ b_0^\varepsilon \in L^\infty(\Omega) : \left\|b_0^\varepsilon - b_0\right\|_{L^1(\Omega)} \leq \varepsilon. \end{cases}$$

Let $\left(b_j^\varepsilon, \mathrm{u}_j^\varepsilon\right)_{j=1}^{N(\varepsilon)}$ *be a solution of the discretized prablem*

$$(DP_{\varepsilon,\psi}) \begin{cases} b_j^\varepsilon \in L^1(\Omega), u_j^\varepsilon : \Omega \to \mathbb{R} \text{ measurable, } T_k\left(u_j^\varepsilon\right) \in W_0^{1,\overrightarrow{p}}(\Omega), \forall k > 0 \\ \displaystyle\int_{\left\{n<\left|u_j^\varepsilon\right|<n+1\right\}} a\left(x, Du_j^\varepsilon\right) \cdot Du_j^\varepsilon \to 0 \text{ as } n \to \infty \\ \displaystyle\int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{t_j^\varepsilon - t_{j-1}^\varepsilon} h\left(u_j^\varepsilon\right)\varphi + \int_\Omega \left(a\left(x, Du_j^\varepsilon\right) + F\left(u_j^\varepsilon\right)\right) \cdot D\left(h\left(u_j^\varepsilon\right)\varphi\right) = \\ = \int_\Omega f_j^\varepsilon h\left(u_j^\varepsilon\right)\varphi \\ \forall \varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega), h \in C_c^1(\mathbb{R}), \\ b_j^\varepsilon \in \beta\left(u_j^\varepsilon\right) \qquad a.e.\ in \Omega,\ for\ all\ j = 1, \ldots\ldots, N(\varepsilon). \end{cases}$$

For $k > 0$, *we define the piecewise constant fonctions* $f_\varepsilon = (0,T] \to L^1(\Omega), b_\varepsilon : [0,T] \to L^1(\Omega)$, *and* $T_k\left(u_\varepsilon\right) : (0,T] \to W_0^{1,\overrightarrow{p}}(\Omega)$ *as follows:* $f_\varepsilon(t) = f_j^\varepsilon, b_\varepsilon(0) = b_0^\varepsilon$, $b_\varepsilon(t) = b_j^\varepsilon$, *and* $T_k\left(u_\varepsilon(t)\right) = T_k\left(u_j^\varepsilon\right)$ *for* $t \in \left(t_j^\varepsilon, t_{j-1}^\varepsilon\right)$ *and* $j = 1, \ldots, N(\varepsilon)$. *If* $t_{N(\varepsilon)}^\varepsilon < T$, $f_\varepsilon, b_\varepsilon$ *and* u_ε *are extended by setting* $f_{\boldsymbol{\varepsilon}}(t) = f_j^{\mathrm{N}(\varepsilon)}, T_k\left(u_\varepsilon(t)\right) = T_k\left(u_{N(\varepsilon)}^\varepsilon\right)$ *and* $b_\varepsilon(t) = b_j^{N(\varepsilon)}$ *far all* $t \in \left(t_\epsilon^{N(\varepsilon)}, T\right]$.

Then the following estimates hold true for all $k > 0$ and $0 < \varepsilon \leq 1$:

i)There exists a constant $C_1\left(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k, T\right) > 0$ not depending an $\varepsilon > 0$ sich that

$$\int_0^T \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i} dxdt \leq C_1\left(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k\right). \tag{9}$$

ii) There exists a canstant $C_2\left(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k, T\right) > 0$ not depending on $\varepsilon > 0$, such that

$$\|T_k(u_\varepsilon)\|_{L^{\vec{p}}\left(0,T,W_0^{1,\vec{p}}(\Omega)\right)} \leq C_2\left(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k\right). \tag{10}$$

iii) There exists a constant $C_3\left(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}\right) > 0$ not depending on $\varepsilon > 0$, such that

$$\sum_{i=1}^N \int_0^T \int_\Omega |a_i(x, DT_k(u_\varepsilon))|^{p_i'} dxdt \leq C_3\left(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}\right). \tag{11}$$

Proof. For $j \in \{1, \ldots, N(\varepsilon)\}$, we take $T_k\left(u_j^\varepsilon\right) h_l\left(u_j^\varepsilon\right), k, l > 0$ as a test function in (DP_ε) to obtain $I_1 + I_2 + I_3 = I_4$, where

$$I_1 = \int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{t_j^\varepsilon - t_{j-1}^\varepsilon} h_l\left(u_j^\varepsilon\right) T_k\left(u_j^\varepsilon\right), \quad I_2 = \int_\Omega a\left(x, Du_j^\varepsilon\right) \cdot D\left(h_l\left(u_j^\varepsilon\right) T_k\left(u_j^\varepsilon\right)\right),$$

$$I_3 = \int_\Omega F\left(u_j^\varepsilon\right) \cdot D\left(h_l\left(u_j^\varepsilon\right) T_k\left(u_j^\varepsilon\right)\right), \qquad I_4 = \int_\Omega f_j^\varepsilon h_l\left(u_j^\varepsilon\right) T_k\left(u_j^\varepsilon\right).$$

By Gauss-Green Theorem, it follows that $I_3 = 0$ for all $l > k$. Applying **(A2)** in I_2, we can pass to the limit with $l \to \infty$ and find

$$\int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{t_j^\varepsilon - t_{j-1}^\varepsilon} T_k\left(u_j^\varepsilon\right) + \lambda \int_\Omega \sum_{i=1}^N \left|\frac{\partial T_k\left(u_j^\varepsilon\right)}{\partial x_i}\right|^{p_i} \leq k \int_\Omega \left|f_j^\varepsilon\right|. \tag{12}$$

If we define the convex, l.s.c., proper function $\phi_{T_k} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi_{T_k}(r) = \begin{cases} \int_0^r T_k\left(\left(\beta^{-1}\right)^0(\sigma)\right) d\sigma, & \text{if } r \in \overline{R(\beta)}, \\ +\infty & \text{otherwise}, \end{cases}$$

then $T_k\left(u_j^\varepsilon\right) \subset \partial\phi_{T_k}\left(b_j^\varepsilon\right)$ for all $j = 1, \ldots, N(\varepsilon)$ and

$$\phi_{T_k}\left(b_j^\varepsilon\right) - \phi_{T_k}\left(b_{j-1}^\varepsilon\right) \leq \left(b_j^\varepsilon - b_{j-1}^\varepsilon\right) T_k\left(u_j^\varepsilon\right) \tag{13}$$

holds almost everywhere in Ω. Therefore, from (12) and (13) it follows that

$$\int_\Omega \frac{\phi_{T_k}\left(b_j^\varepsilon\right)-\phi_{T_k}\left(b_{j-1}^\varepsilon\right)}{t_j^\varepsilon-t_{j-1}^\varepsilon}+\lambda\sum_{i=1}^N\int_\Omega\left|\frac{\partial T_k\left(u_j^\varepsilon\right)}{\partial x_i}\right|^{p_i}dx\le k\int_\Omega\left|f_j^\varepsilon\right|. \tag{14}$$

Integrating (14) over $(t_{j-1},t_j]$ and taking the sum over $j=1,\ldots..,N(\varepsilon)$ yields

$$\int_\Omega \phi_{T_k}\left(b_\varepsilon(T)\right)dx+\lambda\int_0^T\sum_{i=1}^N\int_\Omega\left|\frac{\partial T_k\left(u_\varepsilon\right)}{\partial x_i}\right|^{p_i}dxdt\le\int_\Omega\phi_k\left(b_\varepsilon(0)\right)+k\int_0^T\int_\Omega\left|f_\varepsilon\right|dxdt. \tag{15}$$

According to (D_ε), $b_\varepsilon(0)=b_0^\varepsilon$ converges to b_0 and f_ε converges to f in $L^1\left(0,T,L^1(\Omega)\right)$ as $\varepsilon\downarrow 0$ Therefore, the right-hand side of (15) is bounded by a constant $C_1\left(\|f\|_{L^1(Q_T)},\|b_0\|_{L^1(\Omega)},k\right)>0$ that does not depend on ε. Now, i) and ii) follows from (15) if we neglect the positive term and use **(A2)**.

To prove iii), we use **(A3)** and the same arguments as above. ◄

2.3. Integration-by-parts-formula

In the next Lemma, we prove an integration-by-parts-formula that will be crucial in the sequal. The idea of the proof is the same as in [2] and the generalisations considered in [19].

Lemma 2. *Let $\beta\subset\mathbb{R}\times\mathbb{R}$ be a maximal monotone graph, $u\in V, b\in L^1\left(Q_T\right)$ be such that $b\in\beta(u)$ almost everywhere in $Q_T, b_0\in L^1(\Omega)$ with $b(0,x)=b_0$ almost everywhere in Ω and $u_0:\Omega\to\overline{\mathbb{R}}$ be a measurable function such that $b_0\in\left(u_0\right)$ almost everywhere in Ω. Furthermore, we assume that there exists $G\in L^{\overrightarrow{p}'}(0,T;W_0^{1,\overrightarrow{p}}(\Omega))+L^1\left(Q_T\right)$ satisfying*

$$\int_{Q_T}\left(b-b_0\right)\xi_t=\langle G,\xi\rangle \tag{16}$$

for all $\xi\in D([0,T)\times\Omega)$, where $<.,.>$ denotes the duality pairing between $L^{\overrightarrow{p}'}(0,T;W^{-1,\overrightarrow{p}'}(\Omega))+L^1\left(Q_T\right)$ and $L^{\overrightarrow{p}}(0,T;W_0^{1,\overrightarrow{p}}(\Omega))\cap L^\infty\left(Q_T\right)$. Then,

$$\int_{Q_T}\xi_t\int_{b_0}^{b(t,x)}h\left(\left(\beta^{-1}\right)(\sigma)\right)d\sigma=\langle G,h(u)\xi\rangle \tag{17}$$

for all $h\in C_c^1(\mathbb{R})$ and $\xi\in D([0,T)\times\bar{\Omega})$.

Proof. The proof of following lemma 2 will be omitted since it is very similar to that Lemma 4.2.11, p.63-67 in [23]. the a priori estimates in Lemma 1 naturally lead to an appropriate notion of a renormalized solution to (P,f,b_0). ◄

3. Notions of solutions

Definition 2. *For $f \in L^1(Q_T), b_0 \in L^1(\Omega)$ a weak solution to (P, f, b_0) is a pair of function $(u,b) \in L^{\overrightarrow{p}}\left(0,T,W_0^{1,\overrightarrow{p}}(\Omega)\right) \times L^1(Q_T)$ satisfying $F(u) \in \left(L^{p_i'}(Q_T)\right)^N, b \in \beta(u)$ almost everywhere in $Q_T, b(0,x) = b_0$ almost everywhere in Ω such that*

$$-\int_{Q_T}(b-b_0)\xi_t + \int_{Q_T}(a(x,Du)+F(u))\cdot D(\xi)dxdt = \int_{Q_T} f\xi dxdt \tag{18}$$

holds for all $\xi \in D([0,T)\times\Omega)$.

Definition 3. *For $f \in L^1(Q_T), b_0 \in L^1(\Omega)$, a renormalized solution of (P, f, b_0) is a pair of functions (u,b) satisfying the following conditions:*

(P1) *$u : Q_T \to \mathbb{R}$ is measurable, $b \in L^1(Q_T), u(t,x) \in D(\beta(t,x))$ and $b(t,x) \in \beta(u(t,x))$ for a.e. $(t,x) \in Q_T$*

(P2) *$b(0,x) = b_0(x)$ a.e.in Ω,*

(P3) *For each $k > 0, T_k(u) \in L^{\overrightarrow{p}}\left(0,T,W_0^{1,\overrightarrow{p}}(\Omega)\right)$ and $DT_k(u) \in \prod_{i=1}^{N} L^{p_i}(Q_T)$*

(P4)

$$-\int_{Q_T}\xi_t\int_{b_0}^{b(t,x)} ho\left(\beta^{-1}\right)^0(r)drdxdt + \int_{Q_T}(a(x,Du)+F(u))\cdot D(h(u)\xi)dxdt$$
$$= \int_{Q_T} fh(u)\xi dxdt$$

holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and all $\xi \in D([0,T)\times\Omega)$.

(P5) $\displaystyle\int_{Q_T\cap\{k\le|u|\le k+1\}} a(x,Du)\cdot Dudxdt \to 0$ *as $k \to \infty$.*

Remark 2. *Note that if (u,b) is a renormalized solution to (P,f,b_0) such that $u \in L^\infty(Q_T)$ then (u,b) is a weak solution to (P,f,b_0).*

Indeed, as an immediate consequence from $(P1)$, and $(P3)$, we get $u \in L^{\overrightarrow{p}}\left(0,T,W_0^{1,\overrightarrow{p}}(\Omega)\right)$. Now we fix $\xi \in D([0,T)\times\Omega)$ and choose $h_l(u)\xi$ as a test function in $(P4)$. As usual, we apply the Gauss-Green Theorem and the boundary condition on the convection term $\int_{Q_T} h_l'(u)\xi F(u)\cdot$ Du and $(P5)$ to estimate $\int_{Q_T} h_l'(u)\xi a(x,Du)\cdot Du$. Passing to the limit with $l \to \infty$, we find (18). The remaining conditions for being a weak solution follow from $(P1)$ and $(P2)$.

Proposition 2. *For $f \in L^1(Q_T)$, $b_0 \in L^1(\Omega)$ such that there exists a measurable function $u_0 : \Omega \to \overline{\mathbb{R}}$ with $b_0 \in \beta(u_0)$ almost everywhere in Ω, let (u, b) be a weak solution to (P, f, b_0). Then (u, b) is a renormalized solution to (P, f, b_0).*

Proof. Clearly, (u, b) satisfies $(P1), (P2)$ and $(P5)$. By assumption we have $u \in L^{\overrightarrow{p}}\left(0, T, W_0^{1,\overrightarrow{p}}(\Omega)\right)$, hence u is finite almost everywhere in Q_T and it follows that $|\{k < |u| < k+1\}| \to 0$ as $k \to \infty$. In particular, $|Du|^{p_i} \in L^1(Q_T)$ and therefore $(P3)$ holds. From (10) we get that $(b - b_0)_t \in L^{\overrightarrow{p'}}(0, T; W^{-1,\overrightarrow{p'}}(\Omega)) + L^1(Q_T) + L^1(Q_T)$. Now, $(P4)$ follows from the integration-by parts- formula in Lemma 2. ◄

4. Main Results

Theorem 1. *Let $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$. and A_β being the operator defined as $A_\beta := \{(b, w) \in L^1(\Omega) \times L^1(\Omega) : \exists u : \Omega \to \mathbb{R}$ measurable, $u \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ $b \in \beta(u)$ a.e in Ω, u is a renormalized solution of $-\operatorname{div}(a(x, Du) + F(u)) = w\}$, there exists a weak solution (u, b) to (P, f, b_0). In particular, b is the mild solution of $(ACP)(f, b_0)$.*

Theorem 2. *For each $f \in L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$. With A_β be the operator defined as $A_\beta := \{(b, w) \in L^1(\Omega) \times L^1(\Omega) : \exists u : \Omega \to \mathbb{R}$ measurable, $u \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ $b \in \beta(u)$ a.e in Ω, u is a renormalized solution of $-\operatorname{div}(a(x, Du) + F(u)) = w\}$, there exists a renormalized solution to (P, f, b_0).*

To prove Theorem 2, we will use several approximation procedures. First, we prove existence of weak solutions for L^∞ -data in Theorem 1.

5. Proof of Theorem 1

5.1. Auxiliary Problem

Step 1. Approximate problems. In a first step, for bounded data $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$, we prove existence of a weak solution to our elliptic-parabolic problem with an additional strictly monotone and continuous perturbation $\psi : \mathbb{R} \to \mathbb{R}, \psi(0) = 0$, i.e.

$$(P, \psi, f, b_0) \begin{cases} \beta(u)_t + \psi(u) - \operatorname{div}(a(x, Du) + F(u)) \ni f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ \beta(u(0, .)) \ni b_0 & \text{in } \Omega. \end{cases}$$

To this end, we define the nonlinear operator

$A_{\beta,\psi} := \{(b,w) \in L^1(\Omega) \times L^1(\Omega) : \exists u : \Omega \to \mathbb{R}$ measurable, $b \in \beta(u)$ a.e in Ω, u is a renormalized solution of $-\operatorname{div}(a(x,Du) + F(u)) + \psi(u) = w\}$, where a definition of a renormalized solution to the above problem is obtained from Definition 3.2 (see [1]) upon setting $f = w - \psi(u) - b_0$. Using the same arguments as in Corollary 1 it follows that $A_{\beta,\psi}$ is m-accretive in $L^1(\Omega)$ and $\overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} = \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$, i.e. to each $(f, b_0) \in L^1(Q_T) \times \overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$ there exists a unique mild solution $b \in \mathcal{C}([0,T]; L^1(\Omega))$ of the abstract Cauchy problem

$$(ACP)(f,\psi,b_0)\begin{cases} \dfrac{db}{dt} + A_{\beta,\psi} b \ni f \quad \text{in} \quad (0,T) \\ b(0) = b_0 \end{cases}$$

corresponding to (P,ψ,f,b_0). Moreover, for $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$, b is the uniform limit of piecewise constant functions $b_\varepsilon : (0,T) \to L^1(\Omega)$ define by $b_\varepsilon = b_j^\varepsilon$ on $]t_{j-1}^\varepsilon, t_j^\varepsilon[$, $j = 1, \ldots\ldots, N(\varepsilon)$, $b_\varepsilon(0) = b_0^\varepsilon$, where $\left(u_j^\varepsilon, b_j^\varepsilon\right)_{j=1}^{N(\varepsilon)}$ is a solution of the discretized problem (see [2])

$$(DP_{\varepsilon,\psi})\begin{cases} b_j^\varepsilon \in L^1(\Omega), \quad u_j^\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega) \\ \displaystyle\int_{\{n<|u_j^\varepsilon|<n+1\}} a\left(x, Du_j^\varepsilon\right) \cdot Du_j^\varepsilon \to 0 \text{ as } n \to \infty \\ \displaystyle\int_\Omega \frac{b_j^\varepsilon - b_j^{\varepsilon-1}}{\varepsilon}\varphi + \int_\Omega \left(a\left(x, Du_j^\varepsilon\right) + F\left(u_j^\varepsilon\right)\right) \cdot D\varphi + \int_\Omega \psi\left(u_j^\varepsilon\right)\varphi = \int_\Omega f_j^\varepsilon \varphi \\ \forall \varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \\ b_j^\varepsilon \in \beta\left(u_j^\varepsilon\right) \qquad \text{a.e. in } \Omega \\ j = 1, \ldots\ldots\ldots\ldots, N(\varepsilon) \end{cases}$$

given by an equidistant time discretisation of the form

$$(D_\varepsilon)\begin{cases} t_0^\varepsilon = 0 < t_1^\varepsilon < \ldots. < t_{N(\varepsilon)}^\varepsilon = T \\ t_j^\varepsilon - t_{j-1}^\varepsilon = \varepsilon, \forall j = 1, \ldots\ldots, N(\varepsilon) \\ f_j^\varepsilon \in L^\infty(\Omega), j = 1, \ldots\ldots\ldots, N(\varepsilon) : \displaystyle\sum_{j=1}^{N(\varepsilon)} \int_{t_{j-1}^t}^{t_j^\varepsilon} \left\|f(t) - f_j^\varepsilon\right\|_{L^1(\Omega)} dt \le \varepsilon \\ b_0^\varepsilon \in L^\infty(\Omega) : \|b_0^\varepsilon - b_0\|_{L^1(\Omega)} \le \varepsilon. \end{cases}$$

If we define the piecewise constant function $u_\varepsilon : (0,T) \to W_0^{1,\overrightarrow{p}}(\Omega)$ by $u_\varepsilon(t) = u_j^\varepsilon$ for $t \in \left(t_{j-1}^\varepsilon, t_j^\varepsilon\right]$ and $j = 1, \ldots, N(\varepsilon)$, the following a priori estimates hold:

Step 2. A priori estimates.

Lemma 3. *Let u_ε be defined as above. Then, the following results hold for all $\varepsilon > 0$:*

i)There exists a constant $C_1\left(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}\right) > 0$ not depending on $\varepsilon > 0$, such that

$$\|\psi(u_\varepsilon)\|_{L^\infty(Q_T)} \leq C_1\left(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}\right). \tag{19}$$

ii) There exists a constant $C_2\left(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}, \psi\right) > 0$ not depending on $\varepsilon > 0$, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq C_2\left(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}, \psi\right). \tag{20}$$

iii)There exists a constant $C_3\left(\lambda, \|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}\right) > 0$ not depending on $\varepsilon > 0$, such that

$$\int_0^T \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^{p_i} dxdt \leq C_3\left(\lambda, \|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}\right). \tag{21}$$

Proof. For $j = 1, \ldots, N(\varepsilon)$ we choose $p\left(u_j^\varepsilon\right)$ as a test function in $(DP_{\varepsilon,\psi})$, where $p \in \mathcal{P}_0 = \{p \in \mathcal{C}^\infty(\mathbb{R}); 0 \leq p' < 1,\ \text{supp}\ p' \text{ compact},\ 0 \notin \text{supp}\, p\}$. Upon integrating over (t_{j-1}, t_j) and summing over $j = 1, \ldots.., N(\varepsilon)$ as in [3], we obtain i) and from i) we deduce ii) since ψ is strictly increasing and continuous.

To prove iii), for $j = 1, \ldots\ldots, N(\varepsilon)$ we plug u_j^ε a test function in $(DP_{\varepsilon,\psi})$ to obtain

$$I_1 + I_2 + I_3 + I_4 = I_5,$$

where $I_1 = \displaystyle\int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{\varepsilon} u_j^\varepsilon$, $I_2 = \displaystyle\int_\Omega a\left(x, Du_j^\varepsilon\right) \cdot Du_j^\varepsilon, I_3 = \int_\Omega F\left(u_j^\varepsilon\right) \cdot Du_j^\varepsilon$ $I_4 = \displaystyle\int_\Omega \psi\left(u_j^\varepsilon\right) u_j^\varepsilon$, $I_5 = \displaystyle\int_\Omega f_j^\varepsilon u_j^\varepsilon$. Applying **(A2)** in I_2, using the Lipschitz character of F and stokes formula together with the boundary condition (2), we find

$$\int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{\varepsilon} u_j^\varepsilon + \lambda \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_j^\varepsilon}{\partial x_i}\right|^{p_i} dx + \int_\Omega \psi\left(u_j^\varepsilon\right) u_j^\varepsilon \leq \int_\Omega f_j^\varepsilon u_j^\varepsilon. \tag{22}$$

If we define the convex, l.s.c., proper function $\phi_{id} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi_{id}(r) = \begin{cases} \int_0^r \left(\beta^{-1}\right)^0(\sigma)d\sigma, & \text{if } r \in \overline{R(\beta)}, \\ +\infty & \text{otherwise}, \end{cases}$$

then $u_j^{\varepsilon} \in \partial \phi_{id}\left(b_j^{\varepsilon}\right)$ for all $j=1, \ldots\ldots, N(\varepsilon)$ and

$$\phi_{id}\left(b_j^{\varepsilon}\right)-\phi_{id}\left(b_{j-1}^{\varepsilon}\right) \leq\left(b_j^{\varepsilon}-b_{j-1}^{\varepsilon}\right) u_j^{\varepsilon} \tag{23}$$

holds almost everywhere in Ω. Therefore from (22) and (23) it follows that

$$\int_{\Omega} \frac{\phi_{id}\left(b_j^{\varepsilon}\right)-\phi_{id}\left(b_{j-1}^{\varepsilon}\right)}{\varepsilon}+\lambda \int_{\Omega} \sum_{i=1}^{N}\left|\frac{\partial u_j^{\varepsilon}}{\partial x_i}\right|^{p_i} dx \leq \int_{\Omega} f_j^{\varepsilon} u_j^{\varepsilon}. \tag{24}$$

Integrating (24) over $(t_{j-1}, t_j]$ and taking the sum over $j=1, \ldots.., N(\varepsilon)$ yield

$$\begin{aligned} \int_{\Omega} \phi_{id}\left(b_{\varepsilon}(T)\right)+\lambda \int_0^T \int_{\Omega} \sum_{i=1}^{N}\left|\frac{\partial u_{\varepsilon}}{\partial x_i}\right|^{p_i} dxdt \leq \int_{\Omega} \phi_{id}\left(b_{\varepsilon}(0)\right) \\ +C\left\|f_{\varepsilon}\right\|_{L^{\infty}(Q)}\left(\int_0^T \int_{\Omega} \sum_{i=1}^{N}\left|\frac{\partial u_{\varepsilon}(x)}{\partial x_i}\right|^{p_i} dxdt\right)^{\frac{1}{p_i}} (\text{ due to}(3)). \end{aligned} \tag{25}$$

According to $\left(D_{\varepsilon}\right)$, $b_{\varepsilon}(0)=b_0^{\varepsilon}$ converges to b_0 and f_{ε} converges to f in $L^1\left(0, T, L^1(\Omega)\right)$ as $\varepsilon \downarrow 0$ Therefore, the right-hand side of (25) is bounded by a constant

$$C_1\left(\|f\|_{L^{\infty}\left(Q_T\right)},\left\|b_0\right\|_{L^1(\Omega)}\right)>0$$

that does not depend on ε. Now, (iii) follows from (25) if we neglect the positive term. ◀

5.2. The case where β continuous and non-decreasing.

Lemma 4. *Let β be a continuous and non-decreasing function, $f \in L^{\infty}\left(Q_T\right)$, $b_0 \in \overline{D\left(A_{\beta}\right)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^{\infty}(\Omega)$. For $\varepsilon, \delta>0$ let $\left(D_{\varepsilon}\right),\left(D_{\delta}\right)$ be equidistant time discretisations and $\left(u_i^{\varepsilon}, b_i^{\varepsilon}\right)_{i=1}^{N(\varepsilon)},\left(u_j^{\delta}, b_j^{\delta}\right)_{j=1}^{M(\delta)}$ solutions of the corresponding discretised problems $\left(DP_{\varepsilon, \psi}\right)$ and $\left(DP_{\delta, \psi}\right)$. Assume that the piecewise constant functions $b_{\varepsilon}, b_{\delta}:[0, T] \rightarrow L^1(\Omega)$ defined by $b_{\varepsilon}(0)=b_0^{\varepsilon}, b_{\delta}(0)=b_0^{\delta}, b_{\varepsilon}(t)=b_i^{\varepsilon}, b_{\delta}(t)=b_j^{\delta}$ for $t \in\left(t_{i-1}^{\varepsilon}, t_i^{\varepsilon}\right]$ and $t \in\left(t_{j-1}^{\delta}, t_j^{\delta}\right]$ respectively, $i=1, \ldots, N(\varepsilon), j=1, \ldots, M(\delta)$ converge to a function $b \in \mathcal{C}\left([0, T] ; L^1(\Omega)\right)$ as $\varepsilon, \delta \downarrow 0$ in $L^{\infty}\left(0, T ; L^1(\Omega)\right)$. Then*

$$\lim_{\varepsilon, \delta \downarrow 0} \int_0^T \int_{\Omega}\left|\psi\left(u_{\varepsilon}\right)-\psi\left(u_{\delta}\right)\right|=0 \tag{26}$$

holds for the piecewise constant functions $u_\varepsilon, u_\delta : (0,T) \to W_0^{1,\overrightarrow{p}}(\Omega)$ *defined by* $u_\varepsilon(t) = u_i^\varepsilon$ *for* $t \in \left(t_{i-1}^\varepsilon, t_i^\varepsilon\right]$ *and* $i = 1,\ldots.., N(\varepsilon), u_\delta(t) = u_j^\delta$ *for* $\left(t_{j-1}^\delta, t_j^\delta\right]$ *and* $j = 1,\ldots., M(\delta)$

Proof. Use analogous arguments in [23].

The a priori estimates of Lemma 3 and Lemma 4 imply the following convergence results for the approximate solutions of $(DP_{\varepsilon,\psi})$ for $\varepsilon \downarrow 0$: ◄

Lemma 5. *Let* $\varepsilon > 0$ *take values in a sequence in* $(0,1)$ *tending to* 0. *For* $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$, *let* $u_\varepsilon, b_\varepsilon$ *be the piecewise constant functions defined by* $(DP_{\varepsilon,\psi})$. *Then there exist functions* $b \in C\left([0,T]; L^1(\Omega)\right), u \in L^{\overrightarrow{p}}\left(0,T; W_0^{1,\overrightarrow{p}}(\Omega)\right) \cap L^\infty(Q_T)$ *and* $\Phi \in \prod_{i=1}^N L^{p_i'}(Q_T,)$ *such that for a not relabeled subsequence of* $(u_\varepsilon)_\varepsilon$ *we have the following convergence results for* $\varepsilon \downarrow 0$*:*

i) $u_\varepsilon \to u$ *almost everywhere in* Q_T, *weak-* in* $L^\infty(Q_T)$ *and weak in* $L^{\overrightarrow{p}}\left(0,T; W_0^{1,\overrightarrow{p}}(\Omega)\right)$,

ii) $b_\varepsilon \to b$ *in* $L^\infty\left(0,T,L^1(\Omega)\right)$ *and* $b = \beta(u)$ *almost everywhere in* Q_T,

iii) $Du_\varepsilon \to Du$ *in* $\prod_{i=1}^N L^{p_i}(Q_T)$,

iv) $a(x, Du_\varepsilon) \to \Phi$ *in* $\prod_{i=1}^N L^{p_i'}(Q_T)$.

Proof. If $(\varepsilon_n)_n \subset (0,1)$ is a sequence tending to 0 as $n \to \infty$, applying Lemma 4 with $u_\varepsilon = u_{\varepsilon_n}, u_\delta = u_{\varepsilon_m}$ for $m, n \in \mathbb{N}$ yields that (passing to a subsequence if necessary)

$$|\psi(u_{\varepsilon_n}) - \psi(u_{\varepsilon_m})| \to 0$$

almost everywhere in Q_T as $m, n \to \infty$. Hence, $(u_{\varepsilon_n})_n$ is a Cauchy sequence almost everywhere in Q_T and there exists a measurable function $u : Q_T \to \mathbb{R}$ such that $u_{\varepsilon_n} \to u$ almost everywhere in Q_T as $n \to \infty$. By (20) and (21) it follows that $u \in L^{\overrightarrow{p}}(0,T; W_0^{1,\overrightarrow{p}}(\Omega)) \cap L^\infty(Q_T)$ and i), iii) hold. By definition of the operator $A_{\beta,\psi}$, assuming β to be a continuous, non-decreasing function it follows that $b_\varepsilon(t) = \beta(u_\varepsilon(t))$ a.e. in $(0,T)$ for all $\varepsilon > 0$. Keeping in mind that by nonlinear semigroup theory, $(b_\varepsilon)_\varepsilon$ converges to the mild solution $b \in \mathcal{C}\left([0,T]; L^1(\Omega)\right)$ of $(ACP)(f, \psi, b_0)$ as $\varepsilon \downarrow 0$ and using i) and the continuity of β, ii) holds. Applying (21) and **(A3)** (and passing to a subsequence if necessary), we find that $a(x, Du_\varepsilon) \rightharpoonup \Phi$ in $\prod_{i=1}^N L^{p_i'}(Q_T)$ for some $\Phi \in \prod_{i=1}^N L^{p_i'}(Q_T)$. ◄

Using the convergence results of the preceeding Lemma, we have the following existence result.

Proposition 3. *If $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function, then there exists a weak solution (in the sense of Definition 2 $(u, b = \beta(u))$ to (P, ψ, f, b_0) for any $f \in L^\infty(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$.*

Proof. Let $\tilde{b}_\varepsilon : [0,T] \to L^1(\Omega)$ be the piecewise linear function defined by $\tilde{b}_\varepsilon(t) = b^\varepsilon_{j-1} + \frac{t - t^\varepsilon_{j-1}}{t^\varepsilon_j - t^\varepsilon_{j-1}} \left(b^\varepsilon_j - b^\varepsilon_{j-1}\right)$ for $t \in \left[t^\varepsilon_{j-1}, t^\varepsilon_j\right], j = 1, \ldots\ldots, N(\varepsilon)$. For arbitrary $\xi \in D([0,T) \times \Omega)$ and $t \in [0,T)$ the function $\Omega \ni x \to \xi(t,x)$ is in $D(\Omega)$, hence we can use it as a test function each equation of $(DP_{\varepsilon,\psi})$. Integrating over $\left(t^\varepsilon_{j-1}, t^\varepsilon_j\right)$ and summing over $j = 1, \ldots\ldots, N(\varepsilon)$ we find

$$-\int_0^T \int_\Omega \tilde{b}_\varepsilon \xi_t + \left(a\left(x, Du_\varepsilon\right) + F\left(u_\varepsilon\right)\right) \cdot D\xi + \psi\left(u_\varepsilon\right)\xi - \int_\Omega \tilde{b}_\varepsilon(0)\xi(0,.) = \int_0^T \int_\Omega f_\varepsilon \xi. \tag{27}$$

Since $\tilde{b}_\varepsilon \to b$ in $\mathcal{C}\left([0,T]; L^1(\Omega)\right)$ as $\varepsilon \downarrow 0$ using the convergence results of Lemma 4.3 we can pass to the limit in (27) to obtain

$$-\int_0^T \int_\Omega \left(b - b_0\right)\xi_t + (\Phi + F(u)) \cdot D\xi + \psi(u)\xi = \int_0^T \int_\Omega f\xi \tag{28}$$

for all $\xi \in D([0,T) \times \Omega)$, where $b = \beta(u)$. It remains only to prove that $\Phi = a(x, Du)$. To this end, let κ be a non-negative function in $\mathcal{C}^\infty_c([0,T))$. We discretise κ with respect to $(DP_{\varepsilon,\psi})$ by setting $\kappa_\varepsilon(0) = \kappa(0)$ and $\kappa_\varepsilon(t) = \kappa\left(t^\varepsilon_j\right)$ for $t \in \left(t^\varepsilon_{j-1}, t^\varepsilon_j\right]$ and $j = 1, \ldots\ldots, N(\varepsilon)$. Taking $\kappa\left(t^\varepsilon_j\right) u^\varepsilon_i$ as a test function in $(DP_{\varepsilon,w})$ yields:

$$\kappa\left(t^\varepsilon_j\right) \int_\Omega \frac{b^\varepsilon_j - b^\varepsilon_{j-1}}{\varepsilon} u^\varepsilon_j + \left(a\left(x, Du^\varepsilon_j\right) + F\left(u^\varepsilon_j\right)\right) \cdot Du^\varepsilon_j + \psi\left(u^\varepsilon_j\right) u^\varepsilon_j = \int_\Omega f^\varepsilon_j \kappa\left(t^\varepsilon_j\right) u^\varepsilon_j \tag{29}$$

for all $j = 1, \ldots, , N(\varepsilon)$. If we define $\phi_{id} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi_{id}(r) = \begin{cases} \int_0^r \left(\beta^{-1}\right)^0(\sigma)d\sigma, & \text{if } r \in \overline{R(\beta)}, \\ +\infty & \text{otherwise,} \end{cases} \tag{30}$$

since $b^\varepsilon_j = \beta\left(u^\varepsilon_j\right)$ for all $j = 1, \ldots\ldots, N(\varepsilon)$ it follows that

$$\frac{b^\varepsilon_j - b^\varepsilon_{j-1}}{\varepsilon} u^\varepsilon_j \geq \frac{1}{\varepsilon} \int_{b^\varepsilon_{j-1}}^{b^\varepsilon_j} \left(\beta^{-1}\right)^0(\sigma)d\sigma = \frac{1}{\varepsilon}\left(\phi_{id}\left(b^\varepsilon_j\right) - \phi_{id}\left(b^\varepsilon_{j-1}\right)\right) \tag{31}$$

a.e. in Ω. Now, integration over $\left(t_{j-1}^{\varepsilon}, t_{j}^{\varepsilon}\right)$ in (29) and summation over $j = 1, \ldots, N(\varepsilon)$ yields:

$$\begin{aligned} &\sum_{j=1}^{N(\varepsilon)} \int_{\Omega} \left(\phi_{id}\left(b_j^{\varepsilon}\right) - \phi_{id}\left(b_{j-1}^{\varepsilon}\right)\right) \kappa\left(t_j^{\varepsilon}\right) \\ &+ \int_{Q_T} \kappa_{\varepsilon}\left(\left(a\left(x, Du_{\varepsilon}\right) + F\left(u_{\varepsilon}\right)\right) \cdot Du_{\varepsilon} + \psi\left(u_{\varepsilon}\right) u_{\varepsilon}\right) \leq \int_{Q_T} f_{\varepsilon} u_{\varepsilon} \kappa_{\varepsilon}, \end{aligned} \tag{32}$$

where $u_{\varepsilon} : (0,T) \to W_0^{1,\overrightarrow{p}}(\Omega)$ is a piecewise constant function defined by $u_{\varepsilon}(t) = u_j^{\varepsilon}$, for $t \in \left(t_{j-1}^{\varepsilon}, t_j^{\varepsilon}\right], j = 1, \ldots, N(\varepsilon)$ and $f_{\varepsilon} : (0,T) \to L^{\infty}(\Omega)$ is a piecewise constant function defined by $f_{\varepsilon}(t) = f_j^{\varepsilon}$ for $t \in \left(t_{j-1}^{\varepsilon}, t_j^{\varepsilon}\right], j = 1, \ldots, N(\varepsilon)$. Using summation by parts in the first term of (32) and setting $b_{\varepsilon}(t) = b_j^{\varepsilon}$ for $t \in \left(t_{j-1}^{\varepsilon}, t_j^{\varepsilon}\right], j = 1, \ldots, N(\varepsilon)$, $b_{\varepsilon}(t) = b_0^{\varepsilon}$ for $t \in (-\varepsilon, 0]$ it follows that

$$\begin{aligned} \int_{Q_T} \kappa_{\varepsilon} a\left(x, Du_{\varepsilon}\right) \cdot Du_{\varepsilon} \leq \int_{-\varepsilon}^{T-\varepsilon} \int_{\Omega} \kappa_t(t+\varepsilon) \phi_{id}\left(b_{\varepsilon}\right) + \int_{\Omega} \phi_{id}\left(b_0^{\varepsilon}\right) \kappa_{\varepsilon}(0) \\ - \int_{Q_T} \kappa_{\varepsilon}\left(F\left(u_{\varepsilon}\right) \cdot Du_{\varepsilon} + \left(\psi\left(u_{\varepsilon}\right) - f_{\varepsilon}\right) u_{\varepsilon}\right). \end{aligned} \tag{33}$$

Using the convergence results of Lemma 5, there is no problem to pass to the limit with $\varepsilon \downarrow 0$ on the right-hand side of (33). Moreover,

$$\begin{aligned} &\limsup_{\varepsilon \downarrow 0} \int_{Q_T} \kappa_{\varepsilon} a\left(x, Du_{\varepsilon}\right) \cdot Du_{\varepsilon} \\ &\geq \limsup_{\varepsilon \downarrow 0} \int_{Q_T} \kappa a\left(x, Du_{\varepsilon}\right) \cdot Du_{\varepsilon} + \liminf_{\varepsilon \downarrow 0} \int_{Q_T} \left(\kappa_{\varepsilon} - \kappa\right) a\left(x, Du_{\varepsilon}\right) \cdot Du_{\varepsilon}, \end{aligned} \tag{34}$$

where the second term on the right hand side of (34) is 0 by **(A2)**, (21) and since $\|\kappa_{\varepsilon}\ \kappa\|_{L^{\infty}(0,T)} \to 0$. Combining (33) with (34) and passing to the limit with $\varepsilon \downarrow 0$ we find

$$\begin{aligned} &\limsup_{\varepsilon \downarrow 0} \int_{Q_T} \kappa a\left(x, Du_{\varepsilon}\right) \cdot Du_{\varepsilon} \\ &\leq \int_{Q_T} \kappa_t(t) \phi_{id}(b) + \int_{\Omega} \phi_{id}\left(b_0\right) \kappa(0) - \int_{Q_T} \kappa(F(u) \cdot Du + \psi(u) u - f u). \end{aligned} \tag{35}$$

Since (28) holds, we can apply the integration-by-parts formula of Lemma 2 with $h(u) = u$ and $\xi = \kappa \chi_{\Omega}$ to obtain

$$\int_{Q_T} \kappa_t \int_{b_0}^{b(t,x)} \left(\beta^{-1}\right)^0(\sigma) d\sigma = \int_{Q_T} \kappa(F(u) \cdot Du + \Phi \cdot Du + \psi(u) u - f u) \tag{36}$$

for all $\kappa \in \mathcal{C}_c^\infty([0,T)), \kappa \geq 0$. Combining (35) and (36) we finally get

$$\limsup_{\varepsilon \downarrow 0} \int_{Q_T} \kappa a\left(x, Du_\varepsilon\right) \cdot Du_\varepsilon \leq \int_{Q_T} \kappa \Phi \cdot Du. \tag{37}$$

Therefore it follows that

$$\limsup_{\varepsilon \downarrow 0} \int_{Q_T} \kappa \left(a\left(x, Du_\varepsilon\right) - a(x, Du)\right) \cdot \left(Du_\varepsilon - Du\right) \leq 0 \tag{38}$$

for all $\kappa \in \mathcal{C}_c^\infty([0,T)), \kappa \geq 0$. Using **(A4)** and Minty's monotonicity argument we get $\Phi = a(x, Du)$ from (37)and (38). Moreover, choosing $\kappa = \chi_{(0,\tau)}$ for $0 < \tau < T$, from (38) we obtain $a\left(x, Du_\varepsilon\right) \cdot Du_\varepsilon \to a(x, Du) \cdot Du$ weak in $L^1((0,\tau) \times \Omega)$. ◄

5.3. The general case of multivalued β

Now, let $\beta \subset \mathbb{R} \times \mathbb{R}$ be a maximal monotone graph. To continue the proof of Theorem 2, we proceed as in [20] in the case of a constant exponent and combine the techniques developed in [7] with the approach from [3], so that we do not need the additional assumption that β^{-1} is continuous and defined on $\mathbb{R}$ if we accept one more approximation procedure.

For the first approximation procedure let us regularize β by $\beta_k := \beta + \frac{1}{k} I, k > 0$. Clearly, the results of Subsection 3.7 still apply to the nonlinear operator

$A_{\beta_k,\psi} := \{(b_k, w_k) \in L^1(\Omega) \times L^1(\Omega) : \exists u_k : \Omega \to \mathbb{R}.$ measurable, $b_k \in \beta(u_k)$ a.e in Ω, u_k is a renormalized solution of $-\operatorname{div}(a(x, Du_k) + F(u_k)) + \psi(u_k) = w_k\}$

and therefore there exists a unique mild solution $b^k \in \mathcal{C}\left([0,T]; L^1(\Omega)\right)$ of the abstract Cauchy problem

$$(ACP)_k\left(, \psi, f, b_0^k\right) \begin{cases} \dfrac{db^k}{dt} + A_{\beta_k,\psi} b^k \ni f \text{ in } (0,T) \\ b^k(0) = b_0^k, \end{cases}$$

corresponding to $\left(P_k, \psi, f, b_0^k\right)$ for any given $f \in L^1\left(Q_T\right), b_0^k \in \overline{D\left(A_{\beta_k,\psi}\right)}^{\|\cdot\|_{L^1(\Omega)}}$ and $k > 0$. Moreover, it follows from the results for the elliptic case (see Theorem 4.1 [1]) that for any $f \in L^1(\Omega)$

$$\lim_{k\to\infty} \left\| \left(I + A_{\beta_k,\psi}\right)^{-1} f - \left(I + A_{\beta,\psi}\right)^{-1} f \right\|_{L^1(\Omega)} = 0. \tag{39}$$

Applying the a priori estimates of Lemma 3 we get the following convergence results for the solutions of the discretized problems $\left(DP_{\varepsilon,\psi}^k\right)$:

Lemma 6. *For* $f \in L^{\infty}(Q_T), b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^{\infty}(\Omega)$ *and* $\varepsilon, k > 0$, *let* $\left(b_j^{\varepsilon,k}, u_j^{\varepsilon,k}\right)_{j=1}^{N(\varepsilon)}$ *be a solution of the discretized problem* $\left(DP_{\varepsilon,\psi}^k\right)$. *For* $k > 0$, *let* $b^k \in \mathcal{C}\left([0,T]; L^1(\Omega)\right)$ *be the* $L^{\infty}\left(0,T; L^1(\Omega)\right)$ *-limit of the sequence of piecewise constant functions* $\left(b_\varepsilon^k\right)_\varepsilon$ *defined by* $b_\varepsilon^k : (0,T) \to L^1(\Omega), b_\varepsilon^k(0) = b_0^{\varepsilon,k}, b_\varepsilon^k(t) = b_j^{\varepsilon,k}$ *for* $t \in \left(t_{j-1}^\varepsilon, t_j^\varepsilon\right]$ *and* $j = 1, \ldots, N(\varepsilon)$. *If we define* $u_\varepsilon^k : (0,T) \to W_0^{1,\overrightarrow{p}}(\Omega) \cap L^{\infty}(\Omega)$ *by* $u_\varepsilon^k(t) = u_j^{\varepsilon,k}$ *for* $t \in \left(t_{j-1}^\varepsilon, t_j^\varepsilon\right], j = 1, \ldots, N(\varepsilon)$, *then there exists* $u^k \in L^{\overrightarrow{p}}\left(0,T; W_0^{1,\overrightarrow{p}}(\Omega)\right) \cap L^{\infty}(Q_T)$ *and a subsequence of* $\left(u_\varepsilon^k\right)_\varepsilon$ *such that, as* $\varepsilon \downarrow 0$,

i) $u_\varepsilon^k \to u^k$ *almost everywhere in* Q_T, *weak* $-*$ *in* $L^{\infty}(Q_T)$ *and weak in* $L^{\overrightarrow{p}}\left(0,T, W_0^{1,\overrightarrow{p}}(\Omega)\right)$,

ii) $b_\varepsilon^k \to b^k$ *in* $L^{\infty}\left(0,T; L^1(\Omega)\right)$, *in* $L^1(Q_T)$ *and almost everywhere in* Q_T. *Moreover,* $b^k \in \beta_k\left(u^k\right)$ *almost everywhere in* Q_T,

iii) $Du_\varepsilon^k \rightharpoonup Du^k$ *in* $\prod_{i=1}^N L^{p_i}(Q_T)$,

iv) $a\left(x, Du_\varepsilon^k\right) \rightharpoonup a\left(x, Du^k\right)$ *in* $\prod_{i=1}^N L^{p_i'}(Q_T)$.

Proof. Using the a priori estimates (19), (20) and (21), it follows immediately that there exists $u^k \in L^{\overrightarrow{p}}\left(0,T; W_0^{1,\overrightarrow{p}}(\Omega)\right) \cap L^{\infty}(Q_T)$ such that, passing to a subsequence if necessary, iii) holds and $u_\varepsilon^k \rightharpoonup u^k$ weak-* in $L^{\infty}(Q_T)$ and weak in $L^{\overrightarrow{p}}\left(0,T; W_0^{1,\overrightarrow{p}}(\Omega)\right)$. The convergence of b_ε^k to b^k in $L^{\infty}\left(0,T; L^1(\Omega)\right)$ follows immediately from nonlinear semigroup theory and this implies the other convergence results for a subsequence of $\left(b_\varepsilon^k\right)_\varepsilon$. By $\left(DP_{\varepsilon,\psi}^k\right)$, we have $b_\varepsilon^k \in \beta_k\left(u_\varepsilon^k\right)$ almost everywhere in Q_T.

Since β is a maximal monotone graph, $\left(\beta + \frac{1}{k}I\right)^{-1} = k(k\beta + I)^{-1}$ is single-valued and Lipschitz continuous in $\mathbb{R}$, hence $u_\varepsilon^k := \left(\beta + \frac{1}{k}I\right)^{-1} b_\varepsilon^k$ converges to $u^k = \left(\beta + \frac{1}{k}I\right)^{-1} b^k$ almost everywhere in Q_T. Therefore i) and ii) hold. Finally, iv) follows with the same arguments as in the proof of Lemma 5 and Proposition 3. ◀

Using the convergence results of Lemma 6 we can prove the following result:

Proposition 4. *For any* $k > 0, f \in L^{\infty}(Q_T)$ *and* $b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^{\infty}(\Omega)$ *there exists a weak solution* $\left(u^k, b^k\right)$ *to* $\left(P_k, \psi, f, b_0^k\right)$. *In particular,* b^k *is the mild solution of* $(ACP)_k\left(\psi, f, b_0^k\right)$.

Proof. The assertion follows according to the convergence results of Lemma 6 and by similar arguments as in the proof of Proposition 3.

Next we want to obtain a weak solution (u,b) of (P,ψ,f,b_0) for $f \in L^\infty(Q_T)$ and $b_0 \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ by passing to the limit with $k \to \infty$ in the approximate equations (P_k,ψ,f,b_0). The convergence of the sequence $\left(b^k\right)_k$ is an immediate consequence of nonlinear semigroup theory. ◄

Lemma 7. *If b_0 is in $\overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$ such that there exists $\left(b_0^k\right)_k \subset L^1(\Omega)$ with $b_0^k \in \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$ for all $k>0$ and $b_0^k \to b_0$ in $L^1(\Omega)$ as $k\to\infty$, then b^k converges in $\mathcal{C}\left([0,T];L^1(\Omega)\right)$ to the mild solution b of $(ACP)(\psi,f,b_0)$ as $k\to\infty$.*

Proof. From (39) it follows that $A_{\beta,\psi} \subset \liminf_{k\to\infty} A_{\beta_k,\psi}$ and therefore the assertion follows according to nonlinear semigroup theory (see, e.g. [6]).

The following comparison principle is a corresponding result for multivalued β and was proved in [20] and [7]in the constant exponent case. ◄

Lemma 8. *For $f \in L^1(Q_T)$, $l,k>0$, $b_0^k, b_0^l \in L^1(\Omega)$ such that*

$$\lim_{k,l\to\infty} \left\| b_0^k - b_0^l \right\|_{L^1(\Omega)} = 0,$$

let $\left(u^k,b^k\right), \left(u^l,b^l\right)$ be the weak solutions of $\left(P_k,\psi,f,b_0^k\right), \left(P_l,\psi,f,b_0^l\right)$, respectively. Then,

$$\lim_{k,l\to\infty} \int_\tau^\theta \int_\Omega |\psi(u_k) - \psi(u_l)| = 0$$

holds for all $0<\tau<\theta<T$.

Proof. The proof of this lemma follows the same lines as the proof of the corresponding result in the case of a constant exponent p as stated in [7], Proposition 4.2.2., and (see also [20], Proposition 4.2) and is omitted here in detail. It is based on Kruzhkov's doubling of variable technique (see e.g. [17]) that has been adapted by other authors (see [7], [20]) to prove uniqueness results and comparison principles for elliptic-parabolic problems. In our particular case, we only have to double the time variables: let t,s denote two variables in $[0,T]$. We write the t variable in the weak formulation of $\left(P_k,\psi,f,b_0^k\right)$ and the s variable in the weak formulation of $\left(P_l,\psi,f,b_0^l\right)$. For $\delta>0$, and $r\in\mathbb{R}$ we define the function $r \to \eta_\delta(r)$ by $\eta_\delta(r) := \frac{1}{\delta}T_\delta(r)$. According to the integration-by-parts formula of Lemma 2 we choose $\eta_\delta(u_\delta(t,x) - u_l(s,x) + \delta\pi(x))\phi(t)\rho_n(t-s)$ as a test function in $\left(P_k,\psi,f,b_0^k\right)$ and $\left(P_l,\psi,f,b_0^l\right)$, where $\pi \in D(\Omega)$ such that $0 \le \pi \le 1, \phi \in D([0,T))$ such that $\phi \ge 0$ and $(\rho_n)_n$ is a sequence of mollifiers in

$\mathbb{R}$. There is no problem to pass to the limit with $\delta \downarrow 0$ in the diffusion and the convection term because F is assumed to be locally Lipschitz continuous. ◀

Using this result, we can prove the following:

Proposition 5. *For any $f \in L^\infty(Q_T)$ and $b_0 \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$, there exists $u \in L^{\overrightarrow{p}}\left(0,T;W_0^{1,\overrightarrow{p}}(\Omega)\right) \cap L^\infty(Q_T)$ and $b \in \mathcal{C}\left([0,T];L^1(\Omega)\right)$ such that (u,b) is a weak solution to (P,ψ,f,b_0). In particular, b is the mild solution of $(ACP)(\psi,f,b_0)$.*

Proof. According to Corollary 1 (iii), we have

$$\overline{D(A_{\beta,\Psi})}^{\|\cdot\|_{L^1(\Omega)}} \subset \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$$

for all $k > 0$. Therefore, if b_0 is in $\overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ it is in $\overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ for all $k > 0$ and by Proposition 4, (P_k,ψ,b_0,f) has a weak solution $\left(u^k,b^k\right) \subset \left(L^{\overrightarrow{p}}\left(0,T;W_0^{1,\overrightarrow{p}}(\Omega)\right) \cap L^\infty(Q_T)\right) \times \mathcal{C}\left([0,T];L^1(\Omega)\right)$ for all $k > 0$.
In particular,

$$\int_{Q_T} -\left(b^k - b_0\right)\xi_t + \left(a\left(x,Du^k\right) + F\left(u^k\right)\right)\cdot D\xi + \psi\left(u^k\right)\xi = \int_{Q_T} f\xi \quad (40)$$

holds for any $\xi \in D([0,T) \times \Omega)$. By Lemma 7, $b^k \to b$ as $k \to \infty$ in $\mathcal{C}\left([0,T];L^1(\Omega)\right)$, where $b \in \mathcal{C}\left([0,T];L^1(\Omega)\right)$ is the mild solution of $(ACP)(\psi,f,b_0)$ and therefore $b(0) = b_0$ almost everywhere in Ω. From Lemma 8 it follows with the same arguments as in the proof of Lemma 5 that there exists a measurable function $u : Q_T \to \mathbb{R}$ and a (not relabeled) subsequence of $\left(u^k\right)_k$ such that $u^k \to u$, almost everywhere as $k \to \infty$. Since the a priori estimates of Lemma 3 still hold for u^k, independently of $k > 0$, it follows that, as $k \to \infty$ and up to a non-relabeled subsequence, u^k converges to u weak-* in $L^\infty(Q_T)$ and weak in $L^{\overrightarrow{p}}\left(0,T;W_0^{1,\overrightarrow{p}}(\Omega)\right)$, $Du^k \rightharpoonup Du$ in $\prod_{i=1}^N L^{p_i}(Q_T)$, hence u is in $L^{\overrightarrow{p}}\left(0,T;W_0^{1,\overrightarrow{p}}(\Omega)\right) \cap L^\infty(Q_T)$. Moreover, there exists $\Phi \in \prod_{i=1}^N L^{p_i'}(Q_T)$ such that $a\left(x,Du^k\right) \rightharpoonup \Phi$ weak in $\prod_{i=1}^N L^{p_i'}(Q_T)$ as $k \to \infty$. Using these convergence results, we can pass to the limit in (40) and find that

$$-\int_{Q_T} (b-b_0)\,\xi_t + \int_{Q_T} (\Phi + F(u))\cdot D\xi + \int_{Q_T} \psi(u)\xi = \int_{Q_T} f\xi \tag{41}$$

for all $\xi \in D([0,T)\times\Omega\ )$. Next, we prove $a(x,Du)=\Phi$. To this end, we fix $\sigma \in D([0,T)), \sigma \geq 0$ and $l>0$. Since (40) holds, by Lemma (2) we can use $\sigma T_l\left(u^k\right)$ as a test function and obtain

$$\begin{aligned} &-\int_{Q_T} \sigma_t \int_{b_0}^{b^k} T_l \circ \left(\beta + \frac{1}{k} I\right)^{-1}(s)ds + \int_{Q_T} \sigma a\left(x, Du^k\right)\cdot DT_l\left(u^k\right) \\ &= -\int_{Q_T} \sigma\left(F\left(u^k\right)\cdot DT_l\left(u^k\right) + \left(\psi\left(u^k\right) - f\right) T_l\left(u^k\right)\right). \end{aligned} \tag{42}$$

There is no problem to pass to the limit with $k\to\infty$ on the right-hand side of (42). To pass to the limit in the first term on the left-hand side, we write

$$\int_{b_0}^{b^k} T_l \circ \left(\beta + \frac{1}{k} I\right)^{-1}(s)ds = I_1 + I_2, \tag{43}$$

where, since $b^k \in \left(\beta + \frac{1}{k} I\right) u^k$ almost everywhere in Q_T,

$$I_1 = \int_0^{b^k} T_l \circ \left(\beta + \frac{1}{k} I\right)^{-1}(s)ds = b^k T_l\left(u^k\right) - \int_0^{T_l\left(u^k\right)} \left(\beta^0 + \frac{1}{k} I\right)(s)ds \tag{44}$$

(see [19]]) almost everywhere in Q_T and

$$I_2 = -\int_0^{b_0} T_l \circ \left(\beta + \frac{1}{k} I\right)^{-1}(s)ds. \tag{45}$$

Now, setting $u_0 := \left(\beta^{-1}\right)^0(b_0)$ we have $\left(b_0 + \frac{1}{k}u_0\right) \in \left(\beta + \frac{1}{k} I\right) u_0$, hence

$$I_2 = -b_0 T_l(u_0) + \int_0^{T_l(u_0)} \left(\beta^0 + \frac{1}{k} I\right)(s)ds - \int_{b_0}^{b_0 + \frac{1}{k}u_0} T_l \circ \left(\beta + \frac{1}{k} I\right)^{-1}(s)ds \tag{46}$$

almost everywhere in Ω. Passing to the limit with $k\to\infty$ we find:

$$\lim_{k\to\infty} I_1 = -bT_l(u) + \int_0^{T_l(u)} \left(\beta^0\right)(s)ds = -\int_0^{b} T_l \circ \left(\beta^{-1}\right)^0(s)ds \tag{47}$$

almost everywhere in Q_T and

$$\lim_{k\to\infty} I_2 = -b_0 T_l(u_0) + \int_0^{T_l(u_0)} \left(\beta^0\right)(s)ds = -\int_0^{b_0} T_l \circ \left(\beta^{-1}\right)^0(s)ds \tag{48}$$

almost everywhere in Ω. Now, thanks to Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{k\to\infty} -\int_{Q_T} \sigma_t \int_{b_0^k}^{b^k} T_l \circ \left(\beta + \frac{1}{k} I\right)^{-1}(s)ds = -\int_{Q_T} \sigma_t \int_{b_0}^{b} T_l \circ \left(\beta^{-1}\right)^0 (s)ds, \quad (49)$$

and therefore

$$\limsup_{k\to\infty} \int_{Q_T} \sigma a\left(x, Du^k\right) \cdot DT_l\left(u^k\right) = \int_{Q_T} \sigma_t \int_{b_0}^{b} T_l \circ \left(\beta^{-1}\right)^0 (s)ds - \int_{Q_T} \sigma\left(F(u)\cdot DT_l(u) + (\psi(u) - f)T_l(u)\right). \quad (50)$$

Now we use $\sigma T_l(u)$ as a test function in (41). By Lemma 2 we get

$$\int_{Q_T} \sigma\Phi \cdot DT_l(u) = \int_{Q_T} \sigma_t \int_{b_0}^{b} T_l \circ \left(\beta^{-1}\right)^0 (s)ds - \int_{Q_T} \sigma\left(F(u)\cdot DT_l(u) + (\psi(u) - f)T_l(u)\right). \quad (51)$$

Subtracting (51) from (50) and choosing $l = \|u\|_{L^\infty(Q_T)}$ it follows that

$$\limsup_{k\to\infty} \int_{Q_T} \sigma a\left(x, Du^k\right) \cdot Du^k \le \int_{Q_T} \sigma\Phi \cdot Du \quad (52)$$

for all $\sigma \in D([0,T)), \sigma \ge 0$. Furthermore, using (52) we have

$$\lim_{k\to\infty} \int_{Q_T} \sigma\left(a\left(x, Du^k\right) - a(x, Du)\right) \cdot \left(Du^k - Du\right) = 0 \quad (53)$$

for all $\sigma \in D([0,T)), \sigma \ge 0$. Now, $\Phi = a(x, Du)$ follows from (53) by the Minty monotonicity argument. It is left to prove that $b \in \beta(u)$ almost everywhere in Q_T. Since we have $b^k \in \beta_k\left(u^k\right)$ almost everywhere in Q_T, for any $k > 0$ there exists $B^k \in \beta\left(u^k\right)$ such that $b^k = B^k + \frac{1}{k}u^k$ and since $B^k \to b$ for $k \to \infty$ almost everywhere in Q_T. If we define $j : \mathbb{R} \to \mathbb{R} \sqcup \{+\infty\}$ by $j(r) = \int_0^r \beta^0(\sigma)d\sigma$ if $r \in \overline{D(\beta)}$ and $j(r) = +\infty$ otherwise, it is easy to see that j is a convex, l.s.c, proper function such that $\beta = \partial j$. Therefore,

$$j(r) \ge j\left(u^k\right) + B^k\left(r - u^k\right) \quad (54)$$

holds for any $r \in \mathbb{R}$ and almost everywhere in Q_T. Now, by the almost everywhere convergence of B^k to b and u^k to u, from (54) it follows that $b \in \beta(u)$ almost everywhere in Q_T. ◄

5.4. L^1 -contraction and uniqueness of renormalized solutions

Proposition 6. *For $f_1, f_2 \in L^1(Q_T), b_0^1, b_0^2 \in L^1(\Omega)$, let $(u_1, b_1), (u_2, b_2)$ be renormalized solutions of $(P, f_1, b_0^1), (P, f_2, b_0^2)$ respectively. Then*

$$\int_\Omega (b_1(t) - b_2(t))^+ \leq \int_0^t \int_\Omega (f_1 - f_2)^+ + \int_\Omega (b_0^1 - b_0^2)^+ \tag{55}$$

holds for almost all $t \in (0, T)$.

Proof. We can copy the proof in [7], Theorem 4.1 for the case of a constant exponent with slight modifications such as exchanging the space $L^p\left(0, T; W_0^{1,p(.)}(\Omega)\right)$ by $L^{\overrightarrow{p}}\left(0, T; W_0^{1,\overrightarrow{p}}(\Omega)\right)$. ◄

Remark 3. *The result of Proposition 6 still holds if we replace (P, f_i, b_0^i) by (P, ψ_i, f_i, b_0^i), where $\psi_i : \mathbb{R} \to \mathbb{R}$ is a continuous, non-decreasing function for $i = 1, 2$.*

Remark 4. *Uniqueness of renormalized solutions is a direct consequence of Proposition 6: If (u, b) is a renormalized solution to (P, f, b_0) for $f \in L^1(\Omega)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$, then b is unique. We cannot expect uniqueness of the function u without additional assumptions on β.*

5.5. A comparison principle and weak solutions for L^1 -data

Lemma 9. *Let $b_0, \tilde{b}_0$ be in $L^1(\Omega) f, \tilde{f} \in L^1(Q_T), \psi, \tilde{\psi} : \mathbb{R} \to \mathbb{R}$ be strictly increasing, continuous functions and $(u, b), (\tilde{u}, \tilde{b})$ be weak solutions to (P, ψ, f, b_0) and $\left(P, \tilde{\psi}, \tilde{f}, \tilde{b}_0\right)$, respectively. If we have $b_0 \leq \tilde{b}_0$ almost everywhere in $\Omega, f \leq \tilde{f}$ almost everywhere in Q_T and $\tilde{\psi}(r) \leq \psi(r)$ for all $r \in \mathbb{R}$, then $u \leq \tilde{u}$ holds almost everywhere in Q_T.*

Proof. As in the proof of the corresponding result in the case of a constant exponent ([20], Lemma 4.3.1., p. 120 and [7], Proposition 4.2.).

5.6. Conclusion of the Proof of Theorem 1

For $n \in \mathbb{N}$, we define the continuous, strictly increasing function $\psi_n : \mathbb{R} \to \mathbb{R}$ by

$$\psi_n(r) := \frac{1}{n}\left(\arctan(r) + \frac{\pi}{2}\right), r \in \mathbb{R}.$$

Then, by Proposition 5, there exists a weak solution $(u_n, b_n) \in \left(L^{\overrightarrow{p}}\left(0,T;W_0^{1,\overrightarrow{p}}(\Omega)\right) \cap L^{\infty}(Q_T)\right) \times \mathcal{C}\left([0,T];L^1(Q_T)\right)$ to (P,ψ_n,f,b_0) for any $n \in \mathbb{N}$. Since $A_\beta \subset \liminf_{n\to\infty} A_{\beta,\psi_n}$ and b_n is the mild solution of $(ACP)(\psi_n,f,b_0)$, it follows that b_n con- verges in $\mathcal{C}\left([0,T];L^1(\Omega)\right)$ to the mild solution b of $(ACP)(f,b_0)$ as $n\to\infty$. Since $\psi_n \geq \psi_{n+1}$ in $\mathbb{R}$ for all $n\in\mathbb{N}$, from Lemma 9 it follows that $u_n \leq u_{n+1}$ almost everywhere in Q_T for all $n\in\mathbb{N}$, hence there exists a measurable function $u : Q_T \to \overline{\mathbb{R}}$ such that $u_n \uparrow u$ almost everywhere in Q_T. Moreover, $\arctan(r) - \frac{\pi}{2} \leq \psi_n \leq \arctan(r) + \frac{\pi}{2}$ for all $r\in\mathbb{R}$ and all $n\in\mathbb{N}$ and from Lemma 9 it follows that $u_{\frac{\pi}{2}} \leq u_n \leq u_{-\frac{\pi}{2}}$ almost everywhere in Q_T for all $n\in\mathbb{N}$ where $\left(u_{\frac{\pi}{2}}, b_{\frac{\pi}{2}}\right), \left(u_{-\frac{\pi}{2}}, b_{-\frac{\pi}{2}}\right) \in \left(L^{\overrightarrow{p}}\left(0,T;W_0^{1,\overrightarrow{p}}(\Omega)\right)\cap L^\infty(Q_T)\right)\times \mathcal{C}\left([0,T],L^1(\Omega)\right)$ are the weak solutions to $\left(P, \arctan+\frac{\pi}{2}, f, b_0\right)$ and $\left(P, \arctan-\frac{\pi}{2}, f, b_0\right)$ respectively. Therefore $u\in L^\infty(Q_T)$ and $u_n \to u$ weak-* in $L^\infty(Q_T)$ for a not relabeled subsequence of $(u_n)_n$. For $\delta>0$ we define $\phi_\delta : [0,T]\to\mathbb{R}$ by $\phi_\delta(t) := \min\left(\frac{1}{\delta}\max(T-\delta-t,0),1\right)$. Thanks to the integration-by-parts formula of Lemma 2, can use $\phi_\delta T_k(u_n)$ as a test function and obtain for $k = \|u\|_{L^\infty(QT)}$:

$$\frac{1}{\delta}\int_{T-2\delta}^{T+2\delta}\int_\Omega\int_0^{b_n(t,x)} T_{\|u\|_{L^\infty(Q_T)}}\circ\left(\beta^{-1}\right)^0(\sigma)d\sigma - \int_\Omega\int_0^{b_0} T_{\|u\|_{L^\infty(Q_T)}}\circ\left(\beta^{-1}\right)^0(\sigma)d\sigma$$
$$+\int_{Q_T}\phi_\delta\left(\left((a(x,Du_n)+F(u_n))\cdot Du_n\right)+\frac{1}{n}\arctan(u_n)\cdot u_n\right) = \int_{Q_T}\phi_\delta u_n\left(f-\frac{\pi}{2}\right). \tag{56}$$

We neglect positive terms and use **(A2)**, pass to the limit with $\delta\downarrow 0$ and obtain

$$\int_{Q_T}\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i dxdt \leq C\|u\|_{L^\infty(Q_T)}\left(\left\||f|+\frac{\pi}{2}\right\|_{L^1(Q_T)}+\|b_0\|_{L^1(\Omega)}\right), \tag{57}$$

where $C>0$ is a positive constant not depending on $n\in N$. From (57) we get $u\in L^{\overrightarrow{p}}\left(0,T;W_0^{1,\overrightarrow{p}}(\Omega)\right)$ and there exists a (not relabeled) subsequence of $(u_n)_n$ such that $Du_n \rightharpoonup Du$ weak in $\prod_{i=1}^N L^{p_i}(Q_T)$ and $a(x,Du_n)\rightharpoonup\Phi$ weak in $\prod_{i=1}^N L^{p_i'}(Q_T)$ for a function $\Phi\in\prod_{i=1}^N L^{p'}(Q_T)$. Now we can pass to the limit with $n\to\infty$ in the weak formulation for (P,f,ψ_n,b_0) and obtain

$$-\int_{Q_T}(b-b_0)\xi_t + (\Phi+F(u))\cdot D\xi = \int_{Q_T} f\cdot\xi \tag{58}$$

for all $\xi\in D([0,T)\times\Omega)$. With the same arguments as in the proof of Proposition 5 it follows that $\Phi = a(x,Du)$ (by Minty monotonicity argument) and

$b \in \beta(u)$ (by a subdifferential argument). To prove theorem 2, we will use several approximation procedures.

6. Proof of Theorem 2

As in the proof of Theorem 1 for the elliptic case (see [1]), we will construct monotone sequences of weak solutions for L^∞ -data and show convergence (up to a subsequence) to a renormalized solution. The comparison principles from Lemma 9 and proposition 6 will be a main tool in this approximation procedure.

Step 1. Approximate solutions and a priori estimate.

Let f be in $L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$. For $m,n \in \mathbb{N}$, let $f_{m,n} := \max(\min(f,m),-n), b_0^{m,n} := \max(\min(b_0,m),-n)$. Furthermore, we define $\psi_{m,n} : \mathbb{R} \to \mathbb{R}$ by

$$\psi_{m,n}(r) := \frac{1}{m}\max(r,0) - \frac{1}{n}\max(-r,0) \quad \text{for } r \in \mathbb{R}.$$

By Proposition 5, $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$ has a weak solution $(u_{m,n}, b_{m,n})$ for all $m,n \in \mathbb{N}$. We have $\psi_{m,n} \geq \psi_{m+1,n}$ for all $n \in \mathbb{N}$ and $\psi_{m,n} \leq \psi_{m,n+1}$ on $\mathbb{R}$. By lemma 9 it follows that $u_{m,n} \leq u_{m+1,n}$ and $u_{m,n+1} \leq u_{m,n}$ almost everywhere in Q_T for any $m,n \in \mathbb{N}$ Hence, there exist measurable functions $u^n : Q_T \to \mathbb{R} \cup \{+\infty\}, u : Q_T \to \mathbb{R}$ such that $u_{m,n} \uparrow u^n$ as $m \to \infty$ and $u^n \downarrow u$ as $n \to \infty$ almost everywhere in Q_T. By Lemma 9, it follows that $b_{m,n} \leq b_{m+1,n}$ and $b_{m,n+1} \leq b_{m,n}$ almost everywhere in Q_T for any $n,m \in \mathbb{N}$. Note that we have also $\psi_{m,n}(r) \downarrow \psi^n(r) := -\frac{1}{n}\max(-r,0)$ as $m \to \infty$ and $\psi^n(r) \uparrow 0$ as $n \to \infty$ for all $r \in \mathbb{R}$. Therefore, $A_{\psi_n,\beta} \subset \liminf_{m\to 0} A_{\psi_{m,n},\beta}$ and $A_\beta \subset \liminf_{n\to 0} A_{\psi_n,\beta}$. By nonlinear semigroupe theory it follows that $b_{m,n} \uparrow b^n$ as $m \to \infty$ in $\mathcal{C}([0,T]; L^1(\Omega))$, where b^n is the mild solution of (ACP) (ψ^n, f^n, b_0^n) with $\psi^n(r) := -\frac{1}{n}\max(-r,0), f^n := \max(f,-n)$, and $b_0^n := \max(b_0,-n)$ for $n \in \mathbb{N}$. Moreover, $b^n \downarrow b$ as $n \to \infty$ in $\mathcal{C}([0,T]; L^1(\Omega))$ where b is the mild solution of (ACP) (f, b_0). In the next steps, we will prove that (u,b) is a renormalized solution to (P, f, b_0). Therefore we need the following a priori estimate:

Lemma 10. *For $m,n \in \mathbb{N}, f \in L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$ let $(u_{m,n}, b_{m,n})$ be the weak solution to $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$. Then there exists a constant $C > 0$ not depending on $m,n \in \mathbb{N}$ such that*

$$\int_0^T \int_\Omega \sum_{i=1}^N \left| \frac{\partial T_k(u_{m,n})}{\partial x_i} \right|^{p_i} dxdt \leq Ck\left(\|f\|_{L^1(Q_T)} + \|b_0\|_{L^1(\Omega)}\right) \tag{59}$$

holds for any $k > 0$ and all $m,n \in \mathbb{N}$.

Proof. We fix $k>0$. For $\delta>0$ we define $\phi_\delta:[0,T]\to\mathbb{R}$ by $\phi_\delta(t)=\min\left(\frac{1}{\delta}\max(T-\delta-t,0),1\right)$. Using the integration-by-parts formula of Lemma 2 and density arguments, we plug $\phi_\delta T_k(u_{m,n})$ as a test function into $((P,\psi_{m,n},f_{m,n},b_0^{m,n}))$. Then, for $\delta>0$ small enough, we find

$$I_1+I_2+I_3+I_4=I_5, \tag{60}$$

where

$$\begin{aligned} I_1&=-\int_{T-2\delta}^{T-\delta}\int_\Omega(\phi_\delta)_t\int_{b_0^{m,n}}^{b_{m,n}(t,x)}T_k\circ\left(\beta^{-1}\right)^0(\sigma)d\sigma\\ &=\frac{1}{\delta}\int_{T-2\delta}^{T-\delta}\int_\Omega\int_0^{b_{m,n}(t,x)}T_k\circ\left(\beta^{-1}\right)^0(\sigma)d\sigma-\int_\Omega\int_0^{b_0^{m,n}}T_k\circ\left(\beta^{-1}\right)^0(\sigma)d\sigma. \end{aligned} \tag{61}$$

By **(A2)** we get

$$I_2=\int_{Q_T}\phi_\delta a\left(x,DT_k\left(u_{m,n}\right)\right)\cdot DT_k\left(u_{m,n}\right)\geq\lambda\int_0^{T-2\delta}\int_\Omega\sum_{i=1}^N\left|\frac{\partial T_k\left(u_{m,n}\right)}{\partial x_i}\right|^{p_i}dxdt. \tag{62}$$

Note that applying Gauss-Green Theorem and the boundary condition

$$\int_\Omega F\left(T_k\left(u_{m,n}(t)\right)\right)\cdot DT_k\left(u_{m,n}(t)\right)=0$$

for almost all $t\in(0,T)$ we have

$$I_3=\int_0^T\phi_\delta\int_\Omega F\left(T_k\left(u_{m,n}\right)\right)\cdot DT_k\left(u_{m,n}\right)=0. \tag{63}$$

Moreover, by monotonicity of $\psi_{m,n}$ we have

$$\begin{aligned} I_4&=\int_{Q_T}\psi_{m,n}\left(u_{m,n}\right)T_k\left(u_{m,n}\right)\phi_\delta\geq 0,\\ I_5&=\int_{Q_T}f_{m,n}T_k\left(u_{m,n}\right)\phi_\delta\leq\|f\|_{L^1(Q_T)}k. \end{aligned} \tag{64}$$

Now, plugging (61)-(64) into (60) and neglecting non-negative terms we arrive, at

$$\begin{aligned} \lambda\int_0^{T-2\delta}\int_\Omega\sum_{i=1}^N\left|\frac{\partial T_k\left(u_{m,n}\right)}{\partial x_i}\right|^{p_i}&\leq k\|f\|_{L^1(Q_T)}+\int_\Omega\int_0^{b_0^{m,n}}T_k\circ\left(\beta^{-1}\right)^0(\sigma)d\sigma\\ &\leq k\left(\|f\|_{L^1(Q_T)}+\|b_0\|_{L^1(\Omega)}\right) \end{aligned} \tag{65}$$

for all $k>$ and all $m,n\in\mathbb{N}$. For $\delta\downarrow 0$, the assertion follows. ◄

Remark 5. *There exists a constant $C>0$, not depending on $n, l \in \mathbb{R}$ such that*

$$|\{|u_{m,n}| \geq l\}| \leq C l^{\left(\frac{1}{p}-1\right)}, \tag{66}$$

and from (66) *it follows that*

$$\lim_{l \to \infty} |\{|u| \geq l\}| = 0. \tag{67}$$

Proof. Let $k>0$ large enough

$$\begin{aligned} k\left|\{|u_{m,n}|>k\} \times [0,T]\right| &= \int_0^T \int_{\{|u_{m,n}|>k\}} \mid T_k\left(u_{m,n}\right) \mid dxdt \\ &\leq \int_0^T \int_\Omega \mid T_k\left(u_{m,n}\right|^{p_i} dxdt)^{\frac{1}{p_i}}. \\ &\leq TC\left(\int_Q \sum_{i=1}^N \left|\frac{\partial T_k\left(u_{m,n}\right)}{\partial x_i}\right|^{p_i}\right)^{\frac{1}{p_i}} \leq C k^{\frac{1}{p}}, \end{aligned}$$

which implies that

$$|\{|u_{m,n}|>k\} \times [0,T]| \leq C k^{\frac{1}{p}-1}, \forall k \geq 1.$$

So, we have

$$\lim_{k \to +\infty} |\{|u_{m,n}|>k\} \times [0,T]| = 0.$$

Since $b_{m,n} \in \beta\left(u_{m,n}\right)$ almost everywhere in Q_T, it follows with subdifferential arguments as in the proof of Proposition 5 that $b^n \in \beta\left(u^n\right)$ and $b \in \beta(u)$ almost everywhere in Q_T. ◀

Step 2. Convergence results.

Now applying the diagonal principle and lemma we get the following convergence results:

Lemma 11. *For $m, n \in \mathbb{N}, f \in L^1\left(Q_T\right)$ and $b_0 \in \overline{D\left(A_\beta\right)}^{\|\cdot\|_{L^1(\Omega)}}$ let $(u_{m,n}, b_{m,n})$ be the weak solution to $\left(P, \psi_{m,n}, f_{m,n}, b_0^{m,n}\right)$. Then, there exists a subsequence $(m(n))_n$ such that setting $\psi^n := \psi_{m(n),n}, f_n := f_{m(n),n}, b_{0,n} := b_0^{m(n),n}, b_n := b_{m(n),n}, u_n := u_{m(n),n}$ we have the following convergence results for $n \to \infty$:*

i) $f_n \to f$ in $L^1\left(Q_T\right)$,

ii) $u_n \to u$ almost everywhere in Q_T,

iii) $b_{0,n} \to b_0$ in $L^1(\Omega), b_n \to b$ in $\mathcal{C}\left([0,T], L^1(\Omega)\right)$, and $b \in \beta(u)$ almost everywhere in Q_T, and the uniform renormalized condition

$$\limsup_{l \to \infty} \int_{\{l \leq |u_n| \leq l+1\}} a\left(x, Du_n\right) \cdot Du_n = 0 \tag{68}$$

holds true. Moreover, for any $k > 0$, we have

iv) $T_k(u_n) \rightharpoonup T_k(u)$ *in* $L^{\overrightarrow{p}}\left(0, T, W_0^{1,\overrightarrow{p}}(\Omega)\right)$,

v) $DT_k(u_n) \rightharpoonup DT_k(u)$ *in* $\prod_{i=1}^{N} L^{p_i}(Q_T)$,

vi) $a(x, DT_k(u_n)) \rightharpoonup a(x, DT_k(u))$ *in* $\prod_{i=1}^{N} L^{p_i'}(Q_T)$,

vii) $a(x, DT_k(u_n)) \cdot DT_k(u_n) \rightharpoonup a(x, DT_k(u)) \cdot DT_k(u)$ *weak in* $L^1((0,\tau)\times\Omega)$ *for any* $0 < \tau < T$.

Proof. $i) - v)$ are direct consequences of the approximation procedure, Lemma 10 and Remark 5. To prove the uniform renormalized condition, we take $T_k(u_n)\phi_\delta, \phi_\delta(t) = \min\left(\frac{1}{\delta}\max(T-\delta-t, 0), 1\right)$ as a test function and apply Lemma 2 in the weak formulation for (P, ψ^n, f_n, b_0^n). By Gauss-Green Theorem for Sobolev function and the boundary condition, we have

$$\int_\Omega F(T_k(u_n(t))) \cdot DT_k(u_n(t)) = 0$$

for almost all $t \in (0,T)$, Hence the convection terms vanishes. Then we set $k = l+1$ and after $k = l$. Subtracting the corresponding equalities and neglecting positive terms we obtain

$$\int_{\{l<|u_n|<l+1\}} \phi_\delta a(x, Du_n) \cdot Du_n \le \int_{\{|u_n|>l\}} |f| + \int_\Omega \int_0^{|b_0|} G_l\left(\left(\beta^{-1}\right)^0\right)(s)ds, \quad (69)$$

where $G_l = T_{l+1} - T_l$ and the uniform renormalized condition follows applying (66) in (69). It is left to show that vi) and vii) hold. From Lemma 10 and **(A3)** it follows that for any $k > 0$ there exists $\Phi_k \in \prod_{i=1}^{N} L^{p_i'}(Q_T)$ such that $a(x, DT_k(u_n)) \to \Phi_k$ weak in $\prod_{i=1}^{N} L^{p_i'}(Q_T)$ as $n \to \infty$. To prove that $\Phi_k = a(x, DT_k(u))$, we proceed as in as in [20] for the case of a constant exponent (see also [5]) for variable exponent and [7], Theorem 3.6 [8] for constant exponent. Now, for $\kappa \in D^+([0,T))$ we show that

$$\limsup_{\mu\to\infty} \limsup_{n\to\infty} \int_{Q_T} \kappa a(x, DT_k(u_n)) \cdot \left(T_k(u_n) - (T_k(u))_\mu\right) \le 0. \quad (70)$$

We use the sequence $(T_k(u))_\mu$ of approximations of $T_k(u)$, and plug the test function $\kappa h_l(u_n)\left(T_k(u_n) - (T_k(u))_\mu\right)$ (for $\mu > 0$) in (P, ψ^n, f_n, b_0^n) and obtain

$$I_1 + I_2 + I_3 + I_4 + I_5 = I_6, \quad (71)$$

where

$$I_1 = \left\langle (b_n - b_0^n)_t \, , \kappa h_l\,(u_n) \left(T_k\,(u_n) - (T_k(u))_\mu \right) \right\rangle \tag{72}$$

and $\langle .,. \rangle$ denotes the duality pairing between $L^{p'}\left(0,T,W^{-1,\overrightarrow{p}'}(Q_T)\right) + L^1(Q_T)$ and $L^{\overrightarrow{p}}\left(0,T,W_0^{1,\overrightarrow{p}}(\Omega)\right) \cap L^\infty(Q_T)$,

$$I_2 = \int_{Q_T} \kappa h_l\,(u_n)\, a\,(x, Du_n) \cdot D\left(T_k\,(u_n) - (T_k(u))_\mu\right),$$

$$I_3 = \int_{Q_T} \kappa h_l'\,(u_n) \left(T_k\,(u_n) - (T_k(u))_\mu\right) a\,(x, Du_n) \cdot Du_n,$$

$$I_4 = \int_{Q_T} \kappa F\,(u_n) \cdot D\left(h_l\,(u_n)\left(T_k\,(u_n) - (T_k(u))_\mu\right)\right),$$

$$I_5 = \int_{Q_T} \kappa h_l\,(u_n)\, \psi^n\,(u_n) \left(T_k\,(u_n) - (T_k(u))_\mu\right),$$

$$I_6 = \int_{Q_T} \kappa f_n h_l\,(u_n) \left(T_k\,(u_n) - (T_k(u))_\mu\right).$$

Now we want to pass to the limit with $n \to \infty$ and then with $\mu \to \infty$. To handle I_2, we choose $l > k$ and apply the uniform renormalized condition. It is easy to see that I_6, I_5 and I_4 tend to 0 as $n \to \infty, \mu \to \infty$. Using the uniform renormalized condition (68), it follows that $I_3 \leq w(l,k)$ and $w(l,k) \to 0$ as $l \to \infty$. To handle the parabolic term I_1, we need the following:

Lemma 12.

$$\liminf_{n\to\infty} \liminf_{\mu\to\infty} \left\langle f\,(b_n - b_0^n)_t \, , \kappa h_l\,(u_n)\left(T_k\,(u_n) - (T_k(u))_\mu\right)\right\rangle \geq 0. \tag{73}$$

Proof. The proof is similar to the proof of the corresponding result in the case of a constant exponent (see [5]) for the case of a variable exponent and [7], [8], for the case of a constant exponent). ◄

If (70) holds, then by **(A4)** it follows that

$$\limsup_{n\to\infty} \int_{Q_T} \kappa a\,(x, DT_k\,(u_n)) \cdot D\,(T_k\,(u_n) - (T_k(u))) \leq 0, \tag{74}$$

and vi) follows from (74) by the standard Minty-Browder argument. vii) follows from (74) by choosing κ to be a smooth approximation of $\chi_{(0,T)}$ for $0 < \tau < T$.

Step 3. Conclusion of the proof of Theorem 2.

Now, we are able to conclude the proof of Theorem 2: By Remark 5 and Lemma 11 it follows immediately that $(P1), (P2)$ and $(P3)$ hold for all $k > 0$.

For $h \in C_c^1(\mathbb{R})$ and $\xi \in D([0,T) \times \Omega\ )$ we can plug $h(u_n)\xi$ into (P, ψ^n, f_n, b_0^n) by integration-by parts formula of Lemma 2 and obtain

$$I_1 + I_2 + I_3 + I_4 = I_5, \tag{75}$$

where

$$I_1 = \int_{Q_T} \xi_t \int_{b_0^n}^{b_n} h \circ \left(\beta^{-1}\right)^0 (s) ds, I_2 = \int_{Q_T} a(x, Du_n) \cdot D(h(u_n)\xi),$$

$$I_3 = \int_{Q_T} F(u_n) \cdot D(h(u_n)\xi), I_4 = \int_{Q_T} \psi^n(u_n) h(u_n)\xi, I_5 = \int_{Q_T} f_n h(u_n)\xi.$$

Thanks to the convergence results of Lemma 11, we can pass to limit with $n \to \infty$ in $I_1, \ldots, I_5$: It follows immediately that

$$\lim_{n\to\infty} I_1 = \int_{Q_T} \xi_t \int_{b_0}^{b} h \circ \left(\beta^{-1}\right)^0 (s) ds. \tag{76}$$

Now we choose $m > 0$ such that supp $h \subset [-m, m]$. Next, we write

$$I_2 = I_{2,1} + I_{2,2}, \tag{77}$$

where

$$I_{2,1} = \int_{(0,\tau)\times\Omega} h'(T_m(u_n))\xi a(x, DT_m(u_n)) \cdot DT_m(u_n),$$

for $0 < \tau < T$ is such that $\operatorname{supp}\xi \subset [0,T) \times \Omega\ )$. By Lemma 11, vii), $a(x, DT_m(u_n))$. $DT_m(u_n) \rightharpoonup a(x, DT_m(u)) \cdot DT_m(u)$ weak in $L^1([0,T) \times \Omega)$. Since $h'(u_n)\xi \to h'(u)\xi$ almost everywhere in Q_T and $\|h(u_n)\xi\|_{L^\infty((0,T)\times\Omega)} \leq \|h\|_{L^\infty(Q_T)}\|\xi\|_{L^\infty(Q_T)}$, we may pass to the limit in $I_{2,1}$ and obtain

$$\lim_{n\to\infty} I_{2,1} = \int_{Q_T} h'(u)\xi a(x, Du) \cdot Du. \tag{78}$$

By Lebesgue Dominated Convergence Theorem it follows that $h(T_m(u_n)) \to h(T_m(u))$ as $n \to \infty$ in $L^{p_i}(Q_T, \sigma)$. Using Lemma 11 vi), we can pass to the limit in

$$I_{2,2} = \int_{Q_T} h(T_m(u_n))\, a(x, DT_m(u_n)) \cdot D\xi \tag{79}$$

and find

$$\lim_{n\to\infty} I_{2,2} = \int_{Q_T} h(u) a(x, Du) \cdot D\xi.$$

Next we write $I_3 = I_{3,1} + I_{3,2}$, where $I_{3,1} = \int_{Q_T} h'(T_m(u_n))\xi F(T_m(u_n)) \cdot DT_m(u_n)$, $I_{3,2} = \int_{Q_T} h(T_m(u_n)) F(T_m(u_n)) \cdot D\xi$ Since $h'(T_m(u_n)) F(T_m(u_n))$

converge to $h'(T_m(u))\,F(T_m(u))$ in $\prod_{i=1}^{N} L^{p_i'}(Q_T)$ as $n \to \infty$ using Lemma 11 v) we have

$$\lim_{n\to\infty} I_{3,1} = \int_{Q_T} h'(u)\xi F(u)\cdot Du,$$

and moreover

$$\lim_{n\to\infty} I_{3,2} = \int_{Q_T} h(u)F(u)\cdot D\xi. \tag{80}$$

Note that

$$|I_4| \le \frac{m}{n}\|h\|_{L^\infty(Q_T)}\|\xi\|_{L^\infty(Q_T)} \to 0,$$

for $n \to \infty$. Finally, we have

$$\lim_{n\to\infty} I_5 = \int_{Q_T} f h(u)\xi, \tag{81}$$

and from (75)-(81) it follows that (u, b) satisfies the renormalized formulation $(P4)$. $(P5)$ follows from the uniform renormalized condition (68) and Lemma (11), vii).

7. Conclusion

In this paper, we proved the existence of solutions for some general nonlinear parabolic problems, corresponding to Stefan problems which arise in presence of phase transitions. The novelty of this study is that we extend the results of [23] to anisotropic Sobolev spaces $W^{1,\overrightarrow{p}}(\Omega)$, when the components of vector $\overrightarrow{p} = (p_1, ..., p_N)$ are able to vary. We give sufficient conditions ensuring that the problem (P, f, b_0) admits renormalized solutions.

8. Statements and declarations

8.1. Competing interests.

All authors declare that they have no conflicts of interest.

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