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Stability of Radical Cubic Functional Inequality in Modular Spaces and Fuzzy Banach Space

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Abstract. The aim of this paper is to investigate the Hyers-Ulam stability of radical cubic functional inequality in modular space with Δ_2-condition and in fuzzy Banach space.

Key Words and Phrases: Hyers-Ulam stability, radical cubic functional inequality, modular spaces, fuzzy Banach space, Δ_2-condition.

2010 Mathematics Subject Classifications: 39B82, 39B52

1. Introduction and preliminaries

Nakano established the theory of modulars on linear spaces and the related theory of modular linear spaces in 1950 [17]. After a while, many mathematicians have worked hard to develop this theory, for example, Amemiya [1], Koshi and Shimogaki [11], Yamamuro [27], Orlicz [18], Mazur [14], Musielak [16], Luxemburg [13], and Turpin [23]. The study of interpolation theory [12] and various Orlicz spaces [18] has up till now made extensive use of the notion of modular spaces.

Firstly, we introduce the standard terminologies, notations, definitions and characteristics of the theory of modular spaces.

Definition 1 ([17]). *Let Y be an arbitrary vector space. A functional $\rho : Y \to [0, \infty)$ is called a modular if for arbitrary $x, y \in Y$;*

1. $\rho(x) = 0$ *if and only if* $x = 0$.
2. $\rho(\alpha x) = \rho(x)$ *for every scalar* α *with* $|\alpha| = 1$.
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *if and only if* $\alpha + \beta = 1$ *and* $\alpha, \beta \geq 0$.
   If (3) *is replaced by:*

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4. *$\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y)$ if and only if $\alpha+\beta=1$ and $\alpha,\beta \geq 0$, then we say that ρ is a convex modular.*

5. *It is said that the modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever $x = \rho - \lim_{n\to\infty} x_n$.*

A modular ρ defines a corresponding modular space, i.e., the vector space Y_ρ given by:

$$Y_\rho = \{x \in Y : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

A function modular is said to satisfy the Δ_2-condition if there exists $\tau > 0$ such that $\rho(2x) \leq \tau\rho(x)$ for all $x \in Y_\rho$.

Definition 2. *Let $\{x_n\}$ and x be in Y_ρ. Then:*

1. *The sequence $\{x_n\}$, with $x_n \in Y_\rho$, is ρ-convergent to x (denoted as $x_n \to x$) if $\rho(x_n - x) \to 0$ as $n \to \infty$.*

2. *The sequence $\{x_n\}$, with $x_n \in Y_\rho$, is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n : m \to \infty$.*

3. *Y_ρ is called ρ-complete if every ρ-Cauchy sequence in Y_ρ is ρ-convergent.*

Proposition 1. *In modular space,*

- *If $x_n \overset{\rho}{\to} x$ and a is a constant vector, then $x_n + a \overset{\rho}{\to} x + a$.*
- *If $x_n \overset{\rho}{\to} x$ and $y_n \overset{\rho}{\to} y$, then $\alpha x_n + \beta y_n \overset{\rho}{\to} \alpha x + \beta y$, where $\alpha + \beta \leq 1$ and $\alpha, \beta \geq 1$.*

Remark 1. *Note that $\rho(x)$ is an increasing function, for all $x \in X$. Suppose $0 < a < b$, then property (4) of Definition 1 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in Y$. Moreover, if ρ is a convex modular on X and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$.*

In general, if $\lambda_i \geq 0$, $i = 1, \ldots, n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \leq 1$ then $\rho(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq \lambda_1\rho(x_1) + \lambda_2\rho(x_2) + \cdots + \lambda_n\rho(x_n)$.

If $\{x_n\}$ is ρ-convergent to x, then $\{cx_n\}$ is ρ-convergent to cx, where $|c| \leq 1$. But the ρ-convergence of a sequence $\{x_n\}$ to x does not imply that $\{\alpha x_n\}$ is ρ-convergent to αx_n for scalars α with $|\alpha| > 1$.

If ρ is a convex modular satisfying Δ_2 condition with $0 < \tau < 2$, then $\rho(x) \leq \tau\rho(\frac{1}{2}x) \leq \frac{\tau}{2}\rho(x)$ for all x.

Hence $\rho = 0$. Consequently, we must have $\tau \geq 2$ if ρ is a convex modular.

Ulam [24], who brought up the stability issue of group homomorphisms, is credited with starting the study of the stability of functional equations. The first affirmative response to Ulam's query in the context of a Cauchy functional equation in Banach spaces came from Hyers [9]. The stability of functional equations may be referred to as Hyers-Ulam stability in recognition of Hyers's response to Ulam's initial query. The direct method [9], which Hyers utilized to prove Ulam's problem, has been widely applied to research the stability of numerous functional equations [8]. There are more ways to demonstrate the Hyers-Ulam stability of particular functional equations, such as the method based on sandwich theorems [19], the method of invariant means [21] or the method using the shadowing property [22].

The next point approach [4, 26] is the most used method for demonstrating the stability of functional equations besides the direct method. On the other hand, numerous authors have looked into the stability utilizing Khamsi's [10] fixed point theorem for quasicontraction mappings in modular spaces without Δ_2-condition. Recently, Sadeghi [20] employed Khamsi's fixed point theorem to investigate the stability outcomes of additive functional equations in modular spaces with the Fatou property and Δ_2-condition.

Additionally, Wongkum, Chaipunya and Kumam [25] demonstrated the stability of quadratic functional equations in modular spaces satisfying the Fatou property without utilizing the Δ_2-condition.

In this study consisting of 3 sections, we show the Hyers-Ulam stability of the following inequality:

$$\rho(f(x)+f(y)+f(z)) \leq \rho\left(qf\left(\sqrt{\frac{x^3+y^3+z^3}{q}}\right)\right), \tag{1}$$

where q is a fixed real number with $0 < q < 1$, in modular space satisfying Δ_2-condition with $\tau = 2$, in Section 2. In Section 3, we show the stability of the following inequality:

$$N(f(x)+f(y)+f(z),t) \geq N\left(qf\left(\sqrt{\frac{x^3+y^3+z^3}{q}}\right),t\right) \tag{2}$$

in fuzzy Banach space, using a fixed point method.

2. Stability of (1) in modular spaces satisfying Δ_2-condition with $\tau = 2$

In 2010, Y. J. Chi, C. Park and R. Saadati [6] proved that the generalized Hyers-Ulam stability of the following functional inequality;

$$\|f(x) + f(y) + f(z)| \leq \left\| qf\left(\frac{x+y+z}{q}\right)\right\|$$

in non-Archimedean Banach space. And in 2018, Y. Aribou [2] proved that if a mapping f satisfies

$$\|f(x) + f(y) + f(z)| \leq \left\| qf\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right)\right\|,$$

then $f(\sqrt[3]{x^3+y^3}) = f(x) + f(y)$; $\forall x, y \in \mathbb{R}$. He also studied its stability in non-Archimedean Banch space.

In this work, we study the stability of (1) in modular space.

Lemma 1. *Let $\mathbb{R}$ be a set of real numbers, ρ be a convex modular and Y_ρ be a ρ-modular space. Let $f : \mathbb{R} \to Y_\rho$ a mapping satisfying a functional inequality* (1)*. Then*

$$f\left(\sqrt[3]{x^3+y^3}\right) = f(x) + f(y); \text{ for all } x, y \in \mathbb{R}. \tag{3}$$

Proof. Letting $x = y = z = 0$ in (1), we get

$$\rho(3f(0)) \leq \rho(qf(0)),$$

so

$$\begin{aligned}\rho(f(0)) &\leq \frac{1}{3}\rho(qf(0))\\ &\leq \frac{q}{3}\rho(f(0)).\end{aligned}$$

Since $0 < q < 1, \quad f(0) = 0$.
Letting $z = 0$ and $y = -x$, we get

$$\rho(f(x) + f(-x)) \leq 0.$$

Hence $f(-x) = -f(x)$.
Letting $z = -\sqrt[3]{x^3+y^3}$, we get

$$\rho\left(f(x) + f(y) - f\left(\sqrt[3]{x^3+y^3}\right)\right) \leq 0.$$

Thus we have $f(x)+f(y)=f\left(\sqrt[3]{x^3+y^3}\right)$ for all $x,y\in\mathbb{R}$, and this completes the proof. ◀

Theorem 1. *Let $\mathbb{R}$ be a set of real numbers, ρ be a convex modular, Y_ρ be a ρ-complete modular space satisfying Δ_2-condition with $\tau=2$,and the Fatou property be satisfied Let $\varphi:\mathbb{R}^3\to[0,\infty)$ be a function with*

$$\psi(x,y,z)=\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi(2^{\frac{j-1}{3}}x,2^{\frac{j-1}{3}}y,2^{\frac{j-1}{3}}z)<\infty \tag{4}$$

for all $x,y,z\in\mathbb{R}$. Assume that $f:\mathbb{R}\to Y_\rho$ is a mapping satisfying

$$\rho(f(x)+f(y)+f(z))\le\rho\left(qf\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right)\right)+\varphi(x,y,z) \tag{5}$$

for all $x,y,z\in\mathbb{R}$. Then there exists a unique mapping $T:\mathbb{R}\to Y_\rho$ satisfying (3) *such that*

$$\rho(f(x)-T(x))\le\psi(x,x,-2^{1/3}x). \tag{6}$$

Proof. Letting $y=x$ and $z=-2^{1/3}x$ in (5), we get

$$\rho\left(2f(x)-f(2^{1/3}x)\right)\le\varphi\left(x,x,-2^{1/3}x\right). \tag{7}$$

It follows that

$$\rho\left(\frac{1}{2}f(2^{1/3}x)-f(x)\right)\le\frac{1}{2}\varphi\left(x,x,-2^{1/3}x\right).$$

This implies

$$\rho\left(\frac{1}{2^{k+1}}f\left(2^{\frac{k+1}{3}}x\right)-\frac{1}{2^k}f\left(2^{\frac{k}{3}}x\right)\right)\le\frac{1}{2^{k+1}}\rho\left(2f\left(2^{\frac{k}{3}}x\right)-f\left(2^{\frac{k+1}{3}}x\right)\right).$$

Let m and n be non-negative integers such that $m>n$.

Since $\sum_{k=n+1}^{m}\frac{1}{2^k}<1$, it follows from (7) that

$$\begin{aligned}
&\rho\left(\frac{1}{2^n}f\left(2^{n/3}x\right)-\frac{1}{2^m}f\left(2^{m/3}x\right)\right)\\
&\le\frac{1}{2^{n+1}}\rho\left(2f\left(2^{n/3}x\right)-f\left(2^{\frac{n+1}{3}}x\right)\right)+\frac{1}{2^{n+2}}\rho\left(2f\left(2^{\frac{n+1}{3}}x\right)-f\left(2^{\frac{n+2}{3}}x\right)\right)\\
&+\cdots+\frac{1}{2^m}\rho\left(2f\left(2^{\frac{m-1}{3}}x\right)-f\left(2^{\frac{m}{3}}x\right)\right)
\end{aligned}$$

$$= \sum_{k=1}^{m-n} \frac{1}{2^{n+k}} \rho\left(2f\left(2^{\frac{n+k-1}{3}}x\right) - f\left(2^{\frac{n+k}{3}}x\right)\right)$$
$$\leq \sum_{k=1}^{m-n} \frac{1}{2^{n+k}} \varphi\left(2^{\frac{n+k-1}{3}}x, 2^{\frac{n+k-1}{3}}x, -2^{\frac{n+k}{3}}x\right)$$
$$= \sum_{j=n+1}^{m} \frac{1}{2^j} \varphi\left(2^{\frac{j-1}{3}}x, 2^{\frac{j-1}{3}}x, -2^{j/3}x\right). \tag{8}$$

By (4) and (8), we conclude that $\{\frac{f(2^{\frac{n}{3}}x)}{2^n}\}$ is a ρ-Cauchy sequence in Y_ρ. The ρ-completeness of Y_ρ guarantees its ρ-convergence.

Hence, there exits a mapping $T : \mathbb{R} \to Y_\rho$ defined by

$$T(x) = \rho - \lim \frac{f(2^{\frac{n}{3}}x)}{2^n}; \qquad x \in \mathbb{R}. \tag{9}$$

Moreover, we have

$$\rho(T(x) - f(x)) \leq \lim_{n\to\infty} \inf \rho\left(\frac{1}{2^n} f\left(2^{n/3}x\right) - f(x)\right)$$
$$\leq \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi\left(2^{\frac{j-1}{3}}x, 2^{\frac{j-1}{3}}x, -2^{\frac{j}{3}}x\right) = \psi(x, x, -2^{1/3}x).$$

On the other hand, we have

$$\rho\left(\frac{1}{4}T(x) + \frac{1}{4}T(y) + \frac{1}{4}T(z)\right) \leq \frac{1}{4}\rho\left(T(x) - \frac{f(2^{n/3}x)}{2^n}\right) + \frac{1}{4}\rho\left(T(y) - \frac{f(2^{n/3}y)}{2^n}\right)$$
$$+ \frac{1}{4}\rho\left(T(z) - \frac{f(2^{n/3}z)}{2^n}\right) + \frac{1}{2^{n+2}}\rho\left(f(2^{n/3}x) + f(2^{n/3}y) + f(2^{n/3}z)\right)$$
$$\leq \frac{1}{4}\rho\left(T(x) - \frac{f\left(2^{\frac{n}{3}}x\right)}{2^n}\right) + \frac{1}{4}\rho\left(T(y) - \frac{f\left(2^{\frac{n}{3}}y\right)}{2^n}\right) + \frac{1}{4}\rho\left(T(z) - \frac{f\left(2^{\frac{n}{3}}z\right)}{2^n}\right)$$
$$+ \frac{1}{2^{n+2}}\rho\left(qf\left(2^{n/3}\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right)\right) + \frac{1}{2^{n+2}}\varphi\left(2^{n/3}x, 2^{n/3}y, 2^{n/3}z\right)$$
$$\leq \frac{1}{4}\rho\left(T(x) - \frac{f\left(2^{n/3}x\right)}{2^n}\right) + \frac{1}{4}\rho\left(T(y) - \frac{f\left(2^{n/3}y\right)}{2^n}\right) + \frac{1}{4}\rho\left(T(z) - \frac{f\left(2^{n/3}z\right)}{2^n}\right)$$
$$+ \frac{1}{4}\rho\left(\frac{q}{2^n} f\left(2^{n/3}\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right) - qT\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right)\right)$$

$$+\frac{1}{4}f\left(qT\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right)\right)+\frac{1}{2^{n+2}}\varphi\left(2^{n/3}x,2^{n/3}y,2^{n/3}z\right).$$

Letting $n\to\infty$, we get

$$\rho\left(\frac{1}{4}T(x)+\frac{1}{4}T(y)+\frac{1}{4}T(z)\right)\le\frac{1}{4}\rho\left(qT\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right)\right).$$

And we have

$$\begin{aligned}\rho\left(T(x)+T(y)+T(z)\right)\le&4\rho\left(\frac{1}{4}T(x)+\frac{1}{4}T(y)+\frac{1}{4}T(z)\right)\\ &\le\rho\left(qT\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right)\right).\end{aligned}$$

By Lemma 1, the mapping T satisfies the equation (3).

Finally, to show the uniqueness of T, assume that T_1 and T_2 are radical cubic mappings satisfying (6).

We see that

$$\begin{aligned}\rho\left(\frac{T\left(2^{1/3}x\right)-2T(x)}{2^3}\right)&=\rho\left(\frac{1}{2^3}\left(T\left(2^{1/3}x\right)-\frac{f\left(2^{\frac{n+1}{3}}x\right)}{2^n}\right.\right.\\ &\qquad\left.\left.+\frac{1}{2^2}\left(\frac{f\left(2^{\frac{n+1}{3}}x\right)}{2^{n+1}}-T(x)\right)\right)\right)\\ &\le\frac{1}{2^3}\rho\left(T\left(2^{1/3}x\right)-\frac{f\left(2^{\frac{n+1}{3}}x\right)}{2^n}\right)+\frac{1}{2^2}\rho\left(\frac{f\left(2^{\frac{n+1}{2}}x\right)}{2^{n+1}}-T(x)\right)\end{aligned}\tag{10}$$

for all $x\in\mathbb{R}$. Then by (9), the right-hand side of (10) tends to 0 as $n\to\infty$. Therefore it follows that

$$T\left(2^{1/3}x\right)=2T(x).$$

Next, we write

$$\begin{aligned}
\rho\left(\frac{T_1(x)-T_2(x)}{2}\right) &= \rho\left(\frac{1}{2}\left(\frac{T_1(2^{k/3}x)}{2^k}-\frac{f(2^{k/3}x)}{2^k}\right)\right.\\
&\qquad\left.+\frac{1}{2}\left(\frac{f(2^{k/3}x)}{2^k}-\frac{T_2(2^{k/3})x}{2^k}\right)\right)\\
&\le \frac{1}{2}\rho\left(\frac{T_1(2^{k/3}x)}{2^k}-\frac{f(2^{k/3}x)}{2^k}\right)+\frac{1}{2}\rho\left(\frac{f(2^{k/3}x)}{2^k}-\frac{T_2(2^{k/3}x)}{2^k}\right)\\
&\le \frac{1}{2}\frac{1}{2^k}\{\rho\left(T_1(2^{k/3}x)-f(2^{k/3}x)\right)+\rho\left(f(2^{k/3}x)-T_2(2^{k/3}x)\right)\}\\
&\le \frac{1}{2^k}\psi\left(2^{k/3}x,2^{k/3}x,2^{k/3}x\right)\\
&= \frac{1}{2^k}\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi\left(2^{\frac{j+k-1}{3}}x,2^{\frac{j+k-1}{3}}x,2^{\frac{j+k-1}{3}}x\right)\\
&= \sum_{l=k+1}^{\infty}\varphi\left(2^{\frac{l-1}{3}}x,2^{\frac{l-1}{3}}x,2^{\frac{l-1}{3}}x\right)\to 0 \text{ as } l\to\infty.
\end{aligned}$$

This implies that $T_1 = T_2$.

Now, we have the classical Ulam stability of (1) by putting $\varphi = \varepsilon > 0$.

Corollary 1. *Let* $\mathbb{R}$ *be a linear space,* ρ *be a convex modular and* Y_ρ *be a* ρ*-complete modular space. Assume* $f:\mathbb{R}\mapsto Y_\rho$ *is a mapping such that* $f(0)=0$ *and*

$$\rho(f(x)+f(y)+f(z))\le\rho\left(qf\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right)\right)+\varepsilon \tag{11}$$

for all $x,y,z\in\mathbb{R}$.

Then there exists a unique radical cubic mapping $T:\mathbb{R}\to Y_\rho$ *such that*

$$\rho(f(x)-T(x))\le\varepsilon. \tag{12}$$

Corollary 2. *Let* $\mathbb{R}$ *be a normed linear space,* ρ *be a convex modular space and* Y_ρ *be a* ρ*-complete nodular space. Let* $\theta>0$ *and* $0<p<3$ *be real numbers. Assume that* $f:\mathbb{R}\to Y_\rho$ *is a mapping satisfying*

$$\rho(f(x)+f(y)+f(z))\le\rho\left(qf\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right)\right)+\theta\left(||x||^p+||y||^p+||z||^p\right) \tag{13}$$

for all $x, y, z \in \mathbb{R}$. Then there exists a unique radical cubic mapping $T : \mathbb{R} \to Y_\rho$ such that

$$\rho(f(x) - T(x)) \leq \theta \left(\frac{2 + 2^{p/3}}{2 - 2^{p/3}} \right) ||x||^p \tag{14}$$

3. Fuzzy stability of (2) in fuzzy Banach spaces

Definition 3 ([3]). *Let X be a real space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

1. $N(x, t) = 0$ *for* $t \leq 0$;
2. $x = 0$ *if and only if* $N(x, t) = 1$ *for all* $t > 0$;
3. $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;
4. $N(x + y, s + c) \geq \min\{N(x, s), N(y, c)\}$;
5. $N(x, \cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x, t) = 1$;
6. *for* $x \neq 0, N(x, .)$ *is continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed space.

Example 1. *Let $(X, ||.||)$ be a normed linear space. Then*

$$N(x, t) = \begin{cases} \frac{t}{t+\|x\|}; & x \in X; t > 0 \\ 0 & ; x \in X; t \leq 0 \end{cases}$$

is a fuzzy norm on X.

Definition 4. *Let (X, N) be a fuzzy normed vector space.*

(i) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$ and we denote it by $N - \lim\limits_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim\limits_{n,m\to\infty} N(x_n - x_m, t) = 1$ for all $t > 0$.

If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete, and the fuzzy normed vector space is called a fuzzy Banach space.

The following theorem is a fundamental result in fixed point theory.

Theorem 2 ([7, 5]). *Let (X, d) be a complex generalised metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty$$

for all non-negative integers n or there exists a positive integer n_0 such that

1. $d\left(J^n x, J^{n+1}\right) < \infty$ *for all* $n \geq n_0$*;*
2. *The sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*
3. y^* *is a unique fixed point of* J *in the set* $Y = \{y \in X / d\left(J^{n_0} x, y\right) < \infty\}$*;*
4. $d\left(y, y^*\right) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

We use the definition of fuzzy normed spaces.

Lemma 2. *Let $\mathbb{R}$ be a set of real numbers, (Y, N) be a fuzzy Banach space and $f : \mathbb{R} \mapsto Y$ be a mapping satisfying*

$$N(f(x) + f(y) + f(z), t) \geq N\left(qf\left(\sqrt[3]{\frac{x^3 + y^3 + z^3}{q}}\right), t\right) \tag{15}$$

for all $x, y, z \in \mathbb{R}$ and $0 < q < 1$. Then

$$f(\sqrt[3]{x^3 + y^3}) = f(x) + f(y).$$

Proof. Letting $x = y = z = 0$ in (15), we get

$$N(3f(0), t) \geq N(qf(0), t).$$

So $N\left(f(0), \frac{t}{3}\right) \geq N\left(f(0), \frac{t}{q}\right)$ for all $t > 0$. By (5) and (6), we have $N(f(0), t) = 1$ for all $t > 0$, hence $f(0) = 0$. Letting $z = 0$ and $y = -x$ in (15), we get

$$N(f(x) + f(-x), t) \geq N(0, t) = 1.$$

Hence $f(-x) = -f(x)$.

Letting $z = -\sqrt[3]{x^3 + y^3}$, we get

$$N\left(f(x) + f(y) - f\left(\sqrt[3]{x^3 + y^3}\right), t\right) \geq N(0, t) = 1,$$

so $f\left(\sqrt[3]{x^3 + y^3}\right) = f(x) + f(y)$ and this completes the proof. ◄

Theorem 3. *Let $\mathbb{R}$ be a set of real numbers, and (Y,N) be a fuzzy Banach space. Let $\varphi : \mathbb{R}^3 \to [0,\infty)$ be a function such that $\varphi(0,0,0)=0$ and there exist an $0<L<1$ satisfying*

$$\varphi(x,y,z) \le \frac{L}{2}\varphi\left(2^{1/3}x, 2^{1/3}y, 2^{1/3}z\right)$$

for all $x,y,z\in\mathbb{R}$. Let $f:\mathbb{R}\to Y$ be a mapping that satisfies

$$N(f(x)+f(y)+f(z),t) \ge \min\left\{N\left(qf\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right),t\right), \frac{t}{t+\varphi(x,y,z)}\right\} \tag{16}$$

for all $x,y,z\in X$ and all $t>0$. Then there exists a unique mapping $F:\mathbb{R}\to Y$ satisfying (3) *such that*

$$N(f(x)-F(x),t) \ge \frac{2(1-L)t}{2(1-L)t+L\varphi\left(x,x,-2^{1/3}x\right)}, \tag{17}$$

$x\in\mathbb{R}, t>0$. The mapping F is defined by

$$F(x) = N-\lim_{n\to\infty} 2^n f(\frac{x}{2^{n/3}}).$$

Proof. Letting $y=x$ and $z=-2^{1/3}x$ in (16) we get

$$N\left(2f(x)-f\left(2^{1/3}x\right),t\right) \ge \frac{t}{t+\varphi\left(x,x,-2^{1/3}x\right)}. \tag{18}$$

So $N\left(f(x)-2f(\frac{x}{2^{1/3}}),t\right) \ge \dfrac{t}{t+\varphi\left(\frac{x}{2^{1/3}},\frac{x}{2^{1/3}},-x\right)}$ for all $x\in\mathbb{R}$.

Consider the set

$$S=\{p/p:X\to Y\}$$

and we define on S the generalized metric by

$$d(p,q)=\inf\left\{\mu\in\mathbb{R}^+/N(p(x)-q(x),\mu t) \ge \frac{t}{t+\varphi\left(x,x,-2^{1/3}x\right)}, x\in\mathbb{R}, t>0\right\}.$$

It is easy to show that (S,d) is a complete generalized metric space (see [15]). Then . Now we consider the mapping $J:S\to S$ given by $Jp(x)=2p\left(\frac{x}{2^{1/3}}\right), x\in\mathbb{R}$. Let $p,q\in S$ be given suach that $d(p,q)=\varepsilon$. Then:

$$N(p(x)-q(x),\varepsilon t) \ge \frac{t}{t+\varphi(x,x,-2^{1/3}x)}$$

for all $x \in \mathbb{R}$ and all $t > 0$. Hence:

$$\begin{aligned} N\left(Jp(x) - Jq(x), L\varepsilon t\right) &= N\left(2p\left(\frac{x}{2^{1/3}}\right) - 2q\left(\frac{x}{2^{1/3}}\right), L\varepsilon t\right) \\ &= N\left(p\left(\frac{x}{2^{1/3}}\right) - q\left(\frac{x}{2^{1/3}}\right), \frac{L\varepsilon t}{2}\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi(\frac{x}{2^{1/3}}, \frac{x}{2^{1/3}}, -x)} \\ &\geq \frac{t}{t + \varphi(x, x, -2^{1/3})} \end{aligned}$$

for all $x \in \mathbb{R}$ and all $t > 0$. So $d(p, q) = \varepsilon$ implies that $d(Jp, Jq) \leq L\varepsilon$. This means that $d(Jp, Jq) \leq Ld(p, q)$.
It follows from (18) that

$$\begin{aligned} N\left(f(x) - 2f\left(\frac{x}{2^{1/3}}\right), \frac{Lt}{2}\right) &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2^{1/3}}, \frac{x}{2^{1/3}}, -x\right)} \\ &\geq \frac{t}{t + \varphi(x, x, -2^{1/3})} \end{aligned}$$

for all $x \in \mathbb{R}$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$. By theorem 2, we deduce that there exists a mapping $F : \mathbb{R} \longrightarrow Y$ satisfying:

1. F is a fixed point of J ie: $F\left(\frac{x}{2^{1/3}}\right) = \frac{1}{2}F(x)$ for all $x \in \mathbb{R}$. The mapping F is a unique fixed point of J in the set $M = g \in S; d(f, g) < \infty$. This implies that F is the unique fixed point of J such that there exists a $\mu \in (0, \infty)$ satisfying:
$$N(f(x) - F(x), \mu t) \geq \frac{t}{t + \varphi(x, x, -2^{1/3}x}$$
for all $x \in \mathbb{R}$

2. $d(J^n f, F) \longrightarrow 0$ as $n \longrightarrow \infty$ this implies the equality
$$N - lim_{n \longrightarrow \infty} 2^n f\left(\frac{x}{2^{n/3}}\right) = F(x)$$
for all $x \in \mathbb{R}$

3. $d(f, F) \leq \frac{1}{1-L} d(f, Jf)$ wich implies the inequality $d(f, F) \leq \frac{L}{2(1-L)}$. This implies that the inequality (17) holds.

Now, we show that F satisfies (3).

Replacing (x,y,z) with $\left(\frac{x}{2^{n/3}},\frac{y}{2^{n/3}},\frac{z}{2^{n/3}}\right)$ in (16), we get

$$N\left(2^n\left(f\left(\frac{x}{2^{n/3}}\right)+f\left(\frac{y}{2^{n/3}}\right)+f\left(\frac{z}{2^{n/3}}\right)\right),2^nt\right)\geq$$
$$\min\left\{N\left(2^nqf\left(\frac{1}{2^{n/3}}\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right),2^nt\right),\frac{t}{t+\varphi\left(\frac{x}{2^{n/3}},\frac{y}{2^{n/3}},\frac{z}{2^{n/3}}\right)}\right\}$$

Since

$$\varphi\left(\frac{x}{2^{n/3}},\frac{y}{2^{n/3}},\frac{z}{2^{n/3}}\right)\leq\left(\frac{L}{2}\right)^n\varphi(x,y,z),$$

it follows that

$$N\left(2^n\left(f\left(\frac{x}{2^{n/3}}\right)+f\left(\frac{y}{2^{n/3}}\right)+f\left(\frac{z}{2^{n/3}}\right)\right),t\right)\geq$$
$$\min\left\{N\left(2^nqf\left(\frac{1}{2^{n/3}}\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right),t\right),\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi\left(x,y,z\right)}\right\}$$

for $x,y,z\in\mathbb{R}$, all $t>0$, and all $n\in\mathbb{N}$.
Since

$$\lim_{n\to\infty}\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y,z)}=1,$$

it follows that

$$N(F(x)+F(y)+F(z),t)\geq N\left(qF\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right),t\right)$$

for all $x,y,z\in\mathbb{R}$ and $t>0$.
Then by Lemma 2, the mapping F satisfies (3). Finally, assume that F_1 and F_2 are two mappings satisfying (17). Then

$$N\left(f(x)-F_i(x),t\right)\geq\frac{2(1-L)t}{2(1-L)t+L\varphi\left(x,x,-2^{1/3}x\right)}$$

Consequently

$$N\left(F_2(x)-F_1(x),2t\right)=N\left(2^n\left(F_2(\frac{x}{2^{n/3}})-F_1(\frac{x}{2^{n/3}}),2t\right)\right)$$
$$\geq\min\left\{N\left(2^n\left(f(\frac{x}{2^{n/3}})-F_1(\frac{x}{2^{n/3}})\right),t\right),N\left(2^n\left(f(\frac{x}{2^{n/3}})-F_2(\frac{x}{2^{n/3}})\right),t\right)\right\}$$
$$\geq\frac{2(1-L)\frac{t}{2^n}}{2(1-L)\frac{t}{2^n}+L\varphi\left(\frac{x}{2^{n/3}},\frac{x}{2^{n/3}},-\frac{x}{2^{n-1/3}}\right)}.$$

Since

$$\varphi\left(\frac{x}{2^{n/3}}),(\frac{x}{2^{n/3}}),-(\frac{x}{2^{n-1/3}})\right) \leq \left(\frac{L}{2}\right)^n \varphi\left(x,x,-2^{1/3}x\right),$$

We have:

$$N\left(F_2(x)-F_1(x),2t\right) \geq \frac{2(1-L)\frac{t}{2^n}}{2(1-L)\frac{t}{2^n}+L\left(\frac{L}{2}\right)^n\varphi\left(x,x,-2^{1/3}x\right)} \to 1 \text{ as } n\to\infty.$$

This yields $F_1 = F_2$, as desired.

Corollary 3. *Let* $\mathbb{R}$ *be a real normed space, and* (Y,N) *be a fuzzy Banach space. Let* $\theta > 0$ *and* $0 < r < 1$ *be real numbers, and* $f : X \to Y$ *be a mapping satisfying*

$$N\left(f(x)+f(y)+f(z),t\right) \geq \min\left\{N\left(qf\left(\sqrt[3]{\frac{x^3+y^3+z^3}{q}}\right),t\right),\frac{t}{t+\theta\left(\|x\|^r+\|y\|^r+\|z\|^r\right)}\right\}$$

for all $x,y,z \in \mathbb{R}$ *and* $t > 0$*. Then there exists a unique mapping* $F : \mathbb{R} \to Y$ *satisfying* (3) *such that*

$$N\left(f(x)-F(x),t\right) \geq \frac{2\left(1-2^{r-1}\right)t}{2\left(1-2^{r-1}\right)t+L\theta\|x\|^r\left(2+2^{r/3}\right)}.$$

Proof. Taking $\varphi(x,y,z) = \theta\left(\|x\|^r+\|y\|^r+\|z\|^r\right)$ in Theorem 3, we can choose $L = 2^{r-1}$ to get the result.

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