Azerbaijan Journal of Mathematics
V. 14, No 2, 2024, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2024.2.20

The Absence of Positive Global Periodic Solution of a Second-Order Semi Linear Parabolic Equation With Time-Periodic Coefficients

Sh.G. Bagyrov

Abstract. Second order semilinear parabolic equation $\frac{\partial u}{\partial t} = div(A(x,t)\nabla u) + h(x,t,u)$ with time-periodic coefficients is considered in domain $\Omega \times (-\infty, +\infty)$, where Ω is the exterior of a compact set in R_x^n. Depending on the behavior of the function $h(x,t,u)$ at infinity, the conditions are found under which the positive periodic solution does not exist.

Key Words and Phrases: semilinear parabolic equation, periodic solution, non-existence of global solutions.

2010 Mathematics Subject Classifications: 35A01, 35K10, 35K58, 35B09, 35B10

1. Introduction

Denote $R^+ = [0; +\infty)$, $B_R = \{x : |x| < R\}$, $B'_R = \{x : |x| > R\}$, $S_R = \{x : |x| = R\}$, $B_{R_1,R_2} = \{x : R_1 < |x| < R_2\}$, $Q_T^{R_1,R_2} = B_{R_1,R_2} \times (0,T)$, $Q_T^{R,\infty} = B'_R \times (0,T)$, $Q_T = \Omega \times (0,T)$, $Q = \Omega \times (-\infty; +\infty)$, where Ω is the exterior of a compact set D in R_x^n containing the origin.

Consider the equation

$$\frac{\partial u}{\partial t} = div(A(x,t)\nabla u) + h(x,t,u) \tag{1}$$

in the cylinder Q, where $n \geq 3$, $A(x,t) = (a_{ij}(x,t))_{ij=1}^n$, $h(x,t,u) : \Omega\times(-\infty,+\infty)\times R^+ \to R$, $a_{ij}(x,t)$ are bounded, measurable, T- periodic in t functions, and there exist constants $\nu_1, \nu_2 > 0$ such that

$$\nu_1 |\xi|^2 \leq (A\xi, \xi) \leq \nu_2 |\xi|^2 \tag{2}$$

for every $(x,t) \in Q, \xi = (\xi_1, ..., \xi_n) \in R^n$. Here $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$, $A\nabla u =$ $= \left(\sum_{j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\right)_{i=1}^{n}$, $(A\xi, \eta) = \sum_{i,j=1}^{n} a_{ij}\xi_i\eta_j$, $\xi = (\xi_1, ..., \xi_n)$, $\eta = (\eta_1 ...., \eta_n)$, $div(A\nabla u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j})$.

We will study the existence of a global positive solution of equation (1). The matters of existence and non-existence of global solutions for different classes of differential equations and inequalities play an important role both in theory and applications, that is why they have always been a cause for constant interest of mathematicians. Interest in such problems arose after Fujita's popular paper [16]. After this work, many authors began to investigate the question of existence of global solutions for various types of equations with nonlinearities of various types.

A lot of works have been dedicated to these matters (see [1,2,8,11,13,17,21,22, 23,25,26,28]). For useful reviews of such works, we refer the readers to the article [24], the monograph [27], and the book [30].

In particular, the existence of solutions to the periodic parabolic equations has also been a study object for many researchers (see [10,14,15,18,19,20,29,31]). One of the earliest works dedicated to periodic parabolic equations was Seidman's [31], which treated the existence of non-trivial periodic solution for the following problem:

$$\frac{\partial u}{\partial t} = \Delta u + a_0(x,t) |u|^q, (x,t) \in \Omega' \times (0, +\infty), u/_{\partial\Omega'} = 0, \qquad (*)$$

with $q = 0$, where $a_0(x,t)$ is a periodic in t function and $\Omega' \subset R^n$ is a bounded domain. Since then, many authors have considered the problem (*) for $q > 0$. Beltramo and Hess [10] studied the problem (*) for $q = 1$ and showed that for specially chosen $a_0(x,t)$ it may have non-trivial periodic solutions. Esteban [14, 15] proved that, for every $q > 1$ when $n \le 2$, and for $1 < q < \frac{n}{n-2}$ when $n > 2$ the problem (*) has positive periodic solutions for any kind of $a_0(x,t) > 0$. He also proved that, for $n > 2$, $q \ge \frac{n+2}{n-2}$ this problem has no positive periodic solution. In 2004, Quittner [29] proved, with some restrictions on $a_0(x,t)$, that this problem has positive solutions for $1 < q < \frac{n+2}{n-2}$.

In [4], the equation (1) has been considered in Q with nonlinearity $h(x,t,u) = a_0(x,t) |u|^q$, and it was proved that if $a_0(x,t) \ge c |x|^\sigma$, then there is no positive solution for $2 + \sigma + (2 - n)(q - 1) \ge 0$. In [5], the equation (1) has been again considered in Q and it was proved that if $a_0(x,t) \ge c |x|^\sigma \ln^s |x|$, then there is no positive solution for $2 + \sigma + (2 - n)(q - 1) > 0$, $s \in (-\infty, +\infty)$ and for

$2+\sigma+(2-n)(q-1)=0$, $s \geq -1$. In [6], the equation (1) was considered with a weight, and in [7] a system of such equations was considered.

In the present article, the nonlinearity has a more general form. It is the kind of nonlinearity that was considered in [12], where the existence of non-negative solutions of the equation $\Delta u + h(x,u) = 0$ in Ω with the condition $u/_{\partial\Omega} \geq 0$ was discussed. In this paper, we obtain similar results for parabolic equations with time-periodic coefficients.

2. Main result and its proof

Before giving a definition for solution, we consider the following function space:

$$W_2^{1,\,1/2}(Q_T) =$$

$$= \left\{ u(x,t+T) = u(x,t), u(x,t) \in W_2^{1,0}(Q_T), \sum_{k=-\infty}^{+\infty} |k| \int_\Omega |u_k(x)|^2\, dx < \infty \right\},$$

where

$$u_k(x) = \frac{1}{T}\int_0^T u(x,t) \exp\left\{-ik\frac{2\pi}{T}t\right\} dt.$$

The norm in this space is defined as follows:

$$\|u\|^2_{W_2^{1,\,1/2}(Q_T)} = \|u\|^2_{L_2(Q_T)} + \|\nabla u\|^2_{L_2(Q_T)} + \sum_{k=-\infty}^{+\infty} |k| \int_\Omega |u_k(x)|^2\, dx.$$

By $\overset{\circ}{W}{}_2^{1,1/2}(Q_T)$ we mean a completion of $C^{0,\infty}(Q_T)$ with respect to the norm $\|\cdot\|_{W_2^{1,\,1/2}(Q_T)}$, where $C^{0,\infty}(Q_T)$ is a set of infinitely differentiable functions on Q, which are T periodic in t and vanish in the vicinity of $\partial\Omega$.

A solution of equation (1) is defined as a function
$u(x,t) \in W_{2,loc}^{1,1/2}(Q_T) \bigcap L_{\infty,loc}(Q_T)$ satisfying the corresponding integral identity

$$2\pi \sum_{k=-\infty}^{k=+\infty} ik \int_\Omega u_k(x)\varphi_{-k}(x)dx + \int_Q (A(x,t)\nabla u, \nabla\varphi)dxdt = \int_Q h(x,t,u)\varphi dxdt$$

for each function $\varphi(x,t) \in \overset{0}{W}{}_2^{1,1/2}$.

Denote

$$L_0 u \equiv div(A(x,t)\nabla u) - \frac{\partial u}{\partial t}.$$

Consider the linear equation

$$L_0 u + P(x,t)u = 0 \text{ in } Q_T^{R,\infty}, \tag{3}$$

where

$$P(x,t+T) = P(x,t), P(x,t) \in L_{\infty,loc}(Q_T^{R,\infty}).$$

The following two lemmas are true (see [3]).

Lemma 1. *There exists a constant $C_0 > 0$ depending on n, ν_1, ν_2 and not depending on R such that, if $P(x,t) \geq \frac{C_0}{|x|^2}$, then equation (3) has no positive super solution in $Q_T^{R,\infty}$.*

Note that if $a_{ij}(x,t) = \delta_{ij}$, then $C_0 = \left(\frac{n-2}{2}\right)^2$ (see [9]).

Lemma 2. *Let $n \geq 3$ and $u(x,t) \in W_{2,loc}^{1,1/2}(Q_T^{R,\infty})$ be a continuous and nonnegative function in $\bar{Q}_T^{R,\infty}$ such that $L_0 u \leq 0$ in $Q_T^{R,\infty}$ and $u(x,t) > 0$ on S_R. Then $u(x,t) \geq \beta_0 |x|^{2-n}, (x,t) \in Q_T^{R,\infty}$, $\beta_0 = const > 0$.*

In this article, the nonlinearity has a more general form, namely, we assume that $h(x,t,u) \geq \tilde{h}(x,u) \geq 0$ for all $(x,t) \in Q$ and $\tilde{h} : \Omega \times R^+ \to R^+$ is a function satisfying the following assumption:

(H):

a) for any $x \in B'_e$

$$\frac{\tilde{h}(x,s_1)}{s_1} \geq \frac{\tilde{h}(x,s_2)}{s_2},$$

if $s_1 \geq s_2 > 0$,

b) for any $\tau > 0$,

$$\lim_{|x| \to +\infty} \inf \tilde{h}(x, \tau |x|^{2-n}) |x|^n > C_0,$$

if b) fails, we assume that

b1) there exists $\sigma_1 \in (0,1)$ such that for any $\tau > 0$

$$\lim_{|x| \to +\infty} \inf \tilde{h}(x, \tau |x|^{2-n}) |x|^n (\ln |x|)^{\sigma_1} > 0,$$

or

b2) there exists $\gamma > 1$ such that for any $\tau > 0, \alpha \geq 0$

$$\lim_{|x| \to +\infty} \inf \tilde{h}(x, \tau |x|^{2-n} (\ln |x|)^{\alpha}) |x|^n (\ln |x|)^{-\alpha\gamma+1} > 0,$$

and

b3) there exists $\sigma_2 > 0$ such that for any $\tau > 0$

$$\lim_{|x|\to+\infty} \inf \frac{\tilde{h}(x, \tau |x|^{2-n} (\ln |x|)^{\sigma_2})}{\tau |x|^{-n} (\ln |x|)^{\sigma_2}} > C_0.$$

The main result is the following theorem.

Theorem 1. *Let $n \geq 3$ and $A(x,t)$ satisfy the condition* (2). *Then under the assumption (H) equation* (1) *has no positive solution in Q.*

Proof. Let equation (1) have a positive solution $u(x,t) > 0$. Then for $u(x,t)$ in $Q_T^{R,\infty}$ all the conditions of Lemma 2 are satisfied, and, consequently, $u(x,t) \geq \beta_0 |x|^{2-n}$, where R is such that $D \subset B_R(0)$.

First, consider the cases $(H) - a), b)$

Consider the function $\frac{h(x,t,u)}{u}$. By Lemma 2 and $(H) - a), b)$ for large $|x|$ we obtain

$$\frac{h(x,t,u)}{u} \geq \frac{\tilde{h}(x,u)}{u} \geq \frac{\tilde{h}(x, \beta_0 |x|^{2-n})}{\beta_0 |x|^{2-n}} = \frac{1}{|x|^2} \frac{\tilde{h}(x, \beta_0 |x|^{2-n})}{\beta_0 |x|^{-n}} =$$

$$= \frac{1}{|x|^2} \frac{1}{\beta_0} \tilde{h}(x, \beta_0 |x|^{2-n}) |x|^n \geq \frac{C_0 + \varepsilon}{\beta_0} \frac{1}{|x|^2} > \frac{C_0 + \varepsilon}{|x|^2}.$$

Using this inequality, we get

$$0 = L_0 u + h(x,t,u) \geq L_0 u + \frac{h(x,t,u)}{u} u \geq L_0 u + \frac{C_0 + \varepsilon}{|x|^2} u.$$

And this contradicts the assertion of Lemma 1.

Let b) not be fulfilled, but $b1$) and also $b3$) be fulfilled. Again, suppose that $u(x,t)$ is a positive solution to equation (1). Then, by Lemma 2, $u(x,t) \geq \beta_0 |x|^{2-n}$. Using $b1$), for large $|x|$ we get

$$0 = L_0 u + h(x,t,u) = L_0 u + \frac{h(x,t,u)}{u} u \geq L_0 u + \frac{\tilde{h}(x,u)}{u} u \geq$$

$$\geq L_0 u + \frac{\tilde{h}(x, \beta_0 |x|^{2-n})}{\beta_0 |x|^{2-n}} u \geq L_0 u + \frac{\varepsilon |x|^{-n} \ln^{-\sigma_1} |x|}{\beta_0 |x|^{2-n}} u =$$

$$= L_0 u + \frac{C}{|x|^2 \ln^{\sigma_1} |x|} u\,.$$

Consider the equation

$$L_0 u + \frac{C}{|x|^2 \ln^{\sigma_1} |x|} u = 0 \text{ in } Q_T^{R_0,\infty}, \tag{4}$$

where $R_0 > e$.

Now let us show that the equation (4) has a positive solution $v(x,t)$ in $Q_T^{R_0,\infty}$. To do this, consider the following problem in $Q_T^{R_0,\infty}$:

$$L_0 v + \frac{C}{|x|^2 \ln^{\sigma_1} |x|} v = 0, \tag{5}$$

$$v|_{|x|=R_0} = 1, \quad v|_{|x|=R} = 0, \quad v(x,t+T) = v(x,t). \tag{6}$$

It is known the that problem (5), (6) has solutions $v_R(x,t)$. Let's prove that $0 \le v_R \le 1$. First let's prove that $v_R \le 1$. Consider the function $\psi(x,t) = (v_R - 1)^+ = \max\limits_{Q_T^{R_0,R}} \{v_R - 1, 0\}$. Taking the test function $\psi(x,t)$ in the definition of the solution, we obtain

$$2\pi \sum_{k=-\infty}^{+\infty} (ik) \int_{B_{R_0,R}} v_{R_k} \psi_{-k} dx + \int_{\sup p\psi} (A\nabla v_R, \nabla v_R) dxdt =$$

$$= \int_{\sup p\psi} \frac{C}{|x|^2 \ln^{\sigma_1} |x|} v_R (v_R - 1) dxdt.$$

It is easy to show that the first term on the left-hand side is equal to zero. Then, using (2) and Hardy's inequalities, we obtain

$$\nu_1 \int_{\sup p\psi} |\nabla v_R|^2 \, dxdt + \int_{\sup p\psi} \frac{C}{|x|^2 \ln^{\sigma_1} |x|} v_R dxdt \le$$

$$\le \int_{\sup p\psi} \frac{C}{|x|^2 \ln^{\sigma_1} |x|} v_R^2 dxdt \le C_1 \int_{\sup p\psi} |\nabla v_R|^2 \, dxdt.$$

Take ε such that $C_1 < \nu_1$. As a result, we get

$$(\nu_1 - C_1) \int_{\sup p\psi} |\nabla v_R|^2 \, dxdt + \int_{\sup p\psi} \frac{C}{|x|^2 \ln^{\sigma_1} |x|} v_R dxdt \le 0.$$

Since all integrals in this inequality are positive, this is possible only if $\sup p\psi = 0$. This means that it really is $v_R \le 1$. Similarly, we can show that $v_R \ge 0$.

Since $0 \le v_R \le 1$ for each R and v_R is a solution to problem (5),(6), in each compact set, the sequence of functions v_R is Hölder continuous and uniformly bounded. Then, by the Arzela theorem, in each compact set, the sequence of functions v_R converges uniformly to some function $v(x,t)$, which is a positive solution of equation (3) in $Q_T^{R_0,\infty}$.

Let $v(x,t)$ be a positive solution of equation (3). In the definition of the solution, we take a test function $\varphi(x,t)$ such that $0 \le \varphi(x,t) = \varphi(x) \in C^\infty$, $|\nabla\varphi|^2 \le \frac{C_2}{|x|^2}$, $\varphi(x) = 0$ for $|x| \le R_0$, $|x| > 2\rho$ and $\varphi(x) = 1$ for $2R_0 \le |x| \le \rho$. Then we get the following integral equality:

$$\int_{Q_T^{R_0,2\rho}} \frac{C}{|x|^2 \ln^{\sigma_1}|x|} v\varphi dxdt = \int_{Q_T^{R_0,2\rho}} (A\nabla v, \nabla\varphi) dxdt. \tag{7}$$

Given that $v(x,t) \ge \beta_0 |x|^{2-n}$, we estimate the left-hand side of (7) from below, and the right-hand side from above:

$$C\int_{Q_T^{R_0,2\rho}} \frac{v\varphi}{|x|^2 \ln^{\sigma_1}|x|} dxdt \ge C_2 \int_{Q_T^{2R_0,\rho}} \frac{\beta_0 |x|^{2-n}}{|x|^2 \ln^{\sigma_1}|x|} dxdt = C_3 \int_{2R_0}^{\rho} \frac{dr}{r\ln^{\sigma_1} r} =$$

$$= C_3 \ln^{-\sigma_1+1} r \Big|_{2R_0}^{\rho} \ge C_3 \ln^{1-\sigma_1}\rho - C_3 \ln^{1-\sigma_1} 2R_0 =$$

$$= C_3 \ln^{1-\sigma_1}\rho \left(1 - \frac{\ln^{1-\sigma_1} 2R_0}{\ln^{1-\sigma_1}\rho}\right) \ge C_4 \ln^{1-\sigma_1}\rho, \tag{8}$$

$$\int_{Q_T^{R_0,2\rho}} (A\nabla v, \nabla\varphi) dxdt = \int_{Q_T^{R_0,2R_0}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_j}\frac{\partial \varphi}{\partial x_i} dxdt +$$

$$+ \int_{Q_T^{\rho,2\rho}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial v}{\partial x_j}\frac{\partial \varphi}{\partial x_i} dxdt \le$$

$$\le C_5 + C_6 \left(\int_{Q_T^{\rho,2\rho}} |\nabla v|^2 dxdt\right)^{\frac{1}{2}} \left(\int_{\rho<|x|<2\rho} |\nabla\varphi|^2 dx\right)^{\frac{1}{2}} \le$$

$$\le C_5 + C_7 \left(\int_{Q_T^{\rho,2\rho}} |\nabla v|^2 dxdt\right)^{\frac{1}{2}} \left(\int_{\rho<|x|<2\rho} \frac{1}{|x|^2} dx\right)^{\frac{1}{2}} \le$$

$$\le C_5 + C_8 \rho^{\frac{n-2}{2}} \left(\int_{Q_T^{\rho,2\rho}} |\nabla v|^2 dxdt\right)^{\frac{1}{2}}. \tag{9}$$

Using the Caccioppoli and Harnack inequalities, we estimate the last integral as follows:

$$\left(\int_{Q_T^{\rho,2\rho}} |\nabla v|^2 dxdt\right)^{\frac{1}{2}} \le C_9 \left(\int_{Q_T^{\rho/2,3\rho}} \frac{v^2}{\rho^2} dxdt\right)^{\frac{1}{2}} \le$$

$$\leq C_{10}\left(\int_{Q_T^{\rho/2,3\rho}} \frac{(\min v)^2}{\rho^2}dxdt\right)^{\frac{1}{2}} = C_{11}\rho^{\frac{n-2}{2}} \min_{Q_T^{\rho/2,3\rho}} v\,.$$

Taking into account (8) and (9) in (7), we get

$$C_4 \ln^{1-\sigma_1}\rho \leq C_5 + C_{11}\rho^{n-2} \min_{Q_T^{\rho/2,3\rho}} v\,.$$

From here

$$\min_{Q_T^{\rho/2,3\rho}} v \geq \frac{1}{C_{11}}\rho^{2-n}(C_4 \ln^{1+\sigma_1}\rho - C_5) \geq C_{12}\rho^{2-n}\ln^{1-\sigma_1}\rho.$$

This means that for large $|x|$

$$v(x,t) \geq C_{12}\,|x|^{2-n}\ln^{1-\sigma_1}|x|\,. \tag{10}$$

Taking into account (10) and taking the test function as in (7), we obtain

$$C\int_{Q_T^{R_0,2\rho}} \frac{v\varphi}{|x|^2\ln^{\sigma_1}|x|}dxdt \geq C\int_{Q_T^{2R_0,\rho}} \frac{C_{12}\,|x|^{2-n}\ln^{1-\sigma_1}|x|}{|x|^2\ln^{\sigma_1}|x|}dxdt =$$

$$= C_{13}\int_{2R_0}^{\rho} \frac{\ln^{1-2\sigma_1} rdr}{r} \geq C_{14}\ln^{2(1-\sigma_1)}\rho. \tag{11}$$

Similar to the way we got (10), using (11), we get

$$v(x,t) \geq C_{15}\,|x|^{2-n}\ln^{2(1-\sigma_1)}|x|\,. \tag{12}$$

Then by induction we will have that for any $k \in N$

$$v(x,t) \geq C_{16}\,|x|^{2-n}\ln^{k(1-\sigma_1)}|x|\,. \tag{13}$$

Since there is $k_0 \in N$ such that $k(1-\sigma_1) \geq \sigma_2$ for $k \geq k_0$, from (13) it follows that

$$v(x,t) \geq C_{17}\,|x|^{2-n}\ln^{\sigma_2}|x|\,. \tag{14}$$

Let $v_R(x,t)$ be a solution to problem (3), (4). Since $u(x,t)$ is a super solution of equation (3), it is easy to see that

$$W_R(x,t) = u(x,t) - C_{18}v_R(x,t)$$

satisfies the inequality

$$L_0W_R + \frac{C_0}{|x|^2\ln^{\sigma_1}|x|}W_R \leq 0,$$

where $C_{18}=\frac{1}{2}\min\limits_{|x|=R_0}u(x,t)$, $W_R(x,t)>0$ for $|x|=R_0$, $W_R(x,t)\le 0$ for $|x|=R$ and $W_R(x,t+T)=W(x,t)$.

Just as we have shown that $0\le v_R\le 1$, we can also easily show that $W_R\ge 0$ for any R. This means $u(x,t)\ge C_{18}v_R(x,t)$. If we pass to the limit as $R\to+\infty$, we get $u(x,t)\ge C_{18}v(x,t)$. Then it follows from (14) that for large $|x|$

$$u(x,t)\ge C_{18}\,|x|^{2-n}\ln^{\sigma_2}|x|\,. \tag{15}$$

Since $u(x,t)$ is a positive solution of equation (1), then taking into account (15) and $b3)$, we obtain

$$0=L_0u+h(x,t,u)=L_0u+\frac{h(x,t,u)}{u}u\ge L_0u+\frac{\tilde h(x,u)}{u}u\ge$$

$$\ge L_0u+\frac{\tilde h(x,C_{18}\,|x|^{2-n}\ln^{\sigma_2}|x|)}{C_{18}\,|x|^{2-n}\ln^{\sigma_2}|x|}u\ge L_0u+\frac{1}{|x|^2}\frac{\tilde h(x,C_{18}\,|x|^{2-n}\ln^{\sigma_2}|x|)}{C_{18}\,|x|^{-n}\ln^{\sigma_2}|x|}u\ge$$

$$\ge L_0u+\frac{C_0+\varepsilon}{|x|^2}u.$$

So, $u(x,t)$ is a positive solution to the inequality

$$L_0u+\frac{C_0+\varepsilon}{|x|^2}u\le 0 \ \text{ in } \ Q_T^{R_0,\infty}.$$

And this contradicts Lemma 1.

Let $b1)$ not be fulfilled, but $b2)$ and $b3)$ be fulfilled.

If $u(x,t)$ is a positive solution to equation (1), then again using Lemma 2 and taking $\alpha=0$, $b2)$ we obtain

$$0=L_0u+h(x,t,u)=L_0u+\frac{h(x,t,u)}{u}u\ge L_0u+\frac{\tilde h(x,u)}{u}u\ge$$

$$\ge L_0u+\frac{\tilde h(x,\beta_0\,|x|^{2-n})}{\beta_0\,|x|^{2-n}}u\ge L_0u+C\frac{|x|^{-n}\ln^{-1}|x|}{\beta_0\,|x|^{2-n}}u\ge L_0u+\frac{\beta}{|x|^2\ln|x|}u\,,$$

where $\beta=\frac{C}{\beta_0}$. Hence,

$$L_0u+\frac{\beta}{|x|^2\ln|x|}u\le 0.$$

Consider the following equation:

$$L_0v+\frac{\beta}{|x|^2\ln|x|}v=0 \ \text{ in } \ Q_T^{R_0,\infty}, \tag{16}$$

where $R_0 > e$

Similar to the previous case, this equation has a positive solution $v(x,t)$ and $u(x,t) \geq C_{18} v(x,t)$.

We get a lower estimate for $v(x,t)$ with large $|x|$.

Taking the test function $\varphi(x)$ as in the previous case, from the definition of the solution we obtain

$$\int_{Q_T^{R_0,2\rho}} \frac{\beta v \varphi}{|x|^2 \ln|x|} dxdt \leq C_{19} + C_{20} \rho^{n-2} \min_{Q_T^{\rho,2\rho}} v. \tag{17}$$

Switching to polar coordinates, from here we can write the following:

$$\int_{2R_0}^{\rho} \frac{r^{n-3} m_r(v)}{\ln r} dr \leq \frac{C_{21}}{\beta} + \frac{C_{22}}{\beta} m_\rho(v) \rho^{n-3} \frac{1}{\ln \rho} \rho \ln \rho, \tag{18}$$

where by $m_r(v)$ we denote $m_r(v) = \inf\limits_{|x|=r} v$. Also denote

$$f(\rho) = \int_{2R_0}^{\rho} \frac{r^{n-3} m_r(v)}{\ln r} dr.$$

Then (18) can be written in the following form:

$$f(\rho) \leq \frac{C_{21}}{\beta} + \frac{C_{22}}{\beta} f'(\rho) \rho \ln \rho.$$

From here

$$f'(\rho) - \frac{\beta}{C_{22}} \frac{1}{\rho \ln \rho} f(\rho) + \frac{C_{21}}{C_{22}} \frac{1}{\rho \ln \rho} \geq 0. \tag{19}$$

We multiply each side of (19) by the function

$$e^{-\frac{\beta}{C_{22}} \int \frac{d\rho}{\rho \cdot \ln \rho}} = e^{-\frac{\beta}{C_{22}} \ln \ln \rho} = (\ln \rho)^{-\frac{\beta}{C_{22}}}.$$

As a result, we have

$$\left(f(\rho) (\ln \rho)^{-\frac{\beta}{C_{22}}} - \frac{C_{21}}{\beta} (\ln \rho)^{-\frac{\beta}{C_{22}}} \right)' \geq 0. \tag{20}$$

This means that the function

$$\left(f(\rho) - \frac{C_{21}}{\beta} \right) (\ln \rho)^{-\frac{\beta}{C_{22}}}$$

is non-decreasing. Then for $\rho > R_1$

$$f(\rho) - \frac{C_{21}}{\beta} \geq (\ln \rho)^{\frac{\beta}{C_{22}}} \left(f(R_1) - \frac{C_{21}}{\beta} \right) (\ln R_1)^{-\frac{\beta}{C_{22}}},$$

where R_1 is such that $f(R_1) - \frac{C_{21}}{\beta} > 0$.

Using all this, from (18) we obtain the following inequality:

$$\frac{C_{21}}{\beta} + \frac{C_{22}}{\beta} m_\rho(v)\rho^{n-2} \geq f(\rho) \geq \frac{C_{21}}{\beta} + C_{23}(\ln \rho)^{\frac{\beta}{C_{22}}},$$

where $C_{23} = \left(f(R_1) - \frac{C_{21}}{\alpha}\right)(\ln R_1)^{-\frac{\beta}{C_{22}}}$.

As a result, we have

$$m_\rho(v) \geq C_{24}\rho^{2-n}(\ln \rho)^{\frac{\beta}{C_{22}}}.$$

Hence, by Harnack's inequality

$$v(x,t) \geq C_{24}\left|x\right|^{2-n}\left(\ln |x|\right)^{\frac{\beta}{C_{22}}}.$$

Since $u(x,t) \geq C_{18}v(x,t)$, for large $|x|$ we obtain the following estimate for $u(x,t)$:

$$u(x,t) \geq C_{25}\left|x\right|^{2-n}\left(\ln |x|\right)^{\frac{\beta}{C_{22}}}. \tag{21}$$

Taking the test function $\varphi(x)$ as before, and using the estimate (19) and b2), from the definition of the solution of equation (1) we obtain

$$\int_{Q_T^{R_0,2\rho}} h(x,t,u)\varphi dx \leq C_1 + C_2\rho^{n-2} \min_{Q_T^{\rho,2\rho}} u(x,t). \tag{22}$$

Let's estimate the left-hand side from below:

$$\int_{Q_T^{R_0,2\rho}} h(x,t,u)\varphi dxdt \geq \int_{Q_T^{2R_0,\rho}} \frac{\tilde{h}(x,u)}{u} udxdt \geq$$

$$\geq \int_{Q_T^{2R_0,\rho}} \frac{\tilde{h}\left(x, C_{25}\left|x\right|^{2-n}\left(\ln|x|\right)^{\frac{\beta}{C_{22}}}\right)}{C_{25}\left|x\right|^{2-n}\left(\ln|x|\right)^{\frac{\beta}{C_{22}}}} udxdt \geq$$

$$\geq \int_{Q_T^{2R_0,\rho}} \tilde{h}\left(x, C_{25}\left|x\right|^{2-n}\left(\ln|x|\right)^{\frac{\beta}{C_{22}}}\right) dxdt \geq$$

$$\geq C_{26}\int_{2R_0<|x|<\rho} \left|x\right|^{-n}\left(\ln|x|\right)^{\frac{\beta}{C_{22}}\gamma-1} dx \geq$$

$$\geq C_{26}\int_{2R_0}^{\rho} r^{-n}(\ln r)^{\frac{\beta}{C_{22}}\gamma-1} r^{n-1} dr = C_{26}(\ln r)^{\frac{\beta\gamma}{C_{22}}}\Big|_{2R_0}^{\rho} =$$

$$= C_{26}(\ln\rho)^{\frac{\beta\gamma}{C_{22}}} - C_{26}(\ln(2R_0))^{\frac{\beta\gamma}{C_{22}}} \geq C_{26}(\ln\rho)^{\frac{\beta\gamma}{C_{22}}}.$$

As a result, from (22) we obtain

$$C_{27}(\ln\rho)^{\frac{\beta\gamma}{C_{22}}} \leq C_1 + C_2\rho^{n-2}\min_{Q_T^{\rho;2\rho}} u(x,t).$$

Hence, by Harnack's inequality, we obtain

$$u(x,t) \geq C_{28}\,|x|^{2-n}\,(\ln|x|)^{\frac{\beta\gamma}{C_{22}}}. \tag{23}$$

Doing the same, but this time using estimate (23), we get the following estimate:

$$u(x,t) \geq C_{28}\,|x|^{2-n}\,(\ln|x|)^{\frac{\beta\gamma^2}{C_{22}}}.$$

As a result, due to mathematical induction, we will have

$$u(x,t) \geq C_{29}\,|x|^{2-n}\,(\ln|x|)^{\frac{\beta\gamma^k}{C_{22}}},$$

where k is an arbitrary positive integer.

Since $\gamma > 1$, there exists k_0 such that $\frac{\beta\gamma^k}{C_{22}} > \sigma_2$ for $k \geq k_0$.

Then it is obvious that

$$u(x,t) \geq C_{30}\,|x|^{2-n}\,(\ln|x|)^{\sigma_2} \tag{24}$$

for large $|x|$. Then using (24) and b3) we get the following:

$$0 = L_0u + h(x,t,u) \geq L_0u + \tilde{h}(x,u) = L_0u + \frac{\tilde{h}(x,u)}{u}u \geq$$

$$\geq L_0u + \frac{\tilde{h}\left(x, C_{30}\,|x|^{2-n}\ln|x|^{\sigma_2}\right)}{C_{30}\,|x|^{2-n}\ln|x|^{\sigma_2}}u \geq L_0u + \frac{C_0+\varepsilon}{|x|^2}u.$$

Thus, we have obtained that there exists $\varepsilon > 0$ such that $u(x,t)$ is a super solution of the equation

$$L_0u + \frac{C_0+\varepsilon}{|x|^2}u = 0.$$

This contradicts Lemma 1. The proof of the theorem is finished.

Example 1. *Consider the equation*

$$\frac{\partial u}{\partial t} = div(A(x,t)\nabla u) + a_0(x,t)\,|u|^q$$

in Q*, where* $q > 1$*,* $a_0(x,t)$ *is a* $T-$ *periodic function in* t *and* $a_0(x,t) \geq a_0\,|x|^s \ln^\theta |x|$*,* $s > -2$*. Here* $\tilde{h}(x,u) = a_0\,|x|^s \ln^\theta |x|\,|u|^q$ *and it is clear that the condition* $a)$ *is satisfied. Then for any* $\tau > 0$

$$\tilde{h}(x, \tau\,|x|^{2-n})\,|x|^n > a_0\tau^q\,|x|^{s+(2-n)q+n} \ln^\theta |x| \geq C\,|x|^{s+2+(2-n)(q-1)} \ln^\theta |x|\,.$$

From this, it is clear that at $2+s+(2-n)(q-1) > 0, \theta \in R$ *or* $2+s+(2-n)(q-1) = 0, \theta > 0$ *condition* $b)$ *is satisfied and therefore a positive solution does not exist.*

Let now $2+s+(2-n)(q-1) = 0, -1 \leq \theta \leq 0$*. First consider the case* $-1 < \theta \leq 0$*. Let's check that in this case,* $b1), b3)$ *are fulfilled. Since* $-1 < \theta \leq 0$*, there exists* $\sigma_1 \in (0,1)$ *such that* $\sigma_1 + \theta > 0$*. Then for any* $\tau > 0$

$$\tilde{h}(x, \tau\,|x|^{2-n})\,|x|^n \ln^{\sigma_1} |x| \geq C\,|x|^{s+2+(2-n)(q-1)} \ln^{\sigma_1+\theta} |x| = C \ln^{\sigma_1+\theta} |x|\,.$$

This means that $b1)$ *is fulfilled.*

Since $q > 1$*, there exists such* $\sigma_2 > 0$ *that* $\sigma_2(q-1) > 1$*. Then*

$$\frac{\tilde{h}(x, \tau\,|x|^{2-n} \ln^{\sigma_2} |x|)}{\tau\,|x|^{-n} \ln^{\sigma_2} |x|} \geq C \ln^{\sigma_2(q-1)+\theta} |x|\,.$$

Due to the fact $\sigma_2(q-1+\theta > 0$*, it is clear that* $b3)$ *is fulfilled. Hence, there are no positive solutions in this case either.*

Let at last $\theta = -1$*. Since* $q > 1$*, there exists* $\gamma > 1$ *such that* $q - \gamma > 0$*. Then for any* $\tau > 0$*,* $\alpha > 0$*,* $\tilde{h}(x, \tau\left|x^{2-n}\right| \ln^\alpha |x|)\,|x|^n \ln^{-\gamma\alpha+1} \geq C \ln^{\alpha(q-\gamma)} |x| > 0$*. This means that in this case,* $b2)$ *is fulfilled. Similar to the previous one, it can be shown that* $b3)$ *is also fulfilled. So again, there are no positive solutions either.*

References

[1] S. Alarcon, J. Melian, A. Quaas, *Optimal Liouville theorems for supersolutions of elliptic equations with the Laplacian*, Ann Sc Norm Sup Pisa, **16**, 2016, 129-158.

[2] S. Armstrong, B. Sirakov, *Sharp Liouville results for fully nonlinear equations with power-growth nonlinearities*, Ann Sc Norm Sup Pisa, **10**, 2011, 711-728.

[3] Sh.G. Bagyrov, *On the existence of a positive solution of second-order parabolic equations with time-periodic coefficients*, Vestnik Moskovskogo Universiteta, Seriya 1, Matematika, Mekhanika, **2**, 1996, 86–89.

[4] Sh.G. Bagyrov, *On the existence of a positive solution of a nonlinear second-order parabolic equation with time-periodic coefficients*, Differentsial'nye Uravneniya, **43(4)**, 2007, 562-565.

[5] Sh.G. Bagyrov, *Absence of positive solutions of a second-order semilinear parabolic equation with time-periodic coefficients*, *Differential Equations*, **50(4)**, 2014, 1-6.

[6] Sh.G. Bagyrov, *On non-existence of positive periodic solution for second order semilinear parabolic equation*, Azerbaijan J. of Math, **8(2)**, 2018, 163-180.

[7] Sh.G. Bagyrov, *Nonexistence of global positive solutions of weakly coupled systems of semilinear parabolic equations with time-periodic coefficients*, Differential Equations, **56**, 2020, 721–733.

[8] A. Bahri, P.-L. Lions, *On the existence of positive solutions of semilinear elliptic equations in unbounded domains*, Ann. Inst. H. Poincare Anal. Non Lineaire, **14**, 1997, 365-413.

[9] P. Baras, J.A. Goldstein, *The heat equation with a singular potential*, Transactions of the American Mathematical Society, **284(1)**, 1984, 121-139.

[10] A. Beltramo, P. Hess, *On the principal eigenvalue of a periodic-parabolic operator*, Communications in Partial Differential Equations, **9(9)**, 1984, 919-941.

[11] M. Bidaut-V'eron, S.Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, J D'Analyse Math, **84**, 2001, 1-49.

[12] H. Chen, R. Peng, F. Zhou, *Nonexistence of positive super solution to a class of semilinear elliptic equations and systems in an exterior domain*, Science China Mathematics, **63(7)**, 2019, 1-16.

[13] K. Deng, H.A. Levine, *The role of critical exponents in blow-up theorems: the sequel*, J. Math. Anal. Appl., **243(1)**, 2000, 85-126.

[14] M.J. Esteban, *On periodic solutions of superlinear parabolic problems*, Transactions of the American Mathematical Society, **293(1)**, 1986, 171-189.

[15] M.J. Esteban, *A remark on the existence of positive periodic solutions of superlinear parabolic problems*, Proceedings of the American Mathematical Society, **102(1)**, 1988, 131-136.

[16] H. Fujita, *On the blowing-up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ, Tokyo, Sect. I, **13**, 1966, 109-124.

[17] B. Gidas, J. Spruck, *Global and local behavior of positive solutions of linear elliptic equations*, Comm. Pare. Appl. Math., **34**, 1981, 525-598.

[18] Y. Giga, N. Mizoguchi, *On time periodic solutions of the Dirichlet problem for degenerate parabolic equations of nondivergence type*, Journal of Mathematical Analysis and Applications, **201(2)**, 1996, 396-416.

[19] N. Hirano, N. Mizoguchi, *Positive unstable periodic solutions for superlinear parabolic equations*, Proceedings of the American Mathematical Society, **123(5)**, 1995, 1487-1495.

[20] J. Huska, *Periodic solutions in semilinear parabolic problems*, Acta Math. Univ. Comenianae, **LXXI(1)**, 2002, 19-26.

[21] V. Kondratiev, V. Liskevich, Z. Sobol, *Second-order semilinear elliptic inequalities in exterior domains*, J. Differential Equations, **187**, 2003, 429-455.

[22] A.A. Kon'kov, *On solutions of quasi-linear elliptic inequalities containing terms with lower-order derivatives*, Nonlinear Anal., **90**, 2013, 121-134.

[23] V.V. Kurta, *On the nonexistence of positive solutions to semilinear elliptic equations*, Proceedings of the Steklov Institute of Mathematics, **227**, 1999, 155-162.

[24] H.A. Levine, *The role of critical exponents in blowup theorems*, SIAM Review, **32(2)**, 1990, 262-288.

[25] V. Liskevich, I.I. Skrypnik, I.V. Skrypnik, *Positive super-solutions to general nonlinear elliptic equations in exterior domains*, Manuscripta Math., **115**, 2004, 521-538.

[26] E. Mitidieri, S. Pohozaev, *Absence of positive solutions for quasilinear elliptic problems in RN*, Proc. Steklov Inst. Math., **227**, 1999, 186-216 (in Russian).

[27] E. Mitidieri, S.Z. Pohozhayev, *A priori estimations and no solutions of nonlinear partial equations and inequalities*, Proc. of V.A. Steklov Mathematics Institute of NAS, **234**, 2001, 1-383.

[28] R.G. Pinsky, *Existence and nonexistence of global solutions for $u_t = \Delta u + a(x)\ u^p$ in R^d*, Journal of Differential Equations,**133**, 1997, 152-177.

[29] P. Quittner, *Multiple equilibria, periodic solutions and a priori bounds for solutions in superlinear parabolic problems*, Nonlinear Differential Equations and Applications, **11(2)**, 2004, 237-258.

[30] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, A.P. Mikhaylov, *Blow up of solutions in problems for quasilinear parabolic equations*, Nauka, Moscow, 1987 (in Russian).

[31] T.I. Seidman, *Periodic solutions of a non-linear parabolic equation*, Journal of Differential Equations, **19(2)**, 1975, 242-257.

Shirmayil G. Bagyrov
Baku State University, Baku, Azerbaijan
Institute of Mathematics and Mechanics Baku, Azerbaijan
E-mail: sh_bagirov@yahoo.com

Received 18 May 2023
Accepted 21 August 2023