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Global Existence and Uniqueness of the Solution to a Model of the Pattern Forming Process in E. coli Colonies

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Abstract. In this paper we consider a Parabolic systems modeling bacteria pattern formation proposed by Aotani et al.(Jap. J. Indust. Appl. Math., 27:5–22, 2010). According to Yagi's arguments (in Abstract Parabolic Evolution equations), we reduce them to corresponding evolution equations and show the existence of time global solutions.

Key Words and Phrases: chemotaxis, bacterial colony patterns, nonlinear parabolique system, global existence.

2010 Mathematics Subject Classifications: 35A01, 35K58, 35Q92

1. Introduction

Aotani et al [1] presented a macroscopic continuum model of E. coli pattern formation that incorporates cell diffusion, chemotaxis, population growth and conversion to an inactive state. This model satisfactorily reproduces the observed spot patterns, supporting the view that these patterns are indeed a result of self-organization, and allows us to infer plausible minimal mechanisms that generate the observed patterns. Then they will derive the following system

$$\begin{cases} \partial_t u = D_u \Delta u - \nabla.(u\nabla \frac{\chi v^2}{k_1+v^2}) - \frac{k_2}{k_3+n}u + k_8 g\left(u\right) nu, \ (t,x) \in \mathbb{R}^+ \times \Omega \\ \partial_t v = D_v \Delta v - k_6 v + k_7 u, \ (t,x) \in \mathbb{R}^+ \times \Omega \\ \partial_t n = D_n \Delta n - k_8 g\left(u\right) nu, \ (t,x) \in \mathbb{R}^+ \times \Omega \\ \partial_t w = \frac{k_2}{k_3+n}u, \ (t,x) \in \mathbb{R}^+ \times \Omega, \\ \\ \partial_\nu u = \partial_\nu v = \partial_\nu n = \partial_\nu w = 0 \text{ on } \partial\Omega, \\ \\ u(0,.) = u_0, \ v(0,.) = v_0, \ n(0,.) = n_0, \ w(0,.) = w_0 \text{ in } \Omega. \end{cases} \tag{1}$$

Here $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded domain with C^3 boundary $\partial\Omega$, the initial data u_0, v_0, n_0, w_0 are assumed to be nonnegative and ∂_n denotes the derivative with respect to the outer normal of $\partial\Omega$. This system is a mathematical model describing the bacteria pattern formation, for the active cells u, inactive cells w, chemoattractant v. Denote n by the concentration of nutrient consumed by the cell density . The coefficients $D_u, D_v, D_n, \chi, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$ are the given positive constants. $g(u)\,n$ is the growth rate of the cell population, simplified as follows:

$$g(u) = \frac{1}{2} + \frac{1}{2}\tanh(k_4 u - k_5). \tag{2}$$

We dedicate the first part of this work to the study of the local existence in time in the case of two and three dimensions for the model (1). We want to present existence and uniqueness results in a sophisticated way enough to apply [8] .

In this way, we will consider the Cauchy problem (1) for a semilinear evolution:

$$\begin{cases} \frac{dU}{dt} + AU = F(U), \\ \qquad U(0) = U_0. \end{cases} \tag{3}$$

The equation in (3) contains two operators, a linear operator A and a non-linear operator F. We should prove in a Banach space X, that A is a sectorial operator of X with angle $< \frac{\pi}{2}$.

Meanwhile, the operator F is a nonlinear operator from another Banach space $\mathcal{D}(A^{\frac{\delta}{2}})$ into X, with some exponent $1 \leq \delta < 2$, and F satisfies a Lipschitz condition described by two fractional powers $A^{\frac{1}{2}}$ and $A^{\frac{\delta}{2}}$.

Hence, the existence and uniqueness of a local in time solution U is ensured for any $U_0 \in \mathcal{D}(A^{\frac{1}{2}})$ thanks to [8, Theorem 4.1, p. 178].

Secondly, as [8, Corollary 4.1, p. 185] shows, the a priori estimates for local solutions of (3) with respect to the form $\|A^{\frac{1}{2}}U(t)\| \leq c\left(\|A^{\frac{1}{2}}U_0\|\right), (t \in [0, T_U])$ ensure extension of local solutions without limit so as to construct the global solutions for nonnegative initial data $U_0 \in \mathcal{D}(A^{\frac{1}{2}})$. We recall that, according to Theorem [8, Corollary 4.1, p. 185], the interval $[0, T_{U_0}]$ on which the local solution was constructed is determined by the norms $\|U_0\|_{\mathcal{D}(A^{\frac{1}{2}})}$ only. This fact immediately provides the global existence of solutions which is utilized very often in applications.

The main tool in this second step is the maximum regularity of Sobolev [6] with Sobolev's injections [3].

2. Local in time existence of a solution

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$, and, for $1 \leq p \leq \infty$, $L^p(\Omega)$ be the usual Lebesgue space endowed with the norm $\|.\|_{L^p(\Omega)}$. Let for $s > 0$, $H^s(\Omega)$ be the usual fractional Sobolev space. We define for Ω of C^3 class, and for $\frac{3}{2} < s \leq 3$,

$$H_N^s(\Omega) = \{u \in H^s(\Omega),\ \partial_n u \ = \ 0 \text{ on } \ \partial\Omega\ \},$$

and for $s < \frac{3}{2}$, we set $H_N^s(\Omega) = H^s(\Omega)$, with the norm $\|.\|_{H^s(\Omega)}$.

We denote

$$\begin{aligned}\mathcal{K} \ &= \ \big\{U_0 = (u_0, v_0, n_0, w_0)^t;\ 0 \leq u_0 \in L^2(\Omega), \\ &\quad\ 0 \leq v_0, n_0, w_0 \in H^1(\Omega)\big\}.\end{aligned}$$

The aim of this section is to prove the following theorem:

Theorem 1. *For any $U_0 = (u_0, v_0, n_0, w_0) \in \mathcal{K}$, (1) possesses a unique local solution in the function space*

$$\begin{aligned}u &\in C(]0, T_{U_0}]; H_N^2(\Omega)) \cap C([0, T_{U_0}]; H^1(\Omega)) \cap C^1(]0, T_{U_0}]; L^2(\Omega)), \\ v, n &\in C(]0, T_{U_0}]; H_N^3(\Omega)) \cap C([0, T_{U_0}]; H_N^2(\Omega)) \cap C^1(]0, T_{U_0}]; H^1(\Omega)), \\ w &\in C([0, T_{U_0}]; H^1(\Omega)) \cap C^1(]0, T_{U_0}]; H^1(\Omega)). \qquad (4)\end{aligned}$$

In addition, for all $t \in]0, T_{U_0}]$ the solution satisfies the estimate

$$\|u\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega)} + \|n\|_{H^2(\Omega)} + \|w\|_{H^1(\Omega)} \leq C_{U_0}, \qquad (5)$$

with some constant $T_{U_0}, C_{U_0} > 0$ depending on the norms $\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^2(\Omega)} + \|n_0\|_{H^2(\Omega)} + \|w_0\|_{H^1(\Omega)}$.

2.1. Proof of Theorem 1

We formulate problem (1) as the Cauchy problem for an abstract semilinear equation

$$\begin{cases} \frac{dU}{dt} + AU = F(U), \\ U(0) = U_0, \end{cases} \qquad (6)$$

in a Banach space X as follows:

$$X = \big\{U = (u, v, n, w)^t\ ;\ u \in L^2(\Omega),\ v, n, w \in H^1(\Omega)\big\},$$

endowed with the norm

$$\|(u, v, n, w)^t\| = \|u\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega)} + \|n\|_{H^1(\Omega)} + \|w\|_{H^1(\Omega)}.$$

2.1.1. Basic properties of operator A

We introduce a diagonal matrix operator A acting in X as follows:

$$\begin{aligned} A &= diag\{A_u, A_v, A_n, A_w\} \\ &= diag\{-D_u\triangle + 1, -D_v\triangle + k_6, -D_n\triangle + 1, 1\}. \end{aligned}$$

A is a sectorial operator of X, the spectrum of which is contained in a sectorial domain $\sigma(A) \subset \sum_\omega = \{\lambda \in \mathbb{C}, |\arg\lambda| < \omega_A\}$ with some angle $0 < \omega_A < \frac{\pi}{2}$. According to [8, Theorem 2.4, p. 61], which ensures that the resolvent satisfies for $\lambda \notin \sigma(A)$ the estimate

$$\begin{aligned} &\left\|(\lambda - A)^{-1}\right\| \leq \left\|(\lambda - A_u)^{-1}\right\|_{\mathcal{L}(L^2(\Omega))} + \left\|(\lambda - A_v)^{-1}\right\|_{\mathcal{L}(H^1(\Omega))} \\ &+ \left\|(\lambda - A_n)^{-1}\right\|_{\mathcal{L}(H^1(\Omega))} + \left\|(\lambda - 1)^{-1}\right\|_{\mathcal{L}(H^1(\Omega))} \\ &\leq \left(\tfrac{1+D_u}{\min\{1;D_u\}} + \tfrac{1+D_v}{\min\{k_6;D_v\}} + \tfrac{1+D_n}{\min\{1;D_n\}} + 1\right)\frac{1}{|\lambda|}. \end{aligned}$$

In $L_2(\Omega)$, under the Neumann boundary conditions on $\partial\Omega$, $\mathcal{D}(A_u) = H^2_N(\Omega)$ (see [5, Theorem 3.2.1.3]), and according to [8, Theorem 16.7, p. 547], (see also [4]),

$$\mathcal{D}(A_u^\theta) = \begin{cases} H^{2\theta}(\Omega), & 0 \leq \theta < \frac{3}{4}, \\ H_N^{2\theta}(\Omega), & \frac{3}{4} < \theta \leq 1, \end{cases} \tag{7}$$

with norm equivalence

$$c_\Omega^{-1}\|u\|_{H^{2\theta}(\Omega)} \leq \left\|A_u^\theta u\right\|_{L^2(\Omega)} \leq c_\Omega\|u\|_{H^{2\theta}(\Omega)}, \quad u \in \mathcal{D}(A_u^\theta). \tag{8}$$

Here, A_v and A_n are realizations of $-D_v\triangle + k_6$ and $-D_n\triangle + 1$ under the homogeneous Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$ and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, respectively. Thanks to [8, Theorem 2.9, p.66], A_v and A_n are positive definite self-adjoint operators of $H^1(\Omega)$ with domains $H^3_N(\Omega) = \left\{u \in H^3(\Omega), \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}$. Furthermore, according to [8, Theorem 16.1, p.528], the domains of their fractional powers are characterized by

$$\mathcal{D}(A_v^\theta) = \mathcal{D}(A_n^\theta) = \begin{cases} H^{2\theta+1}(\Omega), & 0 \leq \theta < \frac{1}{4}, \\ H_N^{2\theta+1}(\Omega), & \frac{1}{4} < \theta \leq 1, \end{cases} \tag{9}$$

with norm equivalence for $i = n, v$

$$c_\Omega^{-1}\|u\|_{H^{2\theta+1}(\Omega)} \leq \left\|A_i^\theta u\right\|_{H^1(\Omega)} \leq c_\Omega\|u\|_{H^{2\theta+1}(\Omega)}, \quad u \in \mathcal{D}(A_i^\theta), \tag{10}$$

$c_\Omega > 0$ being determined by Ω.

Moreover, it is clear that $A^\theta = diag\left\{A_u^\theta, A_v^\theta, A_n^\theta, A_w^\theta\right\}$. Thanks to [7], we have $\mathcal{D}(A^\theta) = [X, \mathcal{D}(A)]_\theta$. Then

$$\begin{aligned}
\mathcal{D}(A^\theta) &= \left\{U = (u,v,n,w)^t\,; u \in H^{2\theta}(\Omega), n, v \in H^{2\theta+1}(\Omega), w \in H^1(\Omega)\right\}, 0 < \theta < \frac{1}{4},\\
\mathcal{D}(A^\theta) &= \left\{U = (u,v,n,w)^t\,; u \in H^{2\theta}(\Omega), n, v \in H_N^{2\theta+1}(\Omega), w \in H^1(\Omega)\right\}, \frac{1}{4} < \theta < \frac{3}{4},\\
\mathcal{D}(A^\theta) &= \left\{U = (u,v,n,w)^t\,; u \in H_N^{2\theta}(\Omega), n, v \in H_N^{2\theta+1}(\Omega),\ w \in H^1(\Omega)\right\}, \frac{3}{4} < \theta \le 1.
\end{aligned} \tag{11}$$

2.1.2. Construction of local solutions in Sobolev space

The nonlinear operator F from $D(A^{\frac{\delta}{2}})(\frac{d}{2} \le \delta < 2, d = 2, 3)$ into X is defined by

$$F(U) = \left(-\nabla.(u\nabla\tfrac{\chi v^2}{k_1+v^2}) - \tfrac{k_2 u}{k_3+n} + g(u)\,nu + u,\ k_7 u,\ k_8 g(u)\,nu + n, \tfrac{k_2}{k_3+n}u + w\right)^t,$$

where $g(u) = \frac{1}{2} + \frac{1}{2}\tanh(k_2 u - k_3)$.

Let $U, V \in D(A^{\frac{\delta}{2}})$. Since $U = (u_1, v_1, n_1, w_1)^t$, $V = (u_2, v_2, n_2, w_2)^t$, we have

$$\begin{aligned}
&\|F(U) - F(V)\|\\
&\le \left\|\nabla\cdot\left(\tfrac{\chi v_1^2}{k_1+v_1^2} n_1\nabla v_1 - \tfrac{\chi v_2^2}{k_1+v_2^2} n_2\nabla v_2\right)\right\|_{L^2(\Omega)} + (1+k_7)\|u_1 - u_2\|_{H^1(\Omega)}\\
&\quad + \|w_1 - w_2\|_{H^1(\Omega)} + \|n_1 - n_2\|_{H^1(\Omega)} +\\
&+ 2k_5\|g(u_1)\,n_1 u_1 - g(u_2)\,n_2 u_2\|_{L^2(\Omega)} + 2k_2\left\|\tfrac{u_1}{k_3+|n_1|} - \tfrac{u_2}{k_3+|n_2|}\right\|_{H^1(\Omega)}.
\end{aligned} \tag{12}$$

We apply the Hölder's inequality

$$\begin{aligned}
\left\|\nabla.(\tfrac{\chi v^2}{k_1+v^2} n\nabla v)\right\|_{L^2(\Omega)} &\le \left(\left\|\tfrac{\chi v^2}{k_1+v^2}\right\|_{L^\infty(\Omega)} + \left\|\nabla\tfrac{\chi v^2}{k_1+v^2}\right\|_{L^4(\Omega)}\right)\\
&\left(\|n\|_{L^\infty(\Omega)} + \|\nabla n\|_{L^4(\Omega)}\right)\left(\|\nabla v\|_{L^4(\Omega)} + \|\triangle v\|_{L^2(\Omega)}\right).
\end{aligned}$$

In the sequel, we need the following embeddings: $H_N^\delta(\Omega) \hookrightarrow L^\infty(\Omega)$, $H^{\delta-1}(\Omega) \hookrightarrow L^4(\Omega)$, $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$. This leads us to choose the parameter δ such that $(\frac{d}{2} \le \delta < 2, d = 2, 3)$ in order to ensure for $n \in H^\delta(\Omega)$, $v \in H_N^2(\Omega)$ the following inequality:

$$\left\|\nabla.(\tfrac{\chi v^2}{k_1+v^2} n\nabla v)\right\|_{L^2(\Omega)} \le c_\Omega \left\|\tfrac{\chi v^2}{k_1+v^2}\right\|_{W^{1,4}(\Omega)} \|v\|_{H^2(\Omega)} \|n\|_{H^\delta(\Omega)}. \tag{13}$$

Using the same arguments of [8, (1.93), p. 50], and combining this with the embeddings $H_N^2(\Omega) \hookrightarrow H_N^\delta(\Omega) \hookrightarrow W^{1,4}(\Omega)$ ($\frac{d}{2} \le \delta < 2, d = 2,3$), we get

$$\begin{aligned}
&\left\| \nabla. \left(\tfrac{\chi v_1^2}{k_1+v_1^2} n_1 \nabla v_1 - \tfrac{\chi v_2^2}{k_1+v_2^2} n_2 \nabla v_2 \right) \right\|_{L^2(\Omega)} \\
&\le c_\Omega \left(1 + \|v_1\|_{H^2(\Omega)} \right)^2 \|u_1 - u_2\|_{H^\delta(\Omega)} \\
&+ c_\Omega \|u_2\|_{H^\delta(\Omega)} \left(1 + \|v_1\|_{H^2(\Omega)} + \|v_2\|_{H^2(\Omega)} \right) \|v_1 - v_2\|_{H^2(\Omega)} .
\end{aligned} \tag{14}$$

Since $H_N^2(\Omega) \hookrightarrow H_N^\delta(\Omega) \hookrightarrow L^4(\Omega)$, ($\frac{d}{2} \le \delta < 2, d = 2,3$), we have

$$\begin{aligned}
&\|g(u_1) u_1 n_1 - g(u_2) u_2 n_2\|_{H^1(\Omega)} \\
&\le c_\Omega \left(\|u_1\|_{H^1(\Omega)} + \|u_1\|_{H^1(\Omega)}^2 \right) \|n_1\|_{H^2(\Omega)} \|u_1 - u_2\|_{H^\delta(\Omega)} \\
&+ c_\Omega \|u_2\|_{H^1(\Omega)} \|u_2\|_{H^\delta(\Omega)} \|n_1 - n_2\|_{H^2(\Omega)} .
\end{aligned} \tag{15}$$

In view of $H_N^2(\Omega) \hookrightarrow H_N^\delta(\Omega) \hookrightarrow H^1(\Omega)$ ($\frac{d}{2} \le \delta < 2, d = 2,3$), we obtain that

$$\begin{aligned}
&\left\| \tfrac{k_2}{k_3+|n_1|} u_1 - \tfrac{k_2}{k_3+|n_2|} u_2 \right\|_{H^1(\Omega)} \le \|u_1 - u_2\|_{H^\delta(\Omega)} \|n_2\|_{H^2(\Omega)} \left\| \tfrac{k_2}{(k_3+n_2)^2} \right\|_{L^\infty(\Omega)} \\
&+ \left\| \tfrac{k_2(n_1+n_2)}{(k_3+|n_1|)^2 (k_3+|n_2|)^2} \right\|_{L^\infty(\Omega)} \left(\|n_1\|_{H^2(\Omega)} + \|n_2\|_{H^2(\Omega)} \right) \|n_1 - n_2\|_{H^2(\Omega)} .
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\| \tfrac{k_2}{k_3+|n_1|} u_1 - \tfrac{k_2}{k_3+|n_2|} u_2 \right\|_{H^1(\Omega)} \\
\le\ & \|n_2\|_{H^2(\Omega)} \|u_1 - u_2\|_{H^\delta(\Omega)} + \left(\|n_1\|_{H^2(\Omega)} + \|n_2\|_{H^2(\Omega)} \right) \|n_1 - n_2\|_{H^{1+\delta}(\Omega)} .
\end{aligned} \tag{16}$$

Since $H_N^\delta(\Omega) \hookrightarrow H^1(\Omega)$ ($\frac{d}{2} \le \delta < 2, d = 2,3$), we have

$$\begin{aligned}
&\|u_1 - u_2\|_{H^1(\Omega)} + \|n_1 - n_2\|_{H^1(\Omega)} + \|w_1 - w_2\|_{H^1(\Omega)} \\
\le\ & \|u_1 - u_2\|_{H^\delta(\Omega)} + \|n_1 - n_2\|_{H^{1+\delta}(\Omega)} + \|w_1 - w_2\|_{H^1(\Omega)} .
\end{aligned} \tag{17}$$

We substitute (14), (15), (16) and (17) into (12), for $\frac{d}{2} \le \delta < 2, (d = 2,3)$. Then

$$\begin{aligned}
\|F(U) - F(V)\| &\le c_\Omega \left(1 + \|u_1\|_{H^1(\Omega)} + \|u_2\|_{H^1(\Omega)} + \|v_1\|_{H^2(\Omega)} + \|v_2\|_{H^2(\Omega)} \right)^2 \\
&\left(1 + \|n_1\|_{H^2(\Omega)} + \|n_2\|_{H^2(\Omega)} \right) \Big[(\|u_1 - u_2\|_{H^\delta(\Omega)} + \|n_1 - n_2\|_{H^{1+\delta}(\Omega)} + \\
&\|w_1 - w_2\|_{H^1(\Omega)}) + \left(\|n_2\|_{H^\delta(\Omega)} + \|u_2\|_{H^{1+\delta}(\Omega)} \right) \left(\|v_1 - v_2\|_{H^2(\Omega)} + \|n_1 - n_2\|_{H^2(\Omega)} \right) \Big] .
\end{aligned}$$

Therefore, in view of (11), (8) and (10), for ($\frac{d}{2} \leq \delta < 2, d = 2,3$), we deduce

$$\begin{aligned} \|F(U) - F(V)\| \;\leq\; & c_\Omega(\left\|A^{\frac{1}{2}}U\right\| + \left\|A^{\frac{1}{2}}V\right\| + 1)^3 \left[\left\|A^{\frac{\delta}{2}}(U-V)\right\| \right. \\ & \left. +(\left\|A^{\frac{\delta}{2}}U\right\| + \left\|A^{\frac{\delta}{2}}V\right\|) \left\|A^{\frac{1}{2}}(U-V)\right\|\right]. \end{aligned}$$

[8, Theorem 4.1, p. 178] then provides the existence of local solution. Indeed, for any $U_0 \in \mathcal{K}$, (6) possesses a unique local solution in the function space (4). Furthermore, the solution satisfies the estimate $\|A^{\frac{1}{2}}U\| \leq C_{U_0}$. Here, $C_{U_0}, T_{U_0} > 0$ is determined by the norm $\|U_0\|_{\mathcal{D}(A^{\frac{1}{2}})}$ only.

3. Nonnegativity of local solutions

We shall show that the local solution constructed above is nonnegative for $U_0 \in \mathcal{K}$.

In this section we assume that $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ is a bounded domain with C^3 boundary.

We denote by G the C^1 function defined by $G(s) = \begin{cases} \frac{1}{2}s^2, & s < 0, \\ 0, & s \geq 0. \end{cases}$

Proposition 1. *Under the assumptions of Theorem 1, we have*

$$n(t,x) \geq 0, \; x \in \Omega, \; t \geq 0. \tag{18}$$

Proof. We set $\psi(t) = \int_\Omega G(n)dx$. Using the third equation of system (1), observing that $G'(n) = n$ if $n < 0$ and $G'(n) = 0$ if $n \geq 0$ and $G'(n) \in H^1(\Omega)$ for $n \in H^1(\Omega)$, by the Hölder's inequality, we have

$$\begin{aligned} \psi'(t) &= -D_n \int_\Omega \left|\nabla G'(n)\right|^2 dx - \int_\Omega k_8 g(u)\, unG'(n) \\ &\leq k_8 \|g(u)\, u\|_{L^\infty(\Omega)} \|G'(n)\|^2_{L^2(\Omega)}. \end{aligned}$$

(5) shows that $\|g(u)\,u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} \leq c_{T,U_0}$. Then $\psi'(t) \leq c_{T,U_0}\psi(t)$. By Gronwall's inequality $\psi(t) \leq \psi(0)\exp(t\; c_{T,U_0})$. Then $\psi(0) = \int_\Omega G(n_0(t,x))dx = 0$. Then $\psi(t) = 0$, consequently, $n \geq 0$. ◄

Proposition 2. *Under the assumptions of Theorem 1, we have*

$$u(t,x) \geq 0, \; x \in \Omega, \; t \geq 0. \tag{19}$$

Proof. We note $\psi(t)=\int_\Omega G(u(t,x))dx$. We have $\psi'(t)=\int\limits_\Omega G'(u)u_t dx$. Then

$$\begin{aligned}\psi'(t) &= D_u\int\limits_\Omega G'(u)\triangle u dx-\int\limits_\Omega G'(u)\nabla.(u\nabla\tfrac{\chi v^2}{k_1+v^2})dx\\ &\quad -\int\limits_\Omega \tfrac{k_2 u}{k_3+n}G'(u)dx+\int\limits_\Omega k_8 g\left(u\right)nuG'(u)dx.\end{aligned}$$

Observing that $G'(u)=u$ if $u<0$ and $G'(u)=0$ if $u\geq 0$ and $G'(u)\in H^1(\Omega)$ for $u\in H^1(\Omega)$, and assuming $\frac{\partial v}{\partial n}=0$ in $\partial\Omega$, by the Hölder's inequality, we have

$$\begin{aligned}\psi'(t) &\leq -D_u\|\nabla(G'(u))\|^2_{L^2(\Omega)}+\|\nabla\tfrac{\chi v^2}{k_1+v^2}\|_{L^\infty(\Omega)}\|\nabla G'(u)\|_{L^2(\Omega)}\|G'(n)\|_{L^2(\Omega)}\\ &\quad+\left(\frac{k_2}{k_3}+\|k_8 g\left(u\right)n\|_{L^\infty(\Omega)}\right)\|G'(u)\|^2_{L^2(\Omega)}.\end{aligned}\tag{20}$$

(5) shows that

$$\|\nabla\tfrac{\chi v^2}{k_1+v^2}\|_{L^\infty(\Omega)}+\|k_8 g\left(u\right)n\|_{L^\infty(\Omega)}\leq c_\Omega\|v\|_{H^2(\Omega)}+c_\Omega\|n\|_{H^2(\Omega)}\leq C_{U_0},$$

for $0\leq t\leq T_{U_0}$. Therefore, (20) becomes

$$\begin{aligned}&\|\nabla\tfrac{\chi v^2}{k_1+v^2}\|_{L^\infty(\Omega)}\|\nabla G'(u)\|_{L^2(\Omega)}\|G'(u)\|_{L^2(\Omega)}\\ \leq\ &\tfrac{D_n}{2}\|\nabla(G'(u))\|^2_{L^2(\Omega)}+C_{U_0}\|G'(u)\|^2_{L^2(\Omega)}.\end{aligned}\tag{21}$$

Thus, in view of (21) and (20), $\psi'(t)\leq c_{T,U_0}\psi(t)$. By Gronwall's inequality $\psi(t)\leq\psi(0)\exp\left(t\ c_{T,U_0}\right)$. Then $\psi(0)=\int_\Omega G(u_0(t,x))dx=0$ so that $\psi(t)=0$, hence $u\geq 0$. ◄

Proposition 3. *Under the assumptions of Theorem 1, we have*

$$v\left(t,x\right)\geq 0,\ x\in\Omega,\ t\geq 0.\tag{22}$$

Proof. We set $\psi(t)=\int\limits_\Omega G(v)dx$. Observing that $G'(v)=v$ if $v<0$ and $G'(v)=0$ if $v\geq 0$ and $G'(v)\in H^1(\Omega)$ for $v\in H^1(\Omega)$, and assuming $\frac{\partial v}{\partial n}=0$ in $\partial\Omega$,

$$\psi'(t)=-D_v\int\limits_\Omega\left|\nabla G'(v)\right|^2dx-\int\limits_\Omega k_6 vG'(v)+\int\limits_\Omega k_7 uG'(v).$$

since $G'(v)\leq 0$ and $vG'(v)\geq 0$. Hence, $\psi'(t)\leq 0$. By Gronwall's inequality $\psi(t)\leq\psi(0)$, then $\psi(0)=\int_\Omega G(v_0(t,x))dx=0$, so that $\psi(t)=0$, consequently $v\geq 0$.

4. Global solutions

In this section we assume that $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded domain with C^3 boundary. As [8, Corollary 4.1, p. 185] shows, the a priori estimates for local solutions of (6) with respect to the $A^{\frac{1}{2}}U(t)$ norm ensure extension of local solutions without limit so as to construct the global solutions. ◄

4.1. Preliminaries

For later use we state the following auxiliary results:

Lemma 1. *Under the assumptions of Theorem 1, for* $0 \le t \le T_U$

$$\begin{aligned} & k_7\|n(t)\|_{L^1(\Omega)} + (k_3 + k_5)\|i(t)\|_{L^1(\Omega)} + k_6\|v(t)\|_{L^1(\Omega)} \\ \le \quad & k_7\|n_0\|_{L^1(\Omega)} + (k_3 + k_5)\|i_0\|_{L^1(\Omega)} + k_6\|v_0\|_{L^1(\Omega)}. \end{aligned} \tag{23}$$

Under the same assumption,

$$\|n(t)\|_{L^\infty(\Omega)} \le \|n_0\|_{L^\infty(\Omega)}. \tag{24}$$

Proof. Thanks to the homogeneous boundary condition $\nabla n.\vec{n} = 0$ and $\nabla v.\vec{n} = \nabla i.\vec{n} = 0$ on $\partial\Omega$, we directly integrate and add the three equations to (1), and as $n, v, i \ge 0$, we have $\frac{d}{dt}\left(k_7\|n\|_{L^1(\Omega)} + (k_3 + k_5)\|i\|_{L^1(\Omega)} + k_6\|v\|_{L^1(\Omega)}\right) \le 0$, and mass conservation (23) is satisfied.

Now, n satisfies the variation-of-constants formula,

$$n(t) = e_0^{-tA_n} n_0 + \int_0^t e^{-(t-s)A_n} (1 - k_8 g(u) u) n ds, \; 0 \le t \le T_U,$$

as $n, u \ge 0$, consequently,

$$\|n\|_{L^\infty(\Omega)} \le$$

$$\left\|e^{-tA_v}\right\|_{\mathcal{L}(L^\infty(\Omega))} \|n_0\|_{L^\infty(\Omega)} + \int_0^t \left\|e^{-\frac{(t-s)}{2}A_v}\right\|_{\mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))} \|n\|_{L^\infty(\Omega)} ds,$$

from the estimate in [2, Theorem 12.3, p. 250], and the Gronwall's inequality, we have for $0 \le t \le T_U$:

$$\|n\|_{L^\infty(\Omega)} \le c_\Omega \|n_0\|_{L^\infty(\Omega)},$$

then (24) is satisfied. ◄

Lemma 2. *Let p be a fixed parameter satisfying $1 \le p < \infty$ and suppose that $U_0 = (u_0, v_0, n_0, w_0) \in \mathcal{K}$. There exists a constant $c_\Omega > 0$ (depending on Ω) for any $0 \le t \le T_U$ such that*

$$\|v\|_{L^p(\Omega)} \le c_\Omega(\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^p(\Omega)} + \|n_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)}). \tag{25}$$

Proof. The second equation of (1) is written as the formula,

$$v\,(t) = e^{-tA_v} v_0 + k_7 \int_0^t e^{-(t-s)A_v} u\; ds,\; 0 \le t \le T_U, \tag{26}$$

in $L^p(\Omega)$ $(1 \le p < \infty)$. Then

$$\|v\|_{L^p(\Omega)} \le \left\|e^{-tA_v}\right\|_{\mathcal{L}(L^p(\Omega))} \|v_0\|_{L^p(\Omega)}$$
$$+ k_7 \int_0^t \left\|e^{-\frac{(t-s)}{2}A_v}\right\|_{\mathcal{L}(L^p(\Omega), L^1(\Omega))} \left\|e^{-\frac{(t-s)}{2}A_v}\right\|_{\mathcal{L}(L^1(\Omega))} \|u\|_{L^1(\Omega)}\, ds,$$

. From the estimate in [2, (2.128), p. 102], and the formula $\mu^{-\frac{1}{p}}\Gamma\left(\frac{1}{p}\right) = \int_0^{+\infty} s^{z-1}e^{-\mu s} ds \;\left(Re(z) \in \mathbb{R}_+^*\right)$, it follows that for $0 \le t \le T_U$

$$\|v\|_{L^p(\Omega)} \le c_\Omega \|v_0\|_{L^p(\Omega)} + c_\Omega \mu^{-\frac{1}{p}} \Gamma\left(\frac{1}{p}\right) \|u\|_{L^1(\Omega)}.$$

As given (23), we have (25). ◄

4.1.1. A Priori Estimates

In this section we will prove the following result:

Proposition 4. *Under the assumptions of Theorem 1, we have for any $0 \le t \le T_U$*

$$\|A^{\frac{1}{2}}U\| \le c(\|A^{\frac{1}{2}}U_0\|)\left(t^2 + 1\right). \tag{27}$$

Proof.

1^st^step: We denote $\varphi\,(v) = (k_1 + v)^{\alpha}$ $(\alpha < 0)$, with

$$\begin{cases} \alpha \le -\dfrac{9\chi^2}{k_1^{\frac{3}{2}} D_u D_v};\ \text{if } \dfrac{D_u + 4D_v}{4D_u D_v} \le 1, \\[2ex] \alpha \ge \max\left\{ -\dfrac{1}{\frac{D_u+4D_v}{4D_uD_v} - 1};\ \dfrac{1 - \sqrt{1 + \frac{36\chi^2}{k_1^{\frac{3}{2}} D_u D_v}}}{2} \right\};\ \text{if } \dfrac{D_u + 4D_v}{4D_u D_v} > 1. \end{cases} \tag{28}$$

By multiplying the first and second equations of (1) by $u^3\varphi(v)$ and integrating over Ω, we get

$$\frac{d}{4dt}\int_\Omega u^4\varphi(v)\,dx+\int_\Omega 3D_u u^2|\nabla u|^2\varphi(v)\,dx$$
$$+\int_\Omega \tfrac{k_2u^3\varphi(v)}{k_3+n}dx=-D_v\int_\Omega u^4\varphi''(v)|\nabla v|^2dx$$
$$-(D_u+4D_v)\int_\Omega u^3\varphi'(v)\nabla u.\nabla v dx-\int_\Omega \tfrac{3k_1\chi v}{(k_1+v^2)^2}u^3\varphi(v)\nabla u.\nabla v dx+k_7\int_\Omega u^5\varphi'(v)\,dx$$
$$-k_6\int_\Omega u^4v\varphi'(v)\,dx+\int_\Omega nu^3g(u)\varphi(v)\,dx+\int_\Omega \tfrac{3k_1\chi v}{(k_1+v^2)^2}u^4\varphi'(v)|\nabla v|^2dx.$$

By simplification, we get

$$\frac{d}{4dt}\int_\Omega u^4\varphi(v)\,dx+\int_\Omega 3D_uu^2|\nabla u|^2\varphi(v)\,dx+\int_\Omega \tfrac{k_2u^3\varphi(v)}{k_3+n}dx$$
$$\le-\int_\Omega\left[(D_u+4D_v)\varphi'(v)+\tfrac{3k_1\chi v}{(k_1+v^2)^2}\varphi(v)\right]u^3\nabla u\cdot\nabla v dx$$
$$+\int_\Omega nu^3g(u)\varphi(v)\,dx+k_7\int_\Omega u^5\varphi'(v)\,dx$$
$$-\int_\Omega\left[D_v\varphi''(v)+\tfrac{3k_1\chi v}{(k_1+v^2)^2}\varphi'(v)\right]u^4|\nabla v|^2dx-k_6\int_\Omega u^4v\varphi'(v)\,dx. \tag{29}$$

We apply Hölder and Cauchy inequalities to obtain

$$-k_6\int_\Omega u^4v\varphi'(v)\,dx\le c\int_\Omega u^4\varphi(v)\,dx. \tag{30}$$

Similarly, using (2) and (24) , we get

$$\int_\Omega\left[(D_u+4D_v)\varphi'(v)+\tfrac{3k_1\chi v}{(k_1+v^2)^2}\varphi(v)\right]u^3\nabla u.\nabla v dx$$
$$\le\ D_u\int_\Omega u^2|\nabla u|^2\varphi(v)\,dx+\int_\Omega \tfrac{(D_u+4D_v)(\varphi'(v))^2}{4D_u\varphi(v)}u^4|\nabla v|^2dx$$

$$+\int\limits_{\Omega} \frac{9\chi^2}{k_1^{\frac{3}{2}} D_u} \frac{\varphi(v)}{(k_1+v)^2} u^4 |\nabla v|^2 \, dx. \tag{31}$$

We substitute inequalities (30), (31) in (29), to obtain

$$\begin{aligned} &\frac{d}{4dt}\int\limits_{\Omega} u^4\varphi(v)\,dx + \int\limits_{\Omega} u^4\varphi(v)\,dx - k_7\int\limits_{\Omega} u^5\varphi'(v)\,dx \\ \leq\ & \int\limits_{\Omega}\left(\frac{(D_u+4D_v)(\varphi'(v))^2}{4D_u\varphi(v)} + \frac{9\chi^2}{k_1^{\frac{3}{2}} D_u}\frac{\varphi(v)}{(k_1+v)^2} - D_v\varphi''(v)\right) u^4 |\nabla v|^2\,dx \\ &+ (c+1)\int\limits_{\Omega} u^4\varphi(v)\,dx + \int\limits_{\Omega} nu^3 g(u)\,\varphi(v)\,dx. \end{aligned} \tag{32}$$

According to the inequality (28), Hölder and Cauchy inequalities,

$$\begin{aligned} c\int\limits_{\Omega} u^4\varphi(v)\,dx &= c\int\limits_{\Omega}\left(u\left(-\varphi'(v)\right)^{\frac{1}{5}}\right)^4 \frac{\varphi(v)}{(-\varphi'(v))^{\frac{4}{5}}}\,dx \\ &\leq -\frac{k_7}{4}\int\limits_{\Omega} u^5\varphi'(v)\,dx + c\int\limits_{\Omega}\frac{(\varphi(v))^5}{(-\varphi'(v))^4}\,dx \\ &\leq -\frac{k_7}{4}\int\limits_{\Omega} u^5\varphi'(v)\,dx + c\int\limits_{\Omega}\left(k_1^4+v^4\right)dx. \end{aligned} \tag{33}$$

Similarly, using (2), (24), we get

$$\begin{aligned} &\int\limits_{\Omega} nug(u)\,\varphi(v)\,dx = \int\limits_{\Omega} ng(u)\left(u\left(-\varphi'(v)\right)^{\frac{1}{5}}\right)\frac{\varphi(v)}{(-\varphi'(v))^{\frac{1}{5}}}\,dx \\ \leq\ & -\frac{k_7}{5}\int\limits_{\Omega} u^3\varphi'(v)\,dx + c\int\limits_{\Omega} ng(u)\,\frac{(\varphi(v))^5}{(-\varphi'(v))}\,dx \\ \leq\ & -\frac{k_7}{5}\int\limits_{\Omega} u^3\varphi'(v)\,dx + c\|n_0\|_{L^\infty(\Omega)}\int\limits_{\Omega}(v+k_4)\,dx. \end{aligned} \tag{34}$$

We substitute inequalities (25), (33), (34) in (32) to obtain

$$\begin{aligned} &\frac{d}{dt}\int\limits_{\Omega} u^4\varphi(v)\,dx + \int\limits_{\Omega} u^4\varphi(v)\,dx \\ \leq\ & c_\Omega\left(\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^4(\Omega)} + \|n_0\|_{L^\infty(\Omega)} + \|w_0\|_{L^1(\Omega)}\right). \end{aligned}$$

By solving differential equation, due to (25), (28) we get

$$\int_\Omega u^4\varphi(v)\,dx \le c_\Omega\left(\|u_0\|_{L^4(\Omega)}+\|v_0\|_{L^\infty(\Omega)}+\|n_0\|_{L^\infty(\Omega)}+\|w_0\|_{L^1(\Omega)}\right). \quad (35)$$

Using (2), (22), (25), (32), we get

$$\begin{aligned}\|u\|^2_{L^2(\Omega)} &= \int_\Omega u^2(\varphi(v))^{\frac12}(\varphi(v))^{-\frac12}\,dx\\ &\le c\left(\int_\Omega u^4\varphi(v)\,dx\right)^{\frac12}\left(\int_\Omega k_1^{-\alpha}dx\right)^{\frac12}.\end{aligned}$$

Then, due to (35),

$$\|u\|^2_{L^2(\Omega)} \le c_\Omega\left(\|u_0\|_{L^4(\Omega)}+\|v_0\|_{L^\infty(\Omega)}+\|n_0\|_{L^\infty(\Omega)}+\|w_0\|_{L^1(\Omega)}\right). \quad (36)$$

2^{nd}step: By multiplying the first and second equation of (1) by $\triangle u$, and $\triangle\triangle v$, respectively, and integrating over Ω, we get

$$\begin{aligned}&\tfrac{d}{2dt}\left(\|\nabla u\|^2_{L^2(\Omega)}+\|\triangle v\|^2_{L^2(\Omega)}\right)\\ &+D_u\|\triangle u\|^2_{L^2(\Omega)}+D_v\|\nabla\triangle v\|^2_{L^2(\Omega)}+k_6\|\triangle v\|^2_{L^2(\Omega)}\\ &\le -\int_\Omega \nabla.(u\nabla\tfrac{\chi v^2}{k_1+v^2})\triangle u dx-\int_\Omega \tfrac{k_2u\triangle u}{k_3+n}dx\\ &+k_7\int_\Omega g(u)\,nu\triangle u dx+k_7\int_\Omega \nabla u.\nabla\triangle v dx. \quad (37)\end{aligned}$$

With the help of the Cauchy inequality, using the arguments of [8, (1.93), p. 50], with some exponent $\frac{d}{2}\le\delta<2, (d=2,3)$, we get

$$\int_\Omega -\nabla.(u\nabla\tfrac{\chi v^2}{k_1+v^2})\triangle u dx \le \tfrac{D_u}{4}\|\triangle u\|^2_{L^2(\Omega)}+c_\Omega\|u\|^2_{H^\delta(\Omega)}\|\tfrac{v^2}{k_1+v^2}\|^2_{H^2(\Omega)}. \quad (38)$$

Here,

$$\triangle\left(\tfrac{v^2}{k_1+v^2}\right)=\tfrac{2k_1\left(k_1-3v^2\right)}{(k_1+v^2)^3}|\nabla v|^2+\tfrac{2k_1v}{(k_1+v^2)^2}\triangle v.$$

By (22), using the embedding $H^2_N(\Omega)\hookrightarrow W^{1,4}(\Omega)$, we get

$$
\begin{aligned}
& c_\Omega \|u\|^2_{H^\delta(\Omega)} \left(\|\tfrac{v^2}{k_1+v^2}\|^2_{L^2(\Omega)} + \|\triangle\left(\tfrac{v^2}{k_1+v^2}\right)\|^2_{L^2(\Omega)} \right) \\
\leq \; & c_\Omega \|u\|^2_{H^\delta(\Omega)} \left(\|v\|^4_{L^4(\Omega)} + \|\nabla v\|^4_{L^4(\Omega)} + \|\triangle v\|^2_{L^2(\Omega)} \right) \\
\leq \; & c_\Omega \|u\|^2_{H^\delta(\Omega)} \left(\|v\|^4_{H^2(\Omega)} + \|v\|^2_{H^2(\Omega)} \right).
\end{aligned}
$$

Using the arguments of [8, 2.119, p. 98], by Cauchy's inequality, we have

$$
\begin{aligned}
& c_\Omega \left\| A_u^{\frac{\delta}{2}} u \right\|^2_{L^2(\Omega)} \left(\left\| A_v^{\frac{1}{2}} v \right\|^4_{L^2(\Omega)} + \|A_v v\|^2_{L^2(\Omega)} \right) \\
\leq \; & c_\Omega \|u\|^{\delta-2}_{L^2(\Omega)} \|A_u u\|^{\delta}_{L^2(\Omega)} + c_\Omega \|u\|^{\delta-2}_{L^2(\Omega)} \|A_u u\|^{\delta}_{L^2(\Omega)} \|v\|^{\frac{8}{3}}_{L^2(\Omega)} \left\| A_v^{\frac{3}{2}} v \right\|^{\frac{4}{3}}_{L^2(\Omega)} \\
& + c_\Omega \|u\|^{\delta-2}_{L^2(\Omega)} \|A_u u\|^{\delta}_{L^2(\Omega)} \|v\|^{\frac{2}{3}}_{L^2(\Omega)} \left\| A_v^{\frac{3}{2}} v \right\|^{\frac{4}{3}}_{L^2(\Omega)} \\
\leq \; & c_\Omega \|u\|^2_{L^2(\Omega)} + c_\Omega \|u\|^{\frac{6(\delta-2)}{2-3\delta}}_{L^2(\Omega)} \|v\|^{\frac{16}{2-3\delta}}_{L^2(\Omega)} + c_\Omega \|u\|^{\frac{6(\delta-2)}{2-3\delta}}_{L^2(\Omega)} \|v\|^{\frac{4}{2-3\delta}}_{L^2(\Omega)} \\
& + \tfrac{D_u}{2} \|A_u u\|^2_{L^2(\Omega)} + \tfrac{D_v}{2} \left\| A_v^{\frac{3}{2}} v \right\|^2_{L^2(\Omega)}. \qquad (39)
\end{aligned}
$$

We substitute (38), (39) in (37):

$$
\begin{aligned}
& \tfrac{d}{2dt} \left(\|\nabla u\|^2_{L^2(\Omega)} + \|\triangle v\|^2_{L^2(\Omega)} \right) + D_u \|\triangle u\| + k_6 \|\triangle v\|^2_{L^2(\Omega)} + D_v \|\nabla \triangle v\|^2_{L^2(\Omega)} \\
\leq \; & \|\tfrac{k_2}{k_3+n}\|^2_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)} + k_7 \|g(u)\|^2_{L^\infty(\Omega)} \|n\|^2_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Using (2), (24), (36), by integrating over $[0,t]$, we get

$$
\begin{aligned}
& \left(\|\nabla u\|^2_{L^2(\Omega)} + \|\triangle v\|^2_{L^2(\Omega)} \right) \\
& + \int_0^t \left(D_u \|\triangle u\| + k_6 \|\triangle v\|^2_{L^2(\Omega)} + D_v \|\nabla \triangle v\|^2_{L^2(\Omega)} \right) ds \\
& \leq c(\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^2(\Omega)} + \|n_0\|_{H^2(\Omega)}) t, \qquad (40)
\end{aligned}
$$

By multiplying the third equation of (1) by $\triangle\triangle n$ and integrating over Ω, we get

$$
\tfrac{d}{2dt} \|\triangle n\|^2_{L^2(\Omega)} + D_n \|\nabla \triangle n\|^2_{L^2(\Omega)} \leq - \int_\Omega \nabla(g(u)\, nu).\nabla \triangle n dx. \qquad (41)
$$

By (2), (25), (36), (40), using the embedding $H_N^{1+\delta}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, with some exponent $\frac{d}{2} \leq \delta < 2, (d = 2, 3)$,

$$\begin{aligned} & -\int\limits_\Omega \nabla(g(u)\, nu).\nabla\triangle n dx \\ \leq\ & \tfrac{D_n}{2}\|\nabla\triangle n\|^2_{L^2(\Omega)} + c_\Omega \|g(u)\|^2_{W^{1,\infty}(\Omega)}\|u\|^2_{H^1(\Omega)}\|n\|^2_{W^{1,\infty}(\Omega)} \\ \leq\ & \tfrac{D_n}{2}\|\nabla\triangle n\|^2_{L^2(\Omega)} + \|n\|^2_{H^{1+\delta}(\Omega)} c(\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^1(\Omega)} + \|n_0\|_{H^2(\Omega)})t. \quad (42) \end{aligned}$$

Using the arguments of [8, 2.119, p. 98], we get

$$\|A_n^{\frac{1+\delta}{2}} n\|^2_{L^2(\Omega)} \leq c_\Omega \|n\|^{\frac{4-2\delta}{3}}_{L^2(\Omega)} \left\| A_n^{\frac{3}{2}} n \right\|^{\frac{2(1+\delta)}{3}}_{L^2(\Omega)}. \quad (43)$$

We substitute (42), (43) in (41). Then using (8), (24), by Cauchy's inequality, we get

$$\sup_{0\leq t\leq T_U} \|\triangle n\|^2_{L^2(\Omega)}(t) \leq c(\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^1(\Omega)} + \|n_0\|_{H^2(\Omega)})(t+1)^2. \quad (44)$$

By multiplying the forth equation of (1) by $\triangle w$ and integrating over $\Omega\times[0,t]$, using the embedding $H_N^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^4(\Omega)$, we get

$$\begin{aligned} \|w\|_{H^1(\Omega)} \leq\ & \int\limits_0^t \|\tfrac{k_2}{k_3+n}\|_{L^\infty(\Omega)}\|u\|_{H^1(\Omega)} ds + \int\limits_0^t \|\tfrac{k_2}{(k_3+n)^2}\|_{L^\infty(\Omega)}\|u\|_{L^4(\Omega)}\|\nabla n\|_{L^4(\Omega)} ds \\ \leq\ & \int\limits_0^t \|u\|_{H^1(\Omega)}\left(1 + \|n\|_{H^2(\Omega)}\right) ds, \end{aligned}$$

From (18), with (24), (40), (44), we get

$$\sup_{0\leq t\leq T_U} \|w\|^2_{H^1(\Omega)}(t) \leq c(\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^1(\Omega)} + \|n_0\|_{H^2(\Omega)} + \|w_0\|_{H^2(\Omega)})\left(t^2+1\right). \quad (45)$$

Finally we use (24), (25), (36), (40), (44), (45), to obtain for $0 \leq t \leq T_U$

$$\begin{aligned} & \sup_{0\leq t\leq T_U}\left(\|u(t)\|_{H^1(\Omega)} + \|v(t)\|_{H^2(\Omega)} + \|n(t)\|_{H^2(\Omega)} + \|w(t)\|_{H^1(\Omega)}\right) \\ \leq\ & c(\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^2(\Omega)} + \|n_0\|_{H^2(\Omega)} + \|w_0\|_{H^1(\Omega)})\left(t^2+1\right). \end{aligned}$$

◄

4.1.2. Global solutions

In this section, we will prove the following theorem:

Theorem 2. *For any* $U_0 = (u_0, v_0, n_0, w_0) \in \mathcal{K}$*, there exists a unique global solution of* (1) *in the function space*

$$\begin{aligned} u &\in C(]0,+\infty[\,; H^2_N(\Omega)) \cap C([0,+\infty[\,; H^1(\Omega)) \cap C^1\,]0,+\infty[\,; L^2(\Omega)), \\ v, n &\in C(]0,+\infty[\,; H^3_N(\Omega)) \cap C([0,+\infty[\,; H^2_N(\Omega)) \cap C^1(]0,+\infty[\,; H^1(\Omega)) \\ w &\in C([0,+\infty[\,; H^2_N(\Omega)) \cap C^1(]0,+\infty[\,; H^1(\Omega)). \end{aligned}$$

Proof. Utilizing the a priori estimates (27), we shall construct a global solution to (1).

For $U_0 \in \mathcal{K}$, we know that there exists a unique local solution at least on an interval $[0, T_{U_0}]$.

Let $0 < t_1 < T_{U_0}$. Then, $U_1 = U(t_1) \in \mathcal{K}$. We next consider problem (1) with the initial value U_1 on an interval $[t_1, T]$, where the end time $T > 0$ is any finite time. The Proposition 4 ensures the estimate (27) for any local solution V , i.e,$\left\| A^{\frac{1}{2}} V(t) \right\| \leq c(\|A^{\frac{1}{2}} U_1\|) \left(T^2 + 1\right)$, $t_1 \leq t \leq T_V$. Then, the local solution V can always be extended over an interval $[t_1, T_V + \tau]$ as local solution, $\tau > 0$ being dependent only on $c(\|A^{\frac{1}{2}} U_1\|) \left(T^2 + 1\right)$ and hence being independent of the extreme time T_V (cf. [8, Corollary 4.1, p. 184]). This means that our Cauchy problem possesses a unique global solution on the interval $[t_1, T]$.

This argument is meaningful for any finite time $T > 0$. So, we conclude that for any initial value $U_0 \in \mathcal{K}$, there exists a unique global solution to (1) with $U(t) \in \mathcal{K}, 0 \leq t < \infty$, in the function space

$$U \in C(]0,+\infty[; \mathcal{D}(A)) \cap C([0,+\infty[; \mathcal{D}(A^{\frac{1}{2}})) \cap C^1((]0,+\infty[; X).$$

◄

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