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Positive Integer Powers of the Kronecker Sum of Two Tridiagonal Toeplitz Matrices

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Abstract. In this article, we give an explicit expression for calculating the arbitrary positive integer powers of the Kronecker sum of two tridiagonal Toeplitz matrices.

Key Words and Phrases: Kronecker sum, Toeplitz matrix, tridiagonal, diagonalizable, eigenvalues, eigenvectors, eigenpairs.

2010 Mathematics Subject Classifications: 65F05

1. Introduction

Let a tridiagonal Toeplitz matrix T of size n be given as:

$$T = \begin{pmatrix} b & a & & & \\ c & b & a & & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \\ & & & c & b \end{pmatrix}, \tag{1}$$

where $a \neq 0$ and $c \neq 0$. We denote the matrix M, the Kronecker sum of two tridiagonal Toeplitz matrices T_1 and T_2 given by (1), by

$$M = T_1 \oplus T_2. \tag{2}$$

We find this type of matrices in a wide variety of applications, such as discretization problems of two-dimensional differential equations, including those arising from the two-dimensional Poisson problem [2, 4]. We need to calculate

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the positive integer powers of this type of matrices for solving corresponding difference equations. Salkuyeh in [8] and Rimas in [5, 6, 7] gave an explicit expression for arbitrary positive integer powers of many types of tridiagonal Toeplitz matrices.

In this paper, we present an expression for the positive integer powers of matrix M given by (2) of arbitrary orders.

This paper is organized as follows. We start with the definitions and properties we need in our research in Section 2. In Section 3, an expression for the positive integer powers of the Kronecker sum of two tridiagonal Toeplitz matrices is derived. We give two numerical examples in Section 4.

2. Preliminaries

First, we mention the following definitions and lemma that we will be using in this paper.

Definition 1 (Kronecker product). *[1, 4] Let A and B be two matrices in $\mathbb{R}^{n\times n}$. Then the $n^2 \times n^2$ matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & a_{22}B & ... & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{pmatrix} \tag{3}$$

is the Kronecker product of A and B.

Definition 2 (Kronecker sum). *[4] Let A and B be two matrices in $\mathbb{R}^{n\times n}$. We denote by $A \oplus B$ the Kronecker sum of A and B defined as*

$$A \oplus B = (A \otimes I) + (I \otimes B),$$

where $A \oplus B$ is a matrix of order n^2, and I denotes the identity matrix of order n.

Property 1. *[9, 10] Consider the matrices A, B, C and D with compatible sizes.*

- $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$*, where A and B are invertible matrices,*
- $(\alpha A) \otimes (\beta B) = \alpha\beta(A \otimes B)$*, for any scalars α and β,*
- $(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$*, (the mixed product rule).*

Theorem 2. *[4] Suppose for $n \in \mathbb{N}$ that A and B are two matrices in $\mathbb{R}^{n\times n}$ with eigenpairs $(\lambda_i, u_i), i = 1, ..., n$ and $(\mu_j, v_j), j = 1, ..., n$, respectively. And $M = A \oplus B$. Then $(\lambda_i + \mu_j, u_i \otimes v_j)$ is an eigenpair of the matrix M, $i, j = 1, ..., n$.*

3. Main results

Let T_1 and T_2 be two tridiagonal Toeplitz matrices given by

$$T_1 = \begin{pmatrix} b_1 & a_1 & & & \\ c_1 & b_1 & a_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_1 & b_1 & a_1 \\ & & & c_1 & b_1 \end{pmatrix} \quad ; \quad T_2 = \begin{pmatrix} b_2 & a_2 & & & \\ c_2 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_2 & b_2 & a_2 \\ & & & c_2 & b_2 \end{pmatrix},$$

moreover, (λ_i, u_i) and (μ_j, v_j) be the eigenpairs of T_1 and T_2, respectively, such as [2, 8]:

$$\lambda_i = b_1 + 2a_1\sqrt{\frac{c_1}{a_1}}\cos\frac{i\pi}{n+1}\ ,\ u_i = \begin{pmatrix} \left(\frac{c_1}{a_1}\right)^{\frac{1}{2}} \sin\left(\frac{1i\pi}{n+1}\right) \\ \left(\frac{c_1}{a_1}\right)^{\frac{2}{2}} \sin\left(\frac{2i\pi}{n+1}\right) \\ \left(\frac{c_1}{a_1}\right)^{\frac{3}{2}} \sin\left(\frac{3i\pi}{n+1}\right) \\ \vdots \\ \left(\frac{c_1}{a_1}\right)^{\frac{n}{2}} \sin\left(\frac{ni\pi}{n+1}\right) \end{pmatrix},\ i = 1, 2, .., n, \quad (4)$$

and

$$\mu_j = b_2 + 2a_2\sqrt{\frac{c_2}{a_2}}\cos\frac{j\pi}{n+1}\ ,\ v_j = \begin{pmatrix} \left(\frac{c_2}{a_2}\right)^{\frac{1}{2}} \sin\left(\frac{1j\pi}{n+1}\right) \\ \left(\frac{c_2}{a_2}\right)^{\frac{2}{2}} \sin\left(\frac{2j\pi}{n+1}\right) \\ \left(\frac{c_2}{a_2}\right)^{\frac{3}{2}} \sin\left(\frac{3j\pi}{n+1}\right) \\ \vdots \\ \left(\frac{c_2}{a_2}\right)^{\frac{n}{2}} \sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix},\ j = 1, 2, .., n. \quad (5)$$

i.e., $T_1u_i = \lambda_i u_i$ and $T_2v_j = \mu_j v_j$, moreover, the matrices T_1 and T_2 are diagonalizable, such that $U = (u_1\ u_2,\ ...\ u_n)$ and $V = (v_1\ v_2,\ ...\ v_n)$ diagonalize T_1 and T_2, respectively, i.e.,

$$U^{-1}T_1U = D_1 \quad , \quad V^{-1}T_2V = D_2,$$

where $D_1 = diag(\lambda_1\ \lambda_2\ ...\ \lambda_n)$ and $D_2 = diag(\mu_1\ \mu_2\ ...\ \mu_n)$, otherwise

$$T_1 = UD_1U^{-1} \quad , \quad T_2 = VD_2V^{-1}. \tag{6}$$

Lemma 1. *If T_1 and T_2 are two matrices of the form (6) and $M = T_1 \oplus T_2$, then*

$$M = SDS^{-1},$$

with $S = U \otimes V$ and $D = D_1 \oplus D_2$.

Proof. See [3]. ◄

Now it is easy to compute M^m via the formula SD^mS^{-1}, where m is a positive integer, because M is diagonalizable, and D^m is simply a diagonal matrix [3].

Hence, it is enough to find an explicit expression for S^{-1}. Let

$$\tilde{D}_1 = \left(\left(\frac{c_1}{a_1}\right)^{\frac{1}{2}} \left(\frac{c_1}{a_1}\right)^{\frac{2}{2}} \dots \left(\frac{c_1}{a_1}\right)^{\frac{n}{2}} \right) \tilde{u}_i = \begin{pmatrix} \sin\left(\frac{1i\pi}{n+1}\right) \\ \sin\left(\frac{2i\pi}{n+1}\right) \\ \sin\left(\frac{3i\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{ni\pi}{n+1}\right) \end{pmatrix}, i = 1, 2, .., n,$$

and

$$\tilde{D}_2 = \left(\left(\frac{c_2}{a_2}\right)^{\frac{1}{2}} \left(\frac{c_2}{a_2}\right)^{\frac{2}{2}} \dots \left(\frac{c_2}{a_2}\right)^{\frac{n}{2}} \right) \tilde{v}_j = \begin{pmatrix} \sin\left(\frac{1j\pi}{n+1}\right) \\ \sin\left(\frac{2j\pi}{n+1}\right) \\ \sin\left(\frac{3j\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix}, j = 1, 2, .., n.$$

Then we have: $U = \tilde{D}_1\tilde{U}$ and $V = \tilde{D}_2\tilde{V}$, where $\tilde{U} = (\tilde{u}_1\ \tilde{u}_2,\ \dots\ \tilde{u}_n)$ and $\tilde{V} = (\tilde{v}_1\ \tilde{v}_2,\ \dots\ \tilde{v}_n)$.

Lemma 2. *[8] Suppose $\tilde{U}$ and $\tilde{V}$ are defined as above. Then*

$$\tilde{U}^{-1} = \frac{2}{n+1}\tilde{U} \quad , \quad \tilde{V}^{-1} = \frac{2}{n+1}\tilde{V}.$$

Therefore we have:

$$
\begin{aligned}
S^{-1} &= (U \otimes V)^{-1}, \\
&= U^{-1} \otimes V^{-1}, \\
&= (\tilde{D}_1 \tilde{U})^{-1} \otimes (\tilde{D}_2 \tilde{V})^{-1}, \\
&= (\tilde{U}^{-1} \tilde{D}_1^{-1}) \otimes (\tilde{V}^{-1} \tilde{D}_2^{-1}), \\
&= \left(\frac{2}{n+1} \tilde{U} \tilde{D}_1^{-1} \right) \otimes \left(\frac{2}{n+1} \tilde{V} \tilde{D}_2^{-1} \right), \\
&= \left(\frac{2}{n+1} \right)^2 (\tilde{U} \tilde{D}_1^{-1}) \otimes (\tilde{V} \tilde{D}_2^{-1}), \\
&= \frac{4}{(n+1)^2} (\tilde{U} \otimes \tilde{V})(\tilde{D}_1^{-1} \otimes \tilde{D}_2^{-1}).
\end{aligned}
$$

Theorem 3. *Let* $M = T_1 \oplus T_2$ *be the Kronecker sum of two tridiagonal Toeplitz matrices defined in (1) and*

$$
Z = M^m = \begin{pmatrix} Z_{11} & \dots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{n1} & \dots & Z_{nn} \end{pmatrix},
$$

where m *is a positive integer. Then*

$$
Z_{i,j} = \frac{4}{(n+1)^2} \left(\frac{c_1}{a_1} \right)^{\frac{i-j}{2}} V \sum_{k=1}^{n} (\lambda_k I + D_2)^m \sin\left(\frac{kj\pi}{n+1} \right) \sin\left(\frac{ik\pi}{n+1} \right) \tilde{V} \tilde{D}_2 \ ;
$$

$$
\begin{aligned}
z_{i,j}^{s,t} = \frac{4}{(n+1)^2} \left(\frac{c_1}{a_1} \right)^{\frac{i-j}{2}} \left(\frac{c_2}{a_2} \right)^{\frac{s-t}{2}} \sum_{k=1}^{n} \sum_{l=1}^{n} (\lambda_k + \mu_l)^m \times \\
\times \sin\left(\frac{ik\pi}{n+1} \right) \sin\left(\frac{sl\pi}{n+1} \right) \sin\left(\frac{kj\pi}{n+1} \right) \sin\left(\frac{lt\pi}{n+1} \right);
\end{aligned}
$$

where $\lambda_k = b_1 + 2a_1 \sqrt{c_1/a_1} \cos\left(\frac{k\pi}{n+1} \right)$ *and* $\mu_l = b_2 + 2a_2 \sqrt{c_2/a_2} \cos\left(\frac{l\pi}{n+1} \right)$.

Proof.

$$
\begin{aligned}
M^m &= S D^m S^{-1}, \\
&= \frac{4}{(n+1)^2} (U \otimes V)(D_1 \oplus D_2)^m (\tilde{U} \otimes \tilde{V})(\tilde{D}_1^{-1} \otimes \tilde{D}_2^{-1}).
\end{aligned}
$$

Now by replacing U, V, $\tilde{U}$, and $\tilde{V}$ in the last equation and a bit of math, the desired formula is achieved. ◄

Remark 1. *If M is a non-singular matrix (i.e., all the eigenvalues of M are non-zero), then Theorem 3 can also be used for the negative integer m. We get the inverse of the matrix M in the special case $m = -1$.*

4. Examples

Example 1. *Let*

$$T_1 = \begin{pmatrix} 0.5 & 0.1 & 0 \\ 0.2 & 0.5 & 0.1 \\ 0 & 0.2 & 0.5 \end{pmatrix} \quad ; \quad T_2 = \begin{pmatrix} 0.2 & -0.1 & 0 \\ -0.2 & 0.2 & -0.1 \\ 0 & -0.2 & 0.2 \end{pmatrix}.$$

Here, we have $a_1 = 0.1,\ b_1 = 0.5,\ c_1 = 0.2$ and $a_2 = -0.1,\ b_2 = 0.5,\ c_2 = -0.2$. By using (4) and (5), we have $\lambda_1 = 0.3,\ \lambda_2 = 0.5,\ \lambda_3 = 0.7$ and $\mu_1 = 0,\ \mu_2 = 0.2,\ \mu_3 = 0.4$.

If $M = T_1 \oplus T_2$ is given as

$$M = \begin{pmatrix} 0.7 & -0.1 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ -0.2 & 0.7 & -0.1 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0.7 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0.7 & -0.1 & 0 & 0.1 & 0 & 0 \\ 0 & 0.2 & 0 & -0.2 & 0.7 & -0.1 & 0 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0 & -0.2 & 0.7 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0.7 & -0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & -0.2 & 0.7 & -0.1 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & -0.2 & 0.7 \end{pmatrix},$$

and $m = 10$, then from the formula (3) we have

$$z_{i,j}^{s,t} = 2^{\frac{i-j+s-t-2}{2}} \sum_{k=1}^{3} \sum_{l=1}^{3} (\lambda_k + \mu_l)^{10} \sin\left(\frac{ik\pi}{4}\right) \sin\left(\frac{sl\pi}{4}\right) \sin\left(\frac{kj\pi}{4}\right) \sin\left(\frac{lt\pi}{4}\right)$$

where $i, j, s, t = 1, 2, 3$. We find:

$$M^{10} = \begin{pmatrix} 0.2601 & -0.2056 & 0.0793 & 0.2056 & -0.1586 & 0.0593 & 0.0793 & -0.0593 & 0.0213 \\ -0.4111 & 0.4187 & -0.2056 & -0.3172 & 0.3242 & -0.1586 & -0.1186 & 0.1219 & -0.0593 \\ 0.3172 & -0.4111 & 0.2601 & 0.2373 & -0.3172 & 0.2056 & 0.0853 & -0.1186 & 0.0793 \\ 0.4111 & -0.3172 & 0.1186 & 0.4187 & -0.3242 & 0.1219 & 0.2056 & -0.1586 & 0.0593 \\ -0.6343 & 0.6484 & -0.3172 & -0.6484 & 0.6626 & -0.3242 & -0.3172 & 0.3242 & -0.1586 \\ 0.4746 & -0.6343 & 0.4111 & 0.4877 & -0.6484 & 0.4187 & 0.2373 & -0.3172 & 0.2056 \\ 0.3172 & -0.2373 & 0.0853 & 0.4111 & -0.3172 & 0.1186 & 0.2601 & -0.2056 & 0.0793 \\ -0.4746 & 0.4877 & -0.2373 & -0.6343 & 0.6484 & -0.3172 & -0.4111 & 0.4187 & -0.2056 \\ 0.3412 & -0.4746 & 0.3172 & 0.4746 & -0.6343 & 0.4111 & 0.3172 & -0.4111 & 0.2601 \end{pmatrix}.$$

Example 2. *Let*

$$T_1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad ; \quad T_2 = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

Here, we have $a_1 = -1,\ b_1 = 2,\ c_1 = -1$ *and* $a_2 = -1,\ b_2 = 2,\ c_2 = -2$*. By using* (4) *and* (5)*, we have* $\lambda_1 = 0.5858,\ \lambda_2 = 2,\ \lambda_3 = 3.4142$ *and* $\mu_1 = 0,\ \mu_2 = 2,\ \mu_3 = 4$.

If $M = T_1 \oplus T_2$ *is given as*

$$M = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 4 \end{pmatrix}$$

and the eigenvalues of M *given as* $\lambda_i + \mu_j, \quad i,j = 1,2,3,$ *are non-zero, then the matrix* M *is non-singular.*

For $m = -1$*, from the formula* (3) *we have*

$$z_{i,j}^{s,t} = 2^{\frac{s-t-2}{2}} \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{1}{\lambda_k + \mu_l} \sin\left(\frac{ik\pi}{4}\right) \sin\left(\frac{sl\pi}{4}\right) \sin\left(\frac{kj\pi}{4}\right) \sin\left(\frac{lt\pi}{4}\right);$$

where $i,j,s,t = 1,2,3$*. We find:*

$$M^{-1} = \begin{pmatrix} 0.3643 & 0.1446 & 0.0482 & 0.1681 & 0.1176 & 0.0483 & 0.0727 & 0.0613 & 0.0274 \\ 0.2892 & 0.4608 & 0.1446 & 0.2353 & 0.2647 & 0.1176 & 0.1225 & 0.1275 & 0.0613 \\ 0.1929 & 0.2892 & 0.3643 & 0.1933 & 0.2353 & 0.1681 & 0.1096 & 0.1225 & 0.0727 \\ 0.1681 & 0.1176 & 0.0483 & 0.4370 & 0.2059 & 0.0756 & 0.1681 & 0.1176 & 0.0483 \\ 0.2353 & 0.2647 & 0.1176 & 0.4118 & 0.5882 & 0.2059 & 0.2353 & 0.2647 & 0.1176 \\ 0.1933 & 0.2353 & 0.1681 & 0.3025 & 0.4118 & 0.4370 & 0.1933 & 0.2353 & 0.1681 \\ 0.0727 & 0.0613 & 0.0274 & 0.1681 & 0.1176 & 0.0483 & 0.3643 & 0.1446 & 0.0482 \\ 0.1225 & 0.1275 & 0.0613 & 0.2353 & 0.2647 & 0.1176 & 0.2892 & 0.4608 & 0.1446 \\ 0.1096 & 0.1225 & 0.0727 & 0.1933 & 0.2353 & 0.1681 & 0.1929 & 0.2892 & 0.3643 \end{pmatrix}.$$

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