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Bihyperbolic Numbers of the Fibonacci Type and Triangular Matrices (Tables)

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Abstract. In this paper, we compute paradeterminants and parapermanents of some triangular matrices that give bihyperbolic numbers of the Fibonacci type. Using connections between the paradeterminant of triangular matrix and the lower Hessenberg determinant, we also obtain the general term of these sequences.

Key Words and Phrases: bihyperbolic numbers, Fibonacci type numbers, triangular matrix, Hessenberg matrix.

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1. Introduction

Let $n \geq 0$ be an integer. The nth Fibonacci number F_n is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. There are many numbers given by the second order linear recurrence relations. The nth Pell number P_n is defined by $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with the initial conditions $P_0 = 0$, $P_1 = 1$. The nth Jacobsthal number J_n is defined by $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ and $J_0 = 0$, $J_1 = 1$. These numbers are often called the numbers of the Fibonacci type.

Hyperbolic numbers are two dimensional number system. Hyperbolic imaginary unit, so-called *unipotent*, introduced in 1848 by James Cockle (see [4, 5, 6, 7]), is an element $\mathbf{h}$ such that $\mathbf{h}^2 = 1$ and $\mathbf{h} \neq \pm 1$. The set of hyperbolic numbers is defined as $\mathbb{H} = \{x + y\mathbf{h} : x, y \in \mathbb{R}, \mathbf{h}^2 = 1\}$. Bihyperbolic numbers are a generalization of hyperbolic numbers. Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + j_1 x_1 + j_2 x_2 + j_3 x_3,$$

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where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and j_1, j_2, $j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1.$$

From the above rules, the multiplication of bihyperbolic numbers can be made analogously to the multiplication of algebraic expressions. The addition and the subtraction of bihyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients. The addition and the multiplication on $\mathbb{H}_2$ are commutative and associative, $(\mathbb{H}_2, +, \cdot)$ is a commutative ring. For the algebraic properties of bihyperbolic numbers see [1].

A special kind of bihyperbolic numbers were introduced in [2] as follows. For nonnegative integer n, the nth bihyperbolic Fibonacci number BhF_n, the nth bihyperbolic Pell number BhP_n and the nth bihyperbolic Jacobsthal number BhJ_n are defined as

$$BhF_n = F_n + j_1 F_{n+1} + j_2 F_{n+2} + j_3 F_{n+3},$$

$$BhP_n = P_n + j_1 P_{n+1} + j_2 P_{n+2} + j_3 P_{n+3},$$

$$BhJ_n = J_n + j_1 J_{n+1} + j_2 J_{n+2} + j_3 J_{n+3},$$

respectively. Some combinatorial properties of bihyperbolic numbers of the Fibonacci type can also be found in [3]. For integer $n \geq 2$ we have

$$BhF_n = BhF_{n-1} + BhF_{n-2},$$

with

$$BhF_0 = j_1 + j_2 + 2j_3, \quad BhF_1 = 1 + j_1 + 2j_2 + 3j_3,$$

$$BhP_n = 2BhP_{n-1} + BhP_{n-2},$$

$$BhP_0 = j_1 + 2j_2 + 5j_3, \quad BhP_1 = 1 + 2j_1 + 5j_2 + 12j_3,$$

and

$$BhJ_n = BhJ_{n-1} + 2BhJ_{n-2},$$

where

$$BhJ_0 = j_1 + j_2 + 3j_3, \quad BhJ_1 = 1 + j_1 + 3j_2 + 5j_3.$$

In this paper, we will express bihyperbolic numbers of the Fibonacci type through appropriate paradeterminants and parapermanents of triangular matrices.

2. Main results

A triangular table of numbers from some field K

$$A_n = \begin{bmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ \vdots & \vdots & \ddots & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}_{(n)\times(n)}$$

is called a triangular matrix, and the number n is called its order.

Note that (see [15]) a triangular matrix in the definition is not a matrix in the usual sense because it is triangular rather than rectangular table of numbers.

The functions of triangular matrices, in particular paradeterminants and parapermanents of triangular matrices, are used in algebra, combinatorics, number theory and other branches of mathematics, see for example [10, 16]. For triangular matrices basic concepts and results see [8, 13, 14, 15].

Ganyushkin et al. (see [8]) obtained the following formulas. If A_n is a triangular matrix, then the paradeterminant $\operatorname{ddet}(A_n)$ and parapermanent $\operatorname{pper}(A_n)$ of A_n are

$$\operatorname{ddet}(A_n) = \sum_{r=1}^{n} \sum_{p_1+\cdots+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+\cdots+p_s, p_1+\cdots+p_{s-1}+1}\}$$

and

$$\operatorname{pper}(A_n) = \sum_{r=1}^{n} \sum_{p_1+\cdots+p_r=n} \prod_{s=1}^{r} \{a_{p_1+\cdots+p_s, p_1+\cdots+p_{s-1}+1}\},$$

respectively, where summations are over the set of positive integer solutions of the equality $p_1 + \cdots + p_r = n$ and $\{a_{ij}\} = a_{ij} \cdot a_{i,j+1} \cdots a_{ii}$.

For $n \geq 1$ the following formulae (decompositions of paradeterminant and parapermanent by elements of the last row) hold (see [8, 15]):

$$\operatorname{ddet}(A_n) = \sum_{s=1}^{n} (-1)^{n-s} \{a_{ns}\} \operatorname{ddet}(A_{s-1}), \tag{1}$$

$$\operatorname{pper}(A_n) = \sum_{s=1}^{n} \{a_{ns}\} \operatorname{pper}(A_{s-1}), \tag{2}$$

where $\operatorname{ddet}(A_0) = 1$, $\operatorname{pper}(A_0) = 1$.

For an integer $n \geq 1$, let A_n be any triangular matrix of order n with entries being bihyperbolic numbers.

Theorem 1. *Let $n \geq 0$ be an integer and*

$$A_{n+1} = \begin{bmatrix} j_1 + j_2 + 2j_3 & & & & & \\ -1 - j_2 - j_3 & 1 & & & & \\ 0 & -1 & 1 & & & \\ 0 & 0 & -1 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhF_n = \operatorname{ddet}(A_{n+1})$.

Proof. (by induction on n)
If $n = 0$, then $\operatorname{ddet}(A_1) = j_1 + j_2 + 2j_3 = BhF_0$.
If $n = 1$, then

$$\begin{aligned} \operatorname{ddet}(A_2) &= -(-1 - j_2 - j_3) \cdot 1 + 1 \cdot (j_1 + j_2 + 2j_3) \\ &= 1 + j_2 + j_3 + j_1 + j_2 + 2j_3 \\ &= 1 + j_1 + 2j_2 + 3j_3 = BhF_1. \end{aligned}$$

Now assume that for any $n \geq 0$, we have $BhF_n = \operatorname{ddet}(A_{n+1})$ and $BhF_{n+1} = \operatorname{ddet}(A_{n+2})$. We shall show that $BhF_{n+2} = \operatorname{ddet}(A_{n+3})$. Expanding the paradeterminant $\operatorname{ddet}(A_{n+3})$ by elements of the last row (see (1)), we have

$$\begin{aligned} \operatorname{ddet}(A_{n+3}) &= \sum_{s=1}^{n+3} (-1)^{n+3-s} \{a_{n+3,s}\} \operatorname{ddet}(A_{s-1}) \\ &= 0 + \cdots + 0 + (-1)^{n+3-(n+2)} \{a_{n+3,n+2}\} \operatorname{ddet}(A_{n+2-1}) \\ &\quad + (-1)^{n+3-(n+3)} \{a_{n+3,n+3}\} \operatorname{ddet}(A_{n+3-1}) \\ &= -a_{n+3,n+2} \cdot a_{n+3,n+3} \cdot \operatorname{ddet}(A_{n+1}) + a_{n+3,n+3} \cdot \operatorname{ddet}(A_{n+2}) \\ &= \operatorname{ddet}(A_{n+1}) + \operatorname{ddet}(A_{n+2}) \\ &= BhF_n + BhF_{n+1} = BhF_{n+2}, \end{aligned}$$

which ends the proof. ◄

Theorem 2. *Let $n \geq 0$ be an integer and*

$$A_{n+1} = \begin{bmatrix} j_1 + j_2 + 2j_3 & & & & & \\ 1 + j_2 + j_3 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 0 & 1 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhF_n = \mathrm{pper}(A_{n+1})$.

Proof. Expanding the parapermanent $\mathrm{pper}(A_{n+3})$ by elements of the last row (see (2)) and proceeding analogously as in the proof of the previous theorem, we obtain the desired formula. ◄

In the same way we can prove the next theorems.

Theorem 3. *Let* $n \geq 0$ *be an integer and*

$$A_{n+1} = \begin{bmatrix} j_1 + j_2 + 3j_3 & & & & & \\ -1 - 2j_2 - 2j_3 & 1 & & & & \\ 0 & -2 & 1 & & & \\ 0 & 0 & -2 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -2 & 1 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhJ_n = \mathrm{ddet}(A_{n+1})$.

Theorem 4. *Let* $n \geq 0$ *be an integer and*

$$A_{n+1} = \begin{bmatrix} j_1 + j_2 + 3j_3 & & & & & \\ 1 + 2j_2 + 2j_3 & 1 & & & & \\ 0 & 2 & 1 & & & \\ 0 & 0 & 2 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhJ_n = \mathrm{pper}(A_{n+1})$.

Theorem 5. *Let* $n \geq 0$ *be an integer and*

$$A_{n+1} = \begin{bmatrix} j_1 + 2j_2 + 5j_3 & & & & & \\ -\frac{1}{2} - \frac{1}{2}j_2 - j_3 & 2 & & & & \\ 0 & -\frac{1}{2} & 2 & & & \\ 0 & 0 & -\frac{1}{2} & 2 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 2 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhP_n = \mathrm{ddet}(A_{n+1})$.

Theorem 6. *Let $n \geq 0$ be an integer and*

$$A_{n+1} = \begin{bmatrix} j_1 + 2j_2 + 5j_3 & & & & & \\ \frac{1}{2} + \frac{1}{2}j_2 + j_3 & 2 & & & & \\ 0 & \frac{1}{2} & 2 & & & \\ 0 & 0 & \frac{1}{2} & 2 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 2 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then $BhP_n = \mathrm{pper}(A_{n+1})$.

We recall that a square matrix is said to be lower Hessenberg matrix if all entries above the superdiagonal are zero. Zatorsky and Lishchynskyy in [17] gave the relation between the paradeterminant of triangular matrix and the lower Hessenberg determinant by the following formula:

$$\mathrm{ddet}(A_n) = \det \begin{bmatrix} \{a_{11}\} & 1 & 0 & \cdots & 0 & 0 \\ \{a_{21}\} & \{a_{22}\} & 1 & \cdots & 0 & 0 \\ \{a_{31}\} & \{a_{32}\} & \{a_{33}\} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \{a_{n-1,1}\} & \{a_{n-1,2}\} & \{a_{n-1,3}\} & \cdots & \{a_{n-1,n-1}\} & 1 \\ \{a_{n1}\} & \{a_{n2}\} & \{a_{n3}\} & \cdots & \{a_{n,n-1}\} & \{a_{nn}\} \end{bmatrix}.$$

Using the above formula, we can write the bihyperbolic numbers of the Fibonacci type as the lower Hessenberg determinants as follows:

Corollary 1. *Let $n \geq 0$ be an integer. Then*

$$BhF_n = \det \begin{bmatrix} j_1 + j_2 + 2j_3 & 1 & 0 & \cdots & 0 & 0 \\ -1 - j_2 - j_3 & 1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{(n+1)\times(n+1)},$$

$$BhJ_n = \det \begin{bmatrix} j_1 + j_2 + 3j_3 & 1 & 0 & \cdots & 0 & 0 \\ -1 - 2j_2 - 2j_3 & 1 & 1 & \cdots & 0 & 0 \\ 0 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2 & 1 & 1 \\ 0 & 0 & 0 & \cdots & -2 & 1 \end{bmatrix}_{(n+1)\times(n+1)}$$

and

$$BhP_n = \det \begin{bmatrix} j_1 + 2j_2 + 5j_3 & 1 & 0 & \cdots & 0 & 0 \\ -1 - j_2 - 2j_3 & 2 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{(n+1)\times(n+1)}.$$

3. Concluding remarks

In the literature we can find many papers with some generalizations of numbers of the Fibonacci type or bihyperbolic numbers.

Among generalizations of Fibonacci type numbers, the one given by Horadam in [11] is well-known. Let p, q, n be integers. For $n \geq 0$ Horadam defined the numbers $W_n = W_n(W_0, W_1; p, q)$ by the recursive equation $W_n = p \cdot W_{n-1} - q \cdot W_{n-2}$, for $n \geq 2$ with fixed real numbers W_0, W_1. In particular, $W_n(0, 1; 1, -1) = F_n$, $W_n(0, 1; 1, -2) = J_n$ and $W_n(0, 1; 2, -1) = P_n$.

One of the generalizations of bihyperbolic numbers are the generalized commutative quaternions, introduced quite recently in [12]. Let $\mathbb{H}^c_{\alpha\beta}$ be the set of generalized commutative quaternions $\mathbf{x}$ of the form $\mathbf{x} = x_0 + x_1e_1 + x_2e_2 + x_3e_3$, where quaternionic units e_1, e_2, e_3 satisfy the equalities $e_1^2 = \alpha$, $e_2^2 = \beta$, $e_3^2 = \alpha\beta$, $e_1e_2 = e_2e_1 = e_3$, $e_2e_3 = e_3e_2 = \beta e_1$, $e_3e_1 = e_1e_3 = \alpha e_2$, and $x_0, x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$. If $\alpha > 0$, $\beta = 1$ then we obtain hyperbolic quaternions, in particular, bihyperbolic numbers if $\alpha = 1$. The nth generalized commutative Horadam quaternion $gc\mathcal{H}_n$ is defined (see [12]) as $gc\mathcal{H}_n = W_n + W_{n+1}e_1 + W_{n+2}e_2 + W_{n+3}e_3$.

In [9], Goy used the functions of triangular matrices to obtain new recurrent formulas for Horadam numbers with even (odd) subscripts. The bihyperbolic Fibonacci, Pell and Jacobsthal numbers considered in this paper are special cases of bihyperbolic Horadam numbers. Obtained results are a starting point for finding triangular matrices which paradeterminants or parapermanents are equal to bihyperbolic Horadam numbers or generalized commutative quaternions of the Fibonacci type.

4. Declarations

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