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Investigation of The Resolvent Kernel of a Higher Order Differential Equation With Normal Operator Coefficients On The Semi-Axis

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Abstract. Green function of a 2n-th order differential equation with normal coefficients on the half-axis is studied. We first consider the Green function of our equation with "frozen" coefficients. Using Levi's method, we obtain a Fredholm-type integral equation for the Green function of our problem, whose kernel is a Green function of a problem with constant coefficients. We prove an existence and uniqueness theorem for this integral equation in some Banach spaces of operator-valued functions. The main result of this paper is a theorem stating that the solution of the obtained integral equation is a Green function of our problem.

Key Words and Phrases: operator, operator-differential equations, resolvent, Green function, spectrum, integral equation, eigenvalues, eigenfunctions, Hilbert spaces, Banach spaces.

2010 Mathematics Subject Classifications: 34L10, 30E25, 34B24

1. Introduction and problem statement

The Sturm-Liouville equation with an unbounded self-adjoint operator coefficient with discrete spectrum was first considered by A.G. Kostyuchenko and B.M. Levitan in [10] and I.Ts. Hochberg, M.G. Krein [6], where the asymptotic formula was found for a number of eigenvalues not exceeding the given number λ.

Later, there appeared numerous papers dealing with the spectral properties of differential equations with operator coefficients. B.M.Levitan [11] thoroughly studied the Green function of the Sturm-Liouville equation with a self-adjoint operator coefficient on the whole real axis. A similar problem for more general

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operator-differential equations of second order was considered by E. Abdukadirov [1]. The results obtained in these papers have been generalized and extended by M. Bayramoglu [5] to the higher order operator-differential equations with self-adjoint operator coefficients on the whole axis. We can also mention the works by H.D. Orudzhev, Q.L. Shahbazova [12], G.I. Aslanov [3], A.A. Abudov [2], etc. In case where the coefficient is a normal operator, the Green function of the Sturm-Liouville equation has been studied by E.G. Kleiman [9], M.G. Dushdirov [7], G.I.Kasumova [8]. The Green function of a higher order operator-differential equation with a normal operator coefficient was treated in [4].

Let H be a separable Hilbert space. Consider in the space $H_1 = L_2 (H : [0, \infty))$ the operator generated by the operator-differential expression

$$l (y) = (-1)^n \left(P (x) y^{(n)}\right)^{(n)} + Q (x) y \tag{1}$$

and the boundary conditions

$$y^{(l_1)} (0) = y^{(l_2)} (0) = ... = y^{(l_n)} (0) = 0, \quad 0 \le l_1 < l_2 < ... < l_n \le 2n - 1. \tag{2}$$

where $y \in H_1$ and the derivatives are understood in the strong sense. Let D' be a totality of all functions of the form $\sum\limits_{k=1}^{p} \varphi_k (x) f_k$, where $\varphi_k (x)$ are finite, $2n$-times continuously differentiable functions, and $f_k \in D (Q)$. The domain of definition $D(Q)$ is independent of x.

Let's define the operator L generated by the expression (1) and the boundary conditions (2) with the domain of definition $D^{'}$.

For the operator coefficients $P(x)$ and $Q(x)$, we assume the following:

1. The operator function $P (x)$ is n-times everywhere uniformly differentiable and for all $x \in [0, \infty)$, $h \in H$ the condition

$$m (h, h)_H \le (P (x) h, h) \le M (h, h)_H , \ \ m, M > 0,$$

holds.

2. For all $x \in [0, \infty)$ the operators $Q (x)$ are bounded from below normal operators in H, i.e. for all $f \in H$ $(Q (x) f, f) \ge d (f, f)$, they are inverses of a completely continuous operator. Then the operators $K (x) = P^{-\frac{1}{2}} (x) Q (x) P^{-\frac{1}{2}} (x)$ are also inverses of a completely continuous operator for all $x \in [0, \infty)$. It is assumed that all eigenvalues of the operator $K (x)$ lie outside some domain $\Omega = \{\lambda : \ |\arg \lambda - \pi| < \varepsilon_0, \ \ 0 < \varepsilon_0 < \pi\}$ in the complex plane.

Denote by $\{\beta_k (x)\}$ the eigenvalues of the operator $K (x)$ in an ascending order of their moduli, i.e.

$$|\beta_1 (x)| \le |\beta_2 (x)| \le ... \le |\beta_k (x)| \le ...,$$

and let the following conditions be fulfilled: the series $\sum_{j=1}^{\infty} |\beta_j(x)|^{\frac{1-4n}{2n}}$ converges for all $x \in [0,\infty)$ and its sum $F(x) \in L_1[0,\infty)$.

3. There exist constant numbers $c>0$, $0<a<\frac{2n+1}{2n}$ such that for all x and $|x-\xi| \le 1$ the following inequality is valid:

$$\| [Q(\xi) - Q(x)] Q^{-a}(x) \| \le c_1 |x-\xi| ;$$

4. For $|x-\xi|>1$ we have the inequality

$$\left\| K(\xi) \exp\left(-\frac{Im\varepsilon_1}{2} |x-\xi| \omega \right) \right\|_H \le c_2,$$

where $\omega = \left\{ K(x) + \mu P^{-1}(x) \right\}^{\frac{1}{2n}}$, $\mu>0$, $Im\varepsilon_1 = \min_i \left\{ Im\varepsilon_i > 0,\ \varepsilon_i^{2n} = -1 \right\}$.

5. For all $x, \xi \in [0,\infty)$, the following inequalities are fulfilled:

$$\left\| Q(x)\, P^{-\frac{1}{2}}(x)\, Q^{-1}(x) \right\|_H < c_3, \quad \left\| Q(x)\, P^{-\frac{1}{2}}(x)\, P^{-\frac{1}{2}}(\xi)\, Q^{-1}(\xi) \right\|_H < c_4,$$

where c_1, c_2, c_3, c_4 are positive constants.

The goal of this paper is to study the Green function of the operator L.

The following theorem is true:

Theorem 1. *If the coefficients $P(x)$ and $Q(x)$ of the operator L satisfy conditions 1)-5), then for sufficiently large $\mu>0$ there exists an inverse operator $R_\mu = (L+\mu E)^{-1}$, which is an integral operator with an operator-valued kernel $G(x,\eta;\mu)$. The operator function $G(x,\eta;\mu)$ depends on the variables x,η $(0 \le x,\eta < \infty)$ and the parameter μ, and has the following properties:*

1. *There exist strongly continuous derivatives*

$$\frac{\partial^k G(x,\eta;\mu)}{\partial \eta^k}, \quad k = 0,1,...,2n-2;$$

2. *There exists a strong derivative $\frac{\partial^{2n-1} G(x,\eta;\mu)}{\partial \eta^{2n-1}}$, with*

$$\frac{\partial^{2n-1} G(x,x+0;\mu)}{\partial \eta^{2n-1}} - \frac{\partial^{2n-1} G(x,x-0;\mu)}{\partial \eta^{2n-1}} = (-1)^n P^{-1}(x);$$

3. $(-1)^n \left(G_\eta^{(n)}(x,\eta;\mu)\, P(\eta) \right)^{(n)} + G(x,\eta;\mu) [Q(\eta) + \mu E] = 0;$

4. $\left.\frac{\partial^{l_1} G}{\partial \eta^{l_1}}\right|_{\eta=0} = \left.\frac{\partial^{l_2} G}{\partial \eta^{l_2}}\right|_{\eta=0} = ... = \left.\frac{\partial^{l_n} G}{\partial \eta^{l_n}}\right|_{\eta=0} = 0.$

The theorem will be proved in two steps. First we will construct the Green function of the operator L_1, generated by the expression

$$m_1(y) = (-1)^n \left(P(\xi)\, y^{(n)}\right)^{(n)} + Q(\xi)\, y + \mu\, y \tag{3}$$

and the boundary conditions

$$y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_n)}(0) = 0, \tag{4}$$

where "ξ" is a fixed point in the interval $[0,\infty)$. In the second step the Green function of the operator L determined by variable coefficient differential expression (1) and boundary conditions (2) is constructed.

2. Constructing the Green function of the operator L_1

We will look for the Green function $G(x,\eta,\xi;\mu)$ of the operator L_1 in the following form:

$$G_1(x,\eta,\xi;\mu) = g(x,\eta,\xi;\mu) + V(x,\eta,\xi;\mu), \tag{5}$$

where $g(x,\eta,\xi;\mu)$ is a Green function of the equation $m_1(y) = 0$ on the whole axis. As is known, it can be represented in the following form:

$$g(x,\eta,\xi;\mu) = \frac{1}{2\pi}\int_0^\infty \left[P(\xi)\, S^{2n} + Q(\xi) + \mu\, E\right]^{-1} e^{is(x-\eta)} ds =$$

$$= \frac{1}{2\pi} P^{-\frac{1}{2}}(\xi)\, \omega^{1-2n} \sum_{k=1}^{n} \varepsilon_k \exp\left(i\varepsilon_k \left|x-\eta\right| \omega\right) \cdot P^{-\frac{1}{2}}(\xi), \tag{6}$$

where ε_k's denote the roots of $2n$-th degree roots of (-1), laying in upper half-plane, and $\omega = \left[K(\xi) + \mu P^{-1}(\xi)\right]^{\frac{1}{2n}}$. The function $V(x,\eta,\xi;\mu)$ is a bounded solution of the problem

$$m_1(V) = 0, \tag{7}$$

$$V^{(l_j)}(x,\eta,\xi;\mu)\Big|_{x=0} = -\, g^{(l_j)}(x,\eta,\xi;\mu)\Big|_{x=0}, \quad j = 1,2,...,n, \tag{8}$$

as $x \to \infty$. The solution of equation (7) is represented in the form

$$V(x,\eta,\xi;\mu) = \frac{1}{2ni} P^{-\frac{1}{2}}(\xi)\, \omega^{1-2n} \sum_{p=1}^{n} A_p(\eta,\xi;\mu)\, e^{i\varepsilon_p \omega x} \cdot P^{-\frac{1}{2}}(\xi). \tag{9}$$

The coefficients $A_k(\eta,\xi,\mu)$ are defined from the boundary conditions (8).

We have

$$\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega^{1-2n}\sum_{p=1}^{n}A_p(\eta,\xi;\mu)\,\varepsilon_p^{l_j}e^{i\varepsilon_p\omega x}\cdot P^{-\frac{1}{2}}(\xi)\Big|_{x=0}=$$

$$=-\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega^{1-2n}\sum_{k=1}^{n}\varepsilon_k^{l_j+1}(i\omega)^{l_j}\,e^{i\varepsilon_k\omega|x-\eta|}\Big|_{x=0},\quad j=1,2,...,n.$$

Hence,

$$\sum_{p=1}^{n}A_p(\eta,\xi;\mu)\,\omega^{l_j}=-\sum_{k=1}^{n}\varepsilon_k^{l_j+1}e^{i\varepsilon_k\omega\eta},\quad j=1,2,...,n. \tag{10}$$

Solving the equation (10) by Cramer's rule, we find the coefficients $A_p(\eta,\xi;\mu)$, $p=1,2,...,n$ in the following form:

$$A_p(\eta,\xi;\mu)=-\varepsilon_p e^{i\varepsilon_p\omega\eta}.$$

Substituting the expression $A_p(\eta,\xi;\mu)$ in (9), we get

$$V(x,\eta,\xi;\mu)=-\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega^{1-2n}\sum_{p=1}^{n}\varepsilon_p e^{i\varepsilon_p\omega(x+\eta)}\cdot P^{-\frac{1}{2}}(\xi). \tag{11}$$

Then the Green function of problem (3)-(4) becomes

$$G_1(x,\eta,\xi;\mu)=\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega^{1-2n}\sum_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega|x-\eta|}\cdot P^{-\frac{1}{2}}(\xi)-$$

$$-\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega^{1-2n}\sum_{p=1}^{n}\varepsilon_p e^{i\varepsilon_p\omega(x+\eta)}P^{-\frac{1}{2}}(\xi).$$

We can rewrite the obtained formula as follows:

$$G_1(x,\eta,\xi;\mu)=$$

$$=\begin{cases}\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega^{1-2n}\sum\limits_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega(x-\eta)}\left\{E-e^{2i\varepsilon_k\omega\,\eta}\right\}P^{-\frac{1}{2}}(\xi), & x\geq\eta,\\ \frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega^{1-2n}\sum\limits_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k(x-\eta)}\left\{E-e^{2i\varepsilon_k\omega\,x}\right\}P^{-\frac{1}{2}}(\xi), & x\leq\eta.\end{cases} \tag{12}$$

Since $Re\,(2i\varepsilon_k\omega\eta)<0$, we have $\left\|e^{2i\varepsilon_k\omega\eta}\right\|_H\to 0,\quad \left\|e^{2i\varepsilon_k\omega x}\right\|_H\to 0$ as $\mu\to\infty$.

Thus, from (12) we get

$$G_1(x,\eta,\xi;\mu)=\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega^{1-2n}\sum_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega|x-\eta|}\cdot P^{-\frac{1}{2}}(\xi)\left(E-r(x,\eta,\xi;\mu)\right)=$$

$$=g(x,\eta,\xi;\mu)\,\left[E-r(x,\eta,\xi;\mu)\right], \tag{13}$$

and we have $\|r(x,\eta,\xi,\mu)\|_H=o(1)$ uniformly with respect to (x,η) as $\mu\to\infty$.

3. Constructing the Green function $G(x,\eta,\mu)$ of the operator L.

Now we construct and study some properties of the Green function of the operator L, generated by differential expression (1) and boundary conditions (2).

Let us consider the following integral equation:

$$G(x,\eta,\mu)=G_1(x,\eta,\mu)-\int_0^\infty G(x,\xi,\mu)\left[Q(\xi)-Q(x)\right]G(\xi,\eta;\mu)\,d\xi+$$

$$+\frac{1}{2ni}\int_0^\infty P^{-\frac{1}{2}}(x)\,\omega\sum_{k=1}^{n}\varepsilon_k\exp\{i\varepsilon_k\omega|x-\eta|\}\times$$

$$\times\left[E-r(x,\xi,\mu)\right]P^{-\frac{1}{2}}(x)\left[P(\xi)-P(x)\right]G(x,\xi,\mu)\,d\xi+$$

$$+(-1)^n\sum_{m=1}^{n}C_n^m\int_0^\infty\left(G_1^{(2n-m)}(\xi,\eta;\mu)\right)\cdot P_\xi^{(m)}(\xi)\cdot G(\xi,\eta;\mu)\,d\xi \tag{14}$$

We are going to prove that for sufficiently large values of μ the integral equation (14) is solvable and its solution is a Green function of the operator L. We will consider this equation in Banach spaces $X_1,\ X_2,\ X_3^{(p)},\ X_2^{(s)},\ X_4^{(s)}$ and $X_5\ \ (p\geq 1)$, whose elements are the operator functions $A(x,\eta)$ in the space $H,\ 0\leq x,\eta<\infty$. These spaces were introduced by B.M. Levitan [11].

We first estimate the norm of the operator function $G_1(x,\eta,\mu)$. The inequality $|\lambda+\mu|>\mu\sin\varepsilon_0$ is valid in the complex plane for all λ, outside the sector Λ.

Since

$$\left(\left\{K(x)+\mu P^{-1}(x)\right\}f,f\right)=$$

$$=\left(\{Q(x)+\mu E\}P^{-\frac{1}{2}}(x)f,P^{-\frac{1}{2}}(x)f\right)\geq\gamma(f,f),\quad \gamma=(\mu+d)M^{-1},$$

using spectral representation of a normal operator, we get

$$\left(\omega^{1-2n}\sum_{k=1}^{n}\varepsilon_k \exp\left\{i\varepsilon_k\left|x-\eta\right|\omega\right\}\cdot\left(E-r\left(x,\eta,\mu\right)\right)f,f\right)\le$$

$$\le\gamma^{\frac{1-2n}{2n}}n\left(1+o\left(1\right)\right)\exp\left(-Im\varepsilon_1\left|x-\eta\right|\gamma^{\frac{1}{2n}}\right)\left(f,f\right). \quad (15)$$

From (13) and inequality (15) it follows

$$\left\|G_1\left(x,\eta,\mu\right)\right\|_H\le$$

$$\le\frac{1+0\left(1\right)}{2n}\int_0^\infty\left(\lambda+\mu\right)^{\frac{1-2n}{2n}}\left|P^{-\frac{1}{2}}\left(x\right)\right|\cdot\max\varepsilon_k e^{i\varepsilon_k\left(\lambda+\mu\right)^{\frac{1}{2n}}\left|x-\eta\right|}dE\left(\lambda\right)\cdot\left\|P^{-\frac{1}{2}}\left(x\right)\right\|\le$$

$$\le\frac{1+o\left(1\right)}{2m\delta_0^{\frac{1-2n}{2n}}}\gamma^{\frac{1-2n}{2n}}\exp\left(-2Im\varepsilon_1\left|x-\eta\right|\gamma^{\frac{1}{2n}}\delta_0^{\frac{1}{2n}}\right).$$

Therefore,

$$\int_0^\infty\left\|G_1\left(x,\eta,\mu\right)\right\|_H^2 d\eta\le\frac{\left(1+0\left(1\right)\right)^2}{4m^2\delta_0^{\frac{1-2n}{2n}}}\int_0^\infty e^{-2Im\varepsilon_1\left|x-\eta\right|\gamma^{\frac{1}{2n}}\delta_0^{\frac{1}{2n}}}d\eta\le$$

$$\le\frac{c^2}{8m^2Im\varepsilon_1\delta_0^{\frac{4n-1}{2n}}}\cdot\frac{1}{\gamma^{\frac{4n-1}{2n}}}. \quad (16)$$

From this estimate it follows $G_1\left(x,\eta;\mu\right)\in X_3^{(1)}$.

Using condition 2) of Introduction, we can show that for the norm of the Hilbert-Schmidt operator the following inequality is valid:

$$\int_0^\infty\left\{\int_0^\infty\left\|G_1\left(x,\eta,\mu\right)\right\|_2^2 d\eta\right\}dx<\infty, \quad (17)$$

i.e. $G_1\left(x,\eta;\mu\right)\in X_2$.

Under some additional conditions on operator coefficients, we can show that the operator function $G_1\left(x,\eta;\mu\right)$ also belongs to all of the other Banach spaces mentioned above. Therefore, we can look for the solution of equation (14) in these spaces.

To this end, let's consider in these spaces the operator

$$TA\left(x,\eta\right)=G_1\left(x,\eta,\mu\right)-\int_0^\infty G\left(x,\xi,\mu\right)\left[Q\left(\xi\right)-Q\left(x\right)\right]A\left(\xi,\eta\right)d\xi+$$

$$+\frac{1}{2ni}\int_0^\infty P^{-\frac{1}{2}}(x)\,\omega\sum_{k=1}^n \varepsilon_k \exp\left[i\varepsilon_k\omega\left|x-\eta\right|\right]\times$$

$$\times\left[E-r(x,\xi,\mu)\right]P^{-\frac{1}{2}}(x)\left[P(\xi)-P(x)\right]A(\xi,\eta)\,d\xi+$$

$$+(-1)^n\sum_{m=1}^n C_n^m\int_0^\infty\left(G_{1\xi}^{(2n-m)}(x,\xi,\mu)\right)\cdot P_\xi^{(m)}(\xi)\cdot\,A(\xi,\eta)\,d\xi. \qquad (18)$$

The following lemma plays an important role in this work:

Lemma 1. *If the operator functions $P\ (x)$ and $Q\ (x)$ satisfy the conditions 1)-5) of Introduction, then for sufficiently large $\mu>0$ the operator T is a contraction operator in all above spaces $X_1,\ \ X_2,\ \ X_3^{(p)}, ..., X_5$.*

This lemma is proved in the same way as the corresponding one in [9], therefore we omit the proof here. From this lemma it follows that the equation (14) has a unique solution in all the spaces under consideration if the operator function is an element of the corresponding space. From the estimate (16) it follows that $G_1(x,\eta;\mu)\in X_3^{(2)}$. Then, for sufficiently large $\mu>0$, $G(x,\eta;\mu)$ also belongs to the space $X_3^{(2)}$. The belonging of $G_1(x,\eta;\mu)$ to the space X_2 follows from the estimate (17). Therefore, the function $G(x,\eta;\mu)$ also belongs to the space X_2.

Further, using integral equations (14), we prove the existence of strongly continuous derivatives $\frac{\partial^k G}{\partial\eta^k},\quad k=1,2,...,2n-2$.

For all $x\neq\eta$, there exists a strongly continuous derivative $\frac{\partial^{2n-1}G(x,\eta,\mu)}{\partial\eta^{2n-1}}$, and for $x=\eta$ there is a discontinuity of first kind. We have

$$\frac{\partial^{2n-1}G(x,x+0,\mu)}{\partial\eta^{2n-1}}-\frac{\partial^{2n-1}G(x,x-0,\mu)}{\partial\eta^{2n-1}}=(-1)^n P^{-1}(x)\,.$$

It can be shown that the solution of integral equation (14) satisfies the equation

$$(-1)^n\left(G_\eta^{(n)}(x,\eta,\mu)\,P(\eta)\right)^{(n)}+G(x,\eta,\mu)\left[Q(\eta)+\mu E\right]=0$$

and the boundary conditions

$$\left.\frac{\partial^{l_1}G}{\partial\eta^{l_1}}\right|_{\eta=0}=\left.\frac{\partial^{l_2}G}{\partial\eta^{l_2}}\right|_{\eta=0}=...=\left.\frac{\partial^{l_n}G}{\partial\eta^{l_n}}\right|_{\eta=0}=0.$$

This completes the proof of the theorem. ◄

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