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Transformation Operator for Sturm-Liouville Operators with Growing Potentials

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Abstract. This paper considers a pair of Sturm-Liouville operators $A = -\frac{d^2}{dx^2} + cx^\alpha$ and $B = -\frac{d^2}{dx^2} + cx^\alpha + q(x)$ on the semi-axis $[0, \infty)$, $c = const$, $\alpha \geq 1$. A transformation operator with a condition at infinity is constructed for the operators A and B. An estimate is obtained for the kernel of the transformation operator.

Key Words and Phrases: Sturm-Liouville operator, transformation operator, second-order hyperbolic equation, Riemann function.

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1. Introduction and main result

In many problems of the spectral theory of one-dimensional linear differential equations of the second order, the apparatus of transformation operators is effectively used (see [1] and the references therein). This apparatus is closely related to the theory of generalized shift operators developed by J. Delsarte [2] and B.M. Levitan [3].

For arbitrary Sturm-Liouville equations, transformation operators were constructed by A.Ya. Povzner [4]. B.Ya. Levin [5] introduced transformation operators with a condition on infinity, the existence of which was not obvious from the general theory of generalized shift operators. The role of transformation operators increased significantly after they began to be applied to inverse spectral problems (see [1,6,7]).

Consider the following second-order differential operators:

$$A = -\frac{d^2}{dx^2} + cx^\alpha,\ B = -\frac{d^2}{dx^2} + cx^\alpha + q(x),$$

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given on the half-axis $(0,\infty)$, where $\alpha \geq 1$ and c is a complex number. We will assume that the complex-valued function $q(x)$ satisfies the conditions

$$q(x) \in C^{(1)}(0,\infty), \quad \int_0^\infty \left| x^{2\alpha+3} e^{\omega(x)} q(x) \right| dx < \infty, \tag{1}$$

where

$$\omega(x) = \left(\frac{|c|}{\alpha+1} \right)^{\frac{1}{2}} (2x)^{\frac{\alpha+2}{2}}, \tag{2}$$

which are assumed to be satisfied everywhere below.

We introduce the topological space used in [1,3]. The space E consists of continuous complex-valued functions $f(x)$, defined on interval $(0,\infty)$, which possess a continuous second derivative and satisfy the condition $\sup\limits_{x>0}\{|f(x)| + |f'(x)|\} < \infty$. The topology in E is defined by uniform convergence of functions and their first derivatives in the interval $(0,\infty)$. Recall the definition of the transformation operator (see [1,2]).

Definition 1. *A continuous linear operator X defined on the whole of E is called a transformation operator for a pair of operators A and B if it satisfies the following two conditions:*

1. *X has a continuous inverse X^{-1} on the whole of E;*
2. *There is an operator equality*

$$AX = XB. \tag{3}$$

Note that in the case $c = 0$, the transformation operator with the condition on infinity for the pair of operators A and B has been studied in detail in the works of many authors (see, for example, [1,5,6,7]). At the same time, when constructing a transformation operator with the condition at infinity for Sturm-Liouville operators with growing potentials, significant difficulties arise (see [8]). In this direction, we can mention the works [9,10,11,12,13] in which transformation operators for a pair of operators A and B were obtained for $c = 1$, $\alpha = 1$ and $c = \pm 1$, $\alpha = 2$.

This paper is devoted to the existence of a transformation operator for a pair of operators A and B with a boundary condition at infinity. In this case, the derivation of integral equation for the kernel of the transformation operator, as in [9]-[13], is carried out according to the scheme using the Riemann function. However, in general, in contrast to [9,10,11,12,13], it is not possible to find an explicit form of the Riemann function for the corresponding second-order hyperbolic equation. The proof of the existence of the Riemann function and its evaluation require a number of tricks. The deterioration of the properties of the Riemann function and its derivatives leads to the need for more stringent conditions on the potential $q(x)$.

The results of this work can be used in the study of spectral problems for the operator B. Note that various problems of the spectral theory of the operators A and B were studied in [14,15,16,17,18].

Let us formulate the main result of the work. Let

$$\sigma_0(x) = \int_x^\infty |q(t)| e^{\omega(t)} dt, \sigma_1(x) = \int_x^\infty \sigma_0(t)\, dt. \tag{4}$$

Theorem 1. *For the pair of operators A and B there is a transformation operator X, which can be represented as*

$$Xf(x) = f(x) + \int_x^\infty K(x,t)\ f(t)\, dt. \tag{5}$$

Moreover, the kernel $K(x,t)$ is a continuous function and satisfies the following relations

$$|K(x,t)| \le \frac{1}{2}\sigma_0\left(\frac{x+t}{2}\right) e^{\sigma_1\left(\frac{x+t}{2}\right)}, \tag{6}$$

$$K(x,x) = \frac{1}{2}\int_x^\infty q\ (t)\, dt. \tag{7}$$

2. Proof of the Theorem 1

Following B.M. Levitan (see [1,3]), we first assume that the function $K(x,t)$ satisfies the following conditions:

$$\begin{gathered}\lim_{x+t\to\infty} K(x,t) = 0,\ \lim_{x+t\to\infty} \tfrac{\partial K}{\partial t} = 0,\\ \int_x^\infty |K(x,t)|\, dt < \infty,\ \int_x^\infty \left|\tfrac{\partial K}{\partial t}\right| dt < \infty, \int_x^\infty \left|\tfrac{\partial K}{\partial x}\right| dt < \infty,\\ \int_x^\infty \left|\tfrac{\partial^2 K}{\partial t^2}\right| dt < \infty, \int_x^\infty \left|\tfrac{\partial^2 K}{\partial x^2}\right| dt < \infty.\end{gathered} \tag{8}$$

From (1)-(3) we find that the equality (3) is certainly satisfied if only the kernel $K(x,t)$ satisfies the second-order hyperbolic equation

$$\frac{\partial K(x,t)}{\partial x^2} - \frac{\partial K(x,t)}{\partial t^2} - (cx^\alpha - ct^\alpha - q(t))\ \ K^+(x,t) = 0,\ 0 < x < t, \tag{9}$$

and the condition

$$K\left(x,x\right)=\frac{1}{2}\int_{x}^{\infty}q\ \left(t\right)\ dt. \qquad (10)$$

Let us rewrite here the first of conditions (8) again:

$$\lim_{x+t\to\infty}K\left(x,t\right)=0. \qquad (11)$$

We shall show that the problem (9)-(11) has a unique solution if conditions (1) are fulfilled and the solution, i.e. the function $K(x,t)$, satisfies the rest of conditions (8).

Let us reduce Eq. (9) satisfying the conditions (10), (11) to an integral equation. To this end, we reduce Eq. (8) to the canonical form; to do this, we introduce new variables by the formula

$$\frac{t+x}{2}=\xi,\ \frac{t-x}{2}=\eta.$$

Setting

$$U\left(\xi,\eta\right)=U\left(\frac{t+x}{2},\frac{t-x}{2}\right)=K\left(x,t\right)=K\left(\xi-\eta,\xi+\eta\right),$$

we obtain in the domain $0<\eta<\eta_0\leq\xi_0<\xi$ the following equation for the function $U(\xi,\eta)$:

$$L\left[U\right]\equiv\frac{\partial^2U\left(\xi,\eta\right)}{\partial\xi\partial\eta}-\left[\left(\xi+\eta\right)^{\alpha}-\left(\xi-\eta\right)^{\alpha}\right]U\left(\xi,\eta\right)=U\left(\xi,\eta\right)q\left(\xi+\eta\right), \qquad (12)$$

with the boundary conditions

$$U(\xi,0)=\frac{1}{2}\int_{\xi}^{\infty}q\left(\alpha\right)d\alpha, \qquad (13)$$

$$\lim_{\xi\to\infty}U(\xi,\eta)=0,\ \eta>0. \qquad (14)$$

We introduce the Riemann function $R(\xi,\eta;\xi_0,\eta_0)$) of the equation $L\left[U\right]=\psi(\xi,\eta)$, where $\psi(\xi,\eta)=U(\xi,\eta)q(\xi+\eta)$, i.e., a function satisfying for $0<\eta<\eta_0\leq\xi_0<\xi$ the equation

$$L^{*}(R)\equiv\frac{\partial^2R}{\partial\xi\,\partial\eta}-\left[\left(\xi+\eta\right)^{\alpha}-\left(\xi-\eta\right)^{\alpha}\right]R=0 \qquad (15)$$

and the following conditions on the characteristics:

$$R(\xi,\eta;\xi_0,\eta_0)\left.\right|_{\xi=\xi_0}=1,0\leq\eta\leq\eta_0, \qquad (16)$$

$$R(\xi,\eta;\xi_0,\eta_0)\,|_{\eta=\eta_0}\,=1,\ \xi_0\le\xi<\infty. \tag{17}$$

Let us prove the existence of the function $R(\xi,\eta,\,\xi_0,\eta_0)$ and establish some of its properties. Problem (15)-(17) is obviously equivalent to the integral equation

$$R(\xi,\eta,\,\xi_0,\eta_0)=1-\int_{\xi_0}^{\xi}du\int_{\eta}^{\eta_0}p\,(u,v)\,R(u,v,\,\xi_0,\eta_0)dv, \tag{18}$$

where

$$p\,(u,v)=c\,[(u+v)^{\alpha}-(u-v)^{\alpha}]\,. \tag{19}$$

Let us prove that the equation (18) has a unique solution which can be obtained by the method of successive approximations. In fact, set

$$R_o(\xi,\eta,\,\xi_0,\eta_0)=1\,,$$

$$R_n(\xi,\eta,\,\xi_0,\eta_0)=-\int_{\xi_0}^{\xi}du\int_{\eta}^{\eta_0}p\,(u,v)\,R_{n-1}(u,v,\,\xi_0,\eta_0)dv. \tag{20}$$

It is easy to see that in the domain $\{(\xi,\eta):0\le\eta\le\eta_0\le\xi_0\le\xi\}$ the function $p\,(\xi,\eta)$ satisfies the estimate $|p\,(\xi,\eta)|\le 2^{\alpha}\,|c|\,\xi^{\alpha}$. It follows that

$$|R_1\,(\xi,\eta,\,\xi_0,\eta_0)|\le\frac{2^{\alpha}\,|c|}{\alpha+1}\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)(\eta_0-\eta)\,,$$

$$|R_2\,(\xi,\eta,\,\xi_0,\eta_0)|\le\left(\frac{2^{\alpha}\,|c|}{\alpha+1}\right)^2\frac{\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)^2(\eta_0-\eta)^2}{(2!)^2}$$

and, by induction,

$$|R_n\,(\xi,\eta,\,\xi_0,\eta_0)|\le\left(\frac{2^{\alpha}\,|c|}{\alpha+1}\right)^n\frac{\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)^n(\eta_0-\eta)^n}{(n!)^2}. \tag{21}$$

The estimates show that the series $R\,(\xi,\eta,\,\xi_0,\eta_0)=\sum\limits_{n=0}^{\infty}R_n\,(\xi,\eta,\,\xi_0,\eta_0)$ of continuous functions in the domain $\{(\xi,\eta):0\le\eta\le\eta_0\le\xi_0\le\xi\}$ converges absolutely and uniformly in each of its bounded subsets, and its sum satisfies equation (18). Moreover, by virtue of (21) we have

$$\begin{aligned}|R\,(\xi,\eta,\,\xi_0,\eta_0)|&\le\sum_{n=0}^{\infty}\left(\tfrac{2^{\alpha}|c|}{\alpha+1}\right)^n\frac{\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)^n(\eta_0-\eta)^n}{(n!)^2}=\\&=J_0\left(i\sqrt{\tfrac{2^{\alpha+2}|c|}{\alpha+1}}\left(\xi^{\alpha+1}-\xi_0^{\alpha+1}\right)(\eta_0-\eta)\right),\end{aligned} \tag{22}$$

where $J_0(z) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$ is the Bessel function of the first kind. Taking into account the inequality $|J_0(z)| \leq e^{|Imz|}$ (see [19]), from (22) we finally obtain

$$|R(\xi, \eta, \xi_0, \eta_0)| \leq e^{\omega(\xi)}, \tag{23}$$

where $\omega(\xi)$ is defined by the formula (2).

Next, from (18) we find

$$R_\xi(\xi, \eta, \xi_0, \eta_0) = -\int_\eta^{\eta_0} p(\xi, v)\, R(\xi, v, \xi_0, \eta_0) dv,$$

$$R_\eta(\xi, \eta, \xi_0, \eta_0) = \int_{\xi_0}^{\xi} p(u, \eta)\, R(u, \eta, \xi_0, \eta_0) du.$$

From the last equations and the continuous differentiability of the function $p(u, v)$ it follows that the function $R(\xi, \eta; \xi_0, \eta_0)$ is twice continuously differentiable. Moreover, from the last equations, the equations obtained from them by differentiation and the estimates (23), we obtain for all $0 < \eta \leq \eta_0 \leq \xi_0 < \xi$:

$$\begin{aligned} &|R_\xi(\xi, \eta, \xi_0, \eta_0)| \leq |c|\, 2^\alpha \xi^{\alpha+1} e^{\omega(\xi)}, \\ &|R_\eta(\xi, \eta, \xi_0, \eta_0)| \leq |c|\, 2^\alpha \xi^{\alpha+1} e^{\omega(\xi)}, \\ &|R_{\xi\eta}(\xi, \eta, \xi_0, \eta_0)| \leq |c|\, 2^\alpha \xi^{\alpha} e^{\omega(\xi)}, \\ &|R_{\xi\xi}(\xi, \eta, \xi_0, \eta_0)| \leq |c|\, 2^{\alpha-1} \xi^{\alpha} e^{\omega(\xi)} + (|c|\, 2^\alpha)^2\, \xi^{2\alpha+2} e^{\omega(\xi)}, \\ &|R_{\eta\eta}(\xi, \eta, \xi_0, \eta_0)| \leq |c|\, 2^{\alpha-1} \xi^{\alpha} e^{\omega(\xi)} + (|c|\, 2^\alpha)^2\, \xi^{2\alpha+2} e^{\omega(\xi)}. \end{aligned} \tag{24}$$

Remark 1. *Let $c = 1$ and $\alpha = 1$. Then from relations* (17), (18) *we have*

$$p(u, v) = 2v,$$

$$R_n(\xi, \eta, \xi_0, \eta_0) = (-1)^n \frac{(\xi - \xi_0)^n}{n!} \frac{(\eta_0^2 - \eta^2)^n}{n!}.$$

Whence it follows that

$$\begin{aligned} &R(\xi, \eta, \xi_0, \eta_0) = \textstyle\sum_{n=0}^{\infty} R_n(\xi, \eta, \xi_0, \eta_0) = \\ &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{2\sqrt{(\xi - \xi_0)(\eta_0^2 - \eta^2)}}{2} \right)^{2n} = J_0\left(2\sqrt{(\xi - \xi_0)(\eta_0^2 - \eta^2)}\right) . \end{aligned}$$

Similarly, for $c = 1$, $\alpha = 2$ from (19), (20) we find

$$R(\xi, \eta, \xi_0, \eta_0) = J_0\left(2\sqrt{(\xi^2 - \xi_0^2)(\eta_0^2 - \eta^2)}\right) ,$$

and in both cases, instead of estimate (23), we obtain

$$|R\,(\xi, \eta,\, \xi_0, \eta_0)| \leq 1. \tag{25}$$

In the case $c = -1$, $\alpha = 2$ we have $R\,(\xi, \eta,\, \xi_0, \eta_0) = J_0\left(2\sqrt{(\,\xi_0^2 - \xi^2)(\eta_0^2 -\, \eta^2)}\right)$. In this case, estimate (23) is improved as follows:

$$|R\,(\xi, \eta,\, \xi_0, \eta_0)| \leq e^{2\xi^2}. \tag{26}$$

We are now ready to prove the theorem. Applying the Riemann method (see, for example, [20]) to equation (12), we obtain the following integral equation for $U(\xi_0, \eta_0)$:

$$U(\xi_0, \eta_0) = \frac{1}{2}\int_{\xi_0}^{\infty} R(\xi, 0; \xi_0, \eta_0) q(\xi) d\xi +$$

$$+ \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} U(\xi, \eta) q(\xi + \eta) R(\xi, \eta; \xi_0, \eta_0) d\eta. \tag{27}$$

Consequently, to solve problem (9)-(11) it is sufficient to solve the integral equation (27) with respect to $U(\xi_0, \eta_0)$. Let us show that if conditions (1) are met, this integral equation can be solved using the method of successive approximations. Let

$$U_0(\xi_0, \eta_0) = \frac{1}{2}\int_{\xi_0}^{\infty} R(\xi, 0; \xi_0, \eta_0) q(\xi) d\xi,$$

$$U_n(\xi_0, \eta_0) = \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} U_{n-1}(\xi, \eta) q(\xi + \eta) R(\xi, \eta; \xi_0, \eta_0) d\eta.$$

Using (4), (23), the following estimates are easily obtained:

$$|U_0(\xi_0, \eta_0)| \leq \frac{1}{2}\int_{\xi_0}^{\infty} |\, R(\xi, 0; \xi_0, \eta_0)\,|\; |q(\xi)|\, d\xi \leq$$

$$\leq \frac{1}{2}\int_{\xi_0}^{\infty} e^{\omega(\xi)}\, |q\,(\xi)\,|\; d\xi = \frac{1}{2}\sigma_0\,(\xi_0)\;,$$

$$|U_1\,(\xi_0, \eta_0)| \leq \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} |U_0\,(\xi, \eta)| \cdot |q\,(\xi + \eta)| \cdot R\,(\xi, \eta;\, \xi_0, \eta_0)\, d\eta \leq$$

$$\leq \frac{1}{2}\int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} \sigma_0\,(\xi)\; \left|e^{\omega(\xi+\eta)} q\,(\xi + \eta)\right| d\eta \leq \frac{1}{2}\int_{\xi_0}^{\infty} \sigma_0\,(\xi)\, d\xi \int_0^{\eta_0} \left|e^{\omega(\xi+\eta)} q\,(\xi + \eta)\right|\, d\eta \leq$$

$$\leq \frac{\sigma_0(\xi_0)}{2}\int\limits_{\xi_0}^{\infty} d\xi \int\limits_{0}^{\eta_0}\left|e^{\omega(\xi+\eta)}q(\xi+\eta)\right| d\eta = \frac{\sigma_0(\xi_0)}{2}\int\limits_{\xi_0}^{\infty} d\xi \int\limits_{\xi}^{\eta_0+\xi}\left|e^{\omega(\alpha)}q(\alpha)\right| d\alpha \leq$$

$$\leq \frac{\sigma_0(\xi_0)}{2}\int\limits_{\xi_0}^{\infty} d\xi \int\limits_{\xi}^{\infty}\left|e^{\omega(\alpha)}q(\alpha)\right| d\alpha = \frac{\sigma_0(\xi_0)}{2}\sigma_1(\xi_0).$$

Now let

$$|U_{n-1}(\xi_0,\eta_0)| \leq \frac{1}{2}\sigma(\xi_0)\frac{(\sigma_1(\xi_0))^{n-1}}{(n-1)!}.$$

Then we have

$$|U_n(\xi_0,\eta_0)| \leq \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} |q(\xi+\eta)R(\xi,\eta,\,\xi_0,\eta_0)U_{n-1}(\xi,\eta)|\, d\eta \leq$$

$$\frac{1}{2}\sigma_0(\xi_0)\int_{\xi_0}^{\infty}\frac{(\sigma_1(\xi))^{n-1}}{(n-1)!}\int_{\xi}^{\infty}\left|e^{\omega(\alpha)}q(\alpha)\right| d\alpha d\xi \leq$$

$$\leq, \frac{1}{2}\sigma_0(\xi_0)\int_{\xi_0}^{\infty}\frac{(\sigma_1(\xi))^{n-1}}{(n-1)!}\int_{\xi}^{\infty}\left|e^{\omega(\alpha)}q(\alpha)\right| d\alpha d\xi \leq$$

$$= -\frac{1}{2}\sigma_0(\xi_0)\int_{\xi_0}^{\infty}\frac{(\sigma_1(\xi))^{n-1}}{(n-1)!}d\sigma_1(\xi) = \frac{1}{2}\sigma_0(\xi_0)\frac{(\sigma_1(\xi_0))^{n}}{n!}.$$

From these estimates it follows that the series $U(\xi_0,\eta_0) = \sum\limits_{n=0}^{\infty} U_n(\xi_0,\eta_0)$ converges absolutely and uniformly in the interval $0 \leq \eta_0 \leq \xi_0$. Obviously, the sum of this series, i.e. function $U(\xi_0,\eta_0)$ is a solution to equation (27) and satisfies the inequality

$$|U(\xi_0,\eta_0)| \leq \frac{1}{2}\sigma_0(\xi_0)\, e^{\sigma_1(\xi_0)}. \tag{28}$$

Next, differentiating equations (27) and using (24), we see that the function $U(\xi_0,\eta_0)$ and thus the function $K(x,t) = U\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ are twice continuously differentiable. Moreover, using estimates (23), (24) and equation (27), it is established that for each fixed x relations (8) hold. From here and from (28) it follows that the function $K(x,t) = U\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ satisfies (9)-(11) and estimate (6). This completes the proof of the theorem. ◄

Remark 2. *From the above reasoning and inequality* (25) *it follows that for* $c = 1$, $\alpha = 1$ *or* $\alpha = 2$, *the condition for the potential* $q(x)$ *to decrease can be replaced by the condition*

$$\int_0^{\infty}\left|x^{2\alpha+3}q(x)\right| dx < \infty.$$

In this case, using (28) with respect to the kernel of the transformation operator, we obtain the following estimate:

$$|K(x,t)| \leq \frac{1}{2}\rho_0\left(\frac{x+t}{2}\right) e^{\rho_1\left(\frac{x+t}{2}\right)},$$

where $\rho_0(x) = \int\limits_x^\infty |q(t)|\,dt, \rho_1(x) = \int\limits_x^\infty \rho_0(t)\,dt.$

Remark 3. *From estimate* (6) *it follows that the transformation operator* X *carries out a one-to-one mapping of each of the spaces* $E, L_i(0,\infty), (i = 1, 2, \infty)$ *onto itself and the inverse operator* X^{-1} *has the same form*

$$X^{-1}f(x) = f(x) + \int_x^\infty L(x,t)\ f(t)\,dt.$$

In this case, it is not difficult to estimate the kernel $L(x,t)$ *using the equation*

$$L(x,t) + K(x,t) + \int_x^t L(x,y)\ K(y,t)\,dy = 0.$$

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