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Banach and Knopp's Core Theorems and Classes of Conservative Matrices

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Abstract. The purpose of the present paper is to establish some inequalities between sublinear functionals associated with Knopp's core and Banach core of two real bounded sequences along with their matrix transformations using classes of conservative matrices in order to obtain some inclusion results on these cores involving two transformed sequences.

Key Words and Phrases: conservative matrices, regular matrices, strongly regular matrices, Knopp's core, Banach core.
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1. Introduction

Let m be the Banach space of real bounded sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$. Let m^* be the algebraic dual of m . A sublinear functional $\varphi \in m^*$ is called a Banach limit (see Banach [1]) if it satisfies the following conditions:
(i) $\varphi(x) \geq 0$ if $x_k \geq 0$ for all k;
(ii)$\varphi(x) = \varphi(Sx)$, where S is a shift operator defined by

$$(Sx)_k = x_{k+1};$$

(iii) $\varphi(e) = 1,$ where $e = (1, 1, 1...)$.
A sublinear functional ψ on m is said to generate Banach limit if $\varphi \in m^*$ and $\varphi \leq \psi$ (i.e., $\varphi(x) \leq \psi(x)$ for all $x \in m$) implies that φ is a Banach limit. Also ψ is said to dominate Banach limit if "φ is a Banach limit" implies that $\varphi \leq \psi$. It is known ([3, 4, 6, 14]) that the sublinear functional

$$q(x) = \limsup_n \ \sup_i \frac{1}{n+1} \sum_{k=0}^{n} x_{k+i} \tag{1}$$

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both generates and dominates Banach limits. The sequence x is said to be almost convergent to s if $q(x) = -q(-x) = s$ and in this case we write $s = f - \lim x$, where f is the set of all *almost convergent* sequences defined by Lorentz [11]. Let c be the set of convergent sequences. Consider a class or a method $\mathcal{A} = (A^i)$ as a sequence of matrices $A^i = (a_{nk}(i))$ with real entries. We will write

$$(\mathcal{A}x)_n = (A^i x)_n = \sum_k a_{nk}(i)x_k, \tag{2}$$

if the series converges for each $n, i \in \mathbb{N}$ (summation without limits is from 0 to ∞).
By $\mathcal{A}x$, we denote the sequences of infinite matrices $\{(A^i x)_n\}_{n,i=0}^{\infty}$. The method $\mathcal{A}$ is said to be *conservative* if it maps c into c and $\lim \mathcal{A}x \neq \lim x$ for all $x \in c$ and *regular* if equality holds.

The following definition for a class of conservative matrices is given by Stieglitz ([15]). The method $\mathcal{A}$ is *conservative* if and only if the following conditions hold:
(a) $\|\mathcal{A}\| = \sup_{n,i\geq 0} \sum_k |a_{nk}(i)| < \infty$;
(b) $\exists\ a_k \in \mathbb{C} : \lim_n a_{nk}(i) = a_k$ uniformly in i;
(c) $\exists\ a \in \mathbb{C} : \lim_n \sum_k a_{nk}(i) = a$ uniformly in i.
In above conditions, if $a_k = 0$ and $a = 1$, then $\mathcal{A}$ is said to be *regular*. Let us write

$$\lambda(\mathcal{A}) = a - \sum_k a_k. \tag{3}$$

If $\lambda(\mathcal{A}) = 0$, then the method $\mathcal{A}$ is said to be *conull*, otherwise *co-regular*. Also $\lambda(\mathcal{A}) = 1$ if $\mathcal{A}$ is *regular* (see also [3, 5, 6]).

By considering

$$a_{nk}(i) = \begin{cases} \frac{1}{n+1}, & i \leq k \leq i+n \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

the method $\mathcal{A} = (a_{nk}(i))$ turns into the method f of almost convergent sequences. If $\mathcal{A} = A = (a_{nk})$, it forms the usual summability matrix A. Also, by taking $a_{nk}(i) = \frac{1}{n+1}\sum_{r=i}^{n+i} a_{rk}$, the method $\mathcal{A} = (a_{nk}(i))$ turns into the method of almost summability matrices introduced by King [9]. Thus the method $\mathcal{A}$ is the generalized form of many summability matrices including n, k and i. A method $\mathcal{A} = A = (a_{nk})$ is called *F-conservative* if f -lim $\mathcal{A}x \neq f$- $\lim x$ for each $x \in f$ and said to be *F-regular* if equality holds. $\mathcal{A}$ is called *almost conservative* if it maps c into f and f-lim $\mathcal{A}x \neq \lim x$ for all $x \in c$ and *almost regular* if euality holds. The method $\mathcal{A}$ is called *strongly conservative* if it maps f into c and $\lim \mathcal{A}x \neq f$

-lim x for all $x \in f$ and *stongly regular* if equality holds. Parallel to definitions given by Lorentz [11] and Simons [14], we can set the following definitions.

The method $\mathcal{A}$ is *almost positive* if and only if

$$\lim_n \sum_k |a_{nk}(i)| = 1 \text{ uniformly in } i. \tag{5}$$

Also, $\mathcal{A}$ is *strongly regular* if it is regular and

$$\lim_n \sum_k |a_{nk}(i) - a_{n,k+1}(i)| = 0 \tag{6}$$

uniformly in i.

2. Preliminaries

Knopp's core of real bounded sequence x or $\mathrm{K}-core\{x\}$ is defined by Knopp [10] as

$$\mathrm{K}-core\{x\} = [\liminf x, \limsup x]. \tag{7}$$

Then Knopp [10] and other authors like Das [6], Maddox [12] also showed that for real bounded sequences x

$$\mathrm{K}-core\{Ax\} \subseteq \mathrm{K}-core\{x\}$$

if and only if A is a regular and almost positive matrix, which is well known as Knopp's core theorem. Also, $\mathrm{K}-core\{x\}$ for complex sequences is discussed in [7]. The extensions of Knopp's core theorem were obtained by Choudhary [2] on regular matrix.

Banach core or $\mathrm{B}-core$ of real bounded sequence x is defined as (see [3, 6])

$$\mathrm{B}-core\{x\} = [-q(-x), q(x)], \tag{8}$$

where $q(x)$ is the functional in (1). Since $q(x) \leq \limsup x$, it follows that

$$\mathrm{B}-core\{x\} \subseteq \mathrm{K}-core\{x\}.$$

Also, in [3, 6] it is shown that

$$\mathrm{K}-core\{Ax\} \subseteq \mathrm{B}-core\{x\}$$

if and only if A is an almost positive and strongly regular matrix.

Now we list certain results due to Das [3] to be used in our next section.

Theorem A. ([3], Theorem 2) *Let $\mathcal{A}$ be a conservative method and $\|\mathcal{A}\| < \infty$. For all $x \in m$,*

$$\limsup_n \ \sup_i \sum_k (a_{nk}(i) - a_k)x_k \leq \frac{|\lambda(\mathcal{A})| + \lambda(\mathcal{A})}{2} \limsup x - \frac{|\lambda(\mathcal{A}) - \lambda(\mathcal{A})}{2} \liminf x \tag{9}$$

if and only if

$$\limsup_n \ \sup_i \sum_k |a_{nk}(i) - a_k| = |\lambda(\mathcal{A})|. \tag{10}$$

Theorem B. ([3], Theorem 3) *Let $\mathcal{A}$ be conservative and $\beta \geq 0$. Also let $\beta \geq |\lambda(\mathcal{A})|$. For $x \in m$,*

$$\limsup_n \ \sup_i \sum_k (a_{nk}(i) - a_k)x_k \leq \frac{\beta + \lambda(\mathcal{A})}{2} q(x) + \frac{\beta - \lambda(\mathcal{A})}{2} q(-x) \tag{11}$$

if and only if

$$\limsup_n \ \sup_i \sum_k |a_{nk}(i) - a_k| \leq \beta \tag{12}$$

and

$$\lim_n \sup_i \sum_k |a_{n,k}(i) - a_{n,k+1}(i) - (a_k - a_{k+1})| = 0, \tag{13}$$

and in this case

$$x \in f \Rightarrow \sum_k a_{nk}(i)x_k \longrightarrow \sum_k a_k x_k + \lambda(\mathcal{A}) f\text{-}\lim x. \tag{14}$$

Now we proceed with these inequalities for our main work.

3. The main results

In this section we are going to establish some inequalities between sublinear functionals forming Knopp's core and Banach core of real bounded sequence on classes of different types of conservative matrices which generalise core theorems. From these results we can derive inclusions between two cores of bounded sequences and their matrix transformations.

First we shall show the following inequality by implementing Theorem A and Knopp's core extension theorem [2] together and discuss the inclusions between Knopps core of two different transformed sequences.

Theorem 1. *Let $\mathcal{T} = (t_{nk}(i))$ be conservative such that $\lim_n t_{nk}(i) = t_k$ and $\lim_n \sum_k t_{nk}(i) = t$ uniformly in i. Consider a normal matrix $B = (b_{jk})$ such that its inverse $B^{-1} = (b^{-1}_{jk})$ exists. Let $\mathcal{A} = (a_{nj}(i))$ be any method. Then we have the following:*
(a) For bounded Bx, there exists a bounded sequence $\mathcal{A}x$ and

$$\limsup_n \ \sup_i \sum_j a_{nj}(i)x_j \le \sum_k t_k(Bx)_k + \frac{|\lambda(\mathcal{T})| + \lambda(\mathcal{T})}{2} \limsup \ Bx - \frac{|\lambda(\mathcal{T})| - \lambda(\mathcal{T})}{2} \liminf \ Bx \tag{15}$$

if and only if
(i) $\mathcal{T} = \mathcal{A}B^{-1}$ exists for all k uniformly in i,
(ii) $\limsup_n \ \sup_i \sum_k |t_{nk}(i) - t_k| = |\lambda(\mathcal{T})|$,
(iii) For fixed n, $\sum_{k=0}^{J} \left| \sum_{j=J+1}^{\infty} a_{nj}(i) b^{-1}_{jk} \right| \to 0$ as $J \to \infty$ uniformly in i;
(b) Let $\lambda(\mathcal{T}) > 0$. Then for all $Bx \in m$,

$$\limsup_n \ \sup_i \sum_j a_{nj}(i)x_j \le \sum_k t_k(Bx)_k + \lambda(\mathcal{T}) \limsup \ Bx \tag{16}$$

if and only if conditions (i) and (iii) of (a) hold and

$$\limsup_n \sup_i \sum_k |t_{nk}(i) - t_k| = \lambda(\mathcal{T}); \tag{17}$$

(c)

$$\limsup_n \ \sup_i \sum_j a_{nj}(i)x_j \le \limsup_k \ Bx \tag{18}$$

if and only if conditions (i) and (iii) of (a) hold and

$$\limsup_n \sup_i \sum_k |t_{nk}(i)| = 1. \tag{19}$$

We need the following two lemmas to prove this theorem.

Lemma 1. *Every bounded sequence $y = (y_k)$ can be transformed by the method $\mathcal{D} = (d_{jk}(i))$ into a convergent sequence if and only if*
(a) $\sum_k |d_{jk}(i)| < \infty, \forall\ i$ and for every fixed j,

(b) $d_{jk}(i) \to \delta_k$ *as* $j \to \infty$ *uniformly in* i *and for every fixed* k,
(c) $\sum_k |d_{jk}(i) - \delta_k| \to 0$ *as* $j \to \infty$ *uniformly in* i.
The $\mathcal{D}$*-transform of* $y = (y_k)$ *approaches to* $\sum_k \delta_k y_k$ *provided* $\mathcal{D}$ *satisfies the above conditions. For* $y = Bx$, *we write* $y_k = (Bx)_k$ *for each* k.

Proof. By following Hardy [8] and Choudhary [2], this lemma can be proved in a similar manner. ◄

Lemma 2. *For a fixed* n *and a bounded* Bx, *the sequence* $(\mathcal{A}x)_n$ *should be defined for particular* n *and* i, *if and only if (iii) of Theorem 1 holds and*

$$t_{nk}(i) = \sum_{j=k}^{\infty} a_{nj}(i) b_{jk}^{-1} \text{ exists for all } k \text{ and } i \tag{20}$$

and for each n *and* i,

$$\sum_k |t_{nk}(i)| < \infty. \tag{21}$$

Then for $y = Bx \in m$,

$$(\mathcal{A}x)_n = \sum_k t_{nk}(i) y_k. \tag{22}$$

Proof. Putting $t_{nk}(i)$ and $a_{nk}(i)$ in place of c_{nk} and a_{nk}, respectively, in Lemma 2 of Choudhury [2], we obtain the required result. ◄

Proof of Theorem 1. **(a) Necessity.** Suppose that (15) holds. Taking $y = Bx$ to be bounded and assuming that $(\mathcal{A}x)_n$ exists for each n and i, we get conditions (i) and (iii) of the theorem from Lemma 1 and Lemma 2 . Moreover, for every bounded y, the expression in (22) can be rewritten as

$$(\mathcal{A}x)_n = (\mathcal{T}y)_n. \tag{23}$$

Hence, by (15), we get

$$\limsup_n \sup_i \sum_j a_{nj}(i) x_j = \limsup_n \sup_i \sum_k t_{nk}(i) y_k \leq \sum_k t_k y_k + \frac{|\lambda(\mathcal{T})| + \lambda(\mathcal{T})}{2} \limsup_k y_k - \frac{|\lambda(\mathcal{T})| - \lambda(\mathcal{T})}{2} \liminf_k y_k, \tag{24}$$

i.e.,

$$\limsup_n \sup_i \sum_k (t_{nk}(i) - t_k) y_k \leq \frac{|\lambda(\mathcal{T})| + \lambda(\mathcal{T})}{2} \limsup_k y_k - \frac{|\lambda(\mathcal{T})| - \lambda(\mathcal{T})}{2} \liminf_k y_k. \tag{25}$$

So (ii) follows.

Sufficiency. From conditions (i)-(iii), it is clear that the conditions of above lemmas are satisfied. So (22) holds and $\mathcal{T}y$ is bounded for $y \in m$.

Now (ii) implies that

$$\limsup_n \sup_i \sum_k (t_{nk}(i) - t_k) y_k$$
$$\leq \frac{|\lambda(\mathcal{T})| + \lambda(\mathcal{T})}{2} \limsup_k y_k - \frac{|\lambda(\mathcal{T})| - \lambda(\mathcal{T})}{2} \liminf_k y_k,$$

since y is bounded (see Theorem A). Writing $y = Bx$ and applying (22) we immediately get (15). This proves (a). Now (b) follows immediately from (a) for $\lambda(\mathcal{T}) > 0$. The result of (c) is a consequence of (a) for $\lambda(\mathcal{T}) = 1$ and $t_k = 0$.

This completes the proof of the theorem. ◄

Remark 1. *For a row finite matrix $\mathcal{A}$, the expression in LHS within the modulus in (iii) of Theorem 1 is* 0 *whenever J is sufficiently large (and for all k) uniformly in i. Hence (iii) is necessarily satisfied so that the results are as follows.*

Corollary 1. *Let B be a normal matrix. Then for a row finite method $\mathcal{A}$*

$$\mathrm{K} - core\{\mathcal{A}x\} \subseteq \mathrm{K} - core\{Bx\}$$

for all $x \in m$ if and only if $\mathcal{A}B^{-1}$ is regular and almost positive.

By taking $B = I$ (identity matrix) in Corollary 1, we will get our next corollary.

Corollary 2. *If $\|\mathcal{A}\| < \infty$, then*

$$\mathrm{K} - core\{\mathcal{A}x\} \subseteq \mathrm{K} - core\{x\}$$

$\forall x \in m$ if and only if $\mathcal{A}$ is regular and almost positive.

Proposition 1. *Let $T = (t_{rk})$ be almost conservative, i.e. $\|T\| < \infty$ with $\lim_n \frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk} = \tau_k$ and $\lim_n \sum_k \frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk} = \tau$ uniformly in i. Also let $B = (b_{jk})$ be a normal matrix and its triangular inverse be denoted by $B^{-1} =$*

(b^{-1}_{jk}). *For an arbitrary matrix* $A = (a_{rj})$, *assume that for a bounded* Bx *there exists a bounded sequence* Ax. *Then*

$$\limsup_n \sup_i \sum_j \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rj} x_j \leq \sum_k \tau_k (Bx)_k + \frac{|\lambda(T)| + \lambda(T)}{2} \limsup\ Bx - \frac{|\lambda(T)| - \lambda(T)}{2} \liminf\ Bx \tag{26}$$

if and only if
(i) $T = AB^{-1}$ *exists for all* k *uniformly in* i,
(ii) $\limsup_n \ \sup_i \sum_k |\frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk} - \tau_k| = |\lambda(T)|$,
(iii) $\sum_{k=0}^{J} \left| \sum_{j=J+1}^{\infty} \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rj} b^{-1}_{jk} \right| \to 0$ *as* $J \to \infty$ *for any fixed* n *uniformly in* i.

Proof. Consider fixed n with i and $i \leq r \leq n+i$. According to Lemma 1 and Lemma 2, whenever Bx is bounded, $(Ax)_r$ should be defined for that particular r and it is necessary and sufficient that (iii) of Proposition 1 holds and

$$t_{rk} = \sum_{j=k}^{\infty} a_{rj} b^{-1}_{jk} \text{ exists for all } k,$$

i.e., (i) holds and also for all r,

$$\sum_k |t_{rk}| < \infty.$$

If these conditions are satisfied, then for bounded Bx

$$(Ax)_r = \sum_k t_{rk} y_k.$$

If we define $a_{nj}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rj}$ in Theorem 1, then we can write $t_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk}$ as $(Ax)_r = (Ty)_r$ for $y = Bx$. Also, if $\mathcal{T} = (t_{nk}(i))$ is conservative in Theorem 1, then $T = (t_{rk})$ is almost conservative. Hence the result in Theorem 1(a) can be interpreted as the result of above proposition. ◄

Corollary 3. *Let* $B = (b_{jk})$ *be a normal matrix and* $A = (a_{rj})$ *be any matrix. Assume that for bounded* Bx, *there exists a bounded sequence* Ax . *Then*

$$\mathrm{B} - core\{Ax\} \subseteq \ \mathrm{K} - core\{Bx\}$$

if and only if
(i) and (iii) of Proposition 1 hold,
(iv) $T = AB^{-1}$ *is almost regular,*
(v) $\limsup_n \sup_i \sum_k |\frac{1}{n+1}\sum_{r=i}^{i+n} t_{rk}| = 1.$

Proof. For $\lambda(T) = \tau = 1$ in the above Proposition 1, we have the required result (see also [16]). Also, taking B as the required identity matrix in the above case, we get the following result. ◄

Corollary 4. *For* $\|\mathcal{A}\| < \infty$ *and* $\forall x \in m$,

$$\mathrm{B} - core\{Ax\} \subseteq \mathrm{K} - core\{x\}$$

if and only if A *is almost regular and (i),(iii) of Proposition 1 hold with*

$$\limsup_n \sup_i \sum_k |\frac{1}{n+1}\sum_{r=i}^{i+n} a_{rk}| = 1.$$

Next, we have to show the following inequality by using Theorem B and find inclusions between Knopp's core and Banach core of two different transformed sequences.

Theorem 2. *Let* $\mathcal{T} = (t_{nk}(i))$ *be conservative and* $\lambda(\mathcal{T}) = t - \sum_k t_k$. *Let* $B = (b_{jk})$ *be a normal matrix and its triangular inverse be denoted by* $B^{-1} = (b^{-1}_{jk})$. *Let* $\beta \geq |\lambda(\mathcal{T})|$. *Also assume that for an arbitrary method* $\mathcal{A} = (a_{nj}(i))$, *there exists a sequence* $\mathcal{A}x$ *which is bounded . Then we have the following:*
(a) For $Bx \in m$,

$$\limsup_n \ \sup_i \sum_j a_{nj}(i)x_j \leq \sum_k t_k(Bx)_k + \frac{\beta + \lambda(\mathcal{T})}{2} q(Bx) + \frac{\beta - \lambda(\mathcal{T})}{2} q(-Bx) \qquad (27)$$

if and only if (i), (iii) of Theorem 1 with following two conditions

$$\limsup_n \ \sup_i \sum_k |t_{nk}(i) - t_k| \leq \beta, \qquad (28)$$

and

$$\limsup_n \ \sup_i \sum_k |t_{nk}(i) - t_{n,k+1}(i) - (t_k - t_{k+1})| = 0 \qquad (29)$$

hold;
(b) For $\lambda(\mathcal{T}) > 0$

$$\limsup_n \sup_i \sum_j a_{nj}(i)x_j \leq \sum_k t_k(Bx)_k + \lambda(\mathcal{T})q(Bx) \tag{30}$$

if and only if (i), (iii) of Theorem 1 and the conditions (17) *and* (29) *hold;*
(c)

$$\limsup_n \sup_i \sum_j a_{nj}(i)x_j \leq q(Bx), \quad \textit{for } Bx \in m, \tag{31}$$

if and only if conditions (i) and (iii) of Theorem 1 and (19) *are satisfied, i.e.,* $\mathcal{T}$ *is regular and almost positive and also*

$$\limsup_n \sup_i \sum_k |t_{nk}(i) - t_{n,k+1}(i)| = 0. \tag{32}$$

Proof. (a) **Necessity.** Suppose (27) holds. Taking $y = Bx$ as a bounded sequence and considering the existence of $(\mathcal{A}x)_n$ for each n and i, we get (i) and (iii) of Theorem 1(a) from the statements of the above lemmas. Moreover, for every bounded y, (23) holds.

Hence, by (27) and by hypothesis

$$\limsup_n \sup_i \sum_j a_{nj}(i)x_j = \limsup_n \sup_i \sum_k t_{nk}(i)y_k$$
$$\leq \sum_k t_k y_k + \frac{\beta + \lambda(\mathcal{T})}{2}q(y) + \frac{\beta - \lambda(\mathcal{T})}{2}q(-y) \quad \forall y \in m,$$

i.e., for all bounded y,

$$\limsup_n \sup_i \sum_k (t_{nk}(i) - t_k)y_k \leq \frac{\beta + \lambda(\mathcal{T})}{2}q(y) + \frac{\beta - \lambda(\mathcal{T})}{2}q(-y). \tag{33}$$

So (28) and (29) hold by following Theorem B.

Sufficiency. Observe that the conditions (i)-(iii) of Theorem 1 provide the required conditions of the above lemmas. So (22) holds and $(\mathcal{T}y)$ is bounded whenever $y \in m$.

Now (28) and (29) imply that

$$\limsup_n \sup_i \sum_k (t_{nk}(i) - t_k)y_k \leq \frac{\beta + \lambda(\mathcal{T})}{2}q(y) + \frac{\beta - \lambda(\mathcal{T})}{2}q(-y) \quad \forall y \in m$$

(see Theorem B). Writing $y = Bx$ and applying (22), the inequality (27) immediately follows, whence the result in (a), and consequently we obtain (b) of the theorem for $\lambda(\mathcal{T}) > 0$. Whenever $\lim_n t_{nk}(i) = t_k = 0$ uniformly in i for fixed k and $\lambda(\mathcal{T}) = 1$, we obtain (c) of the above theorem. ◀

Corollary 5. *Let B be a normal matrix and $\mathcal{A}$ be a row finite method with $\|\mathcal{A}\| < \infty$. Then $\forall x \in m$,*

$$\mathrm{K} - core\{\mathcal{A}x\} \subseteq \ \mathrm{B} - core\{Bx\}$$

if and only if $\mathcal{A}B^{-1}$ is strongly regular and almost positive.

Proof. This corollary is a consequence of Theorem 2(b), which follows by using the argument given in Remark. ◀

Corollary 6. *For $\|\mathcal{A}\| < \infty$ and $\forall x \in m$,*

$$\mathrm{K} - core\{\mathcal{A}x\} \subseteq \ \mathrm{B} - core\{x\}$$

if and only if $\mathcal{A}$ is strongly regular and almost positive.

This immediately follows from Corollary 5 taking $B = I$.

Proposition 2. *Let $T = (t_{rk})$ be almost conservative, i.e. $\|T\| < \infty$ with $\lambda(T) = \tau - \sum_k \tau_k$. Also, for an arbitrary matrix $A = (a_{rj})$ and a normal matrix $B = (b_{jk})$, assume that there exists a bounded sequence Ax for every bounded Bx. Then*

$$\limsup_n \sup_i \sum_j \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rj} x_j \leq \sum_k \tau_k (Bx)_k + \frac{|\lambda(T)| + \lambda(T)}{2} q(Bx) + \frac{|\lambda(T)| - \lambda(T)}{2} q(-Bx) \quad (34)$$

if and only if
(i),(ii) and (iii) of Proposition 1 hold and

$$\limsup_n \sup_i \sum_k |(\frac{1}{n+1} \sum_{r=i}^{i+n} (t_{rk} - t_{r,k+1})) - \tau_k + \tau_{k+1}| = 0. \quad (35)$$

Proof. If we write $a_{nj}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rj}$ in Theorem 2(a), then we get $t_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk}$, so that $(Ax)_r = (Ty)_r$ for every bounded $y = Bx$. This happens if and only if (i) and (iii) of Proposition 1 hold, so the result in Theorem 2(a) can be interpreted as the above result for $\beta = \lambda(T)$.

Note 1. *Let $T = (t_{rk})$ be almost conservative and (35) hold. Then T is said to be F-conservative. The definition of F-regular matrix is similar (see [13]). Some results of Yardimci [16] and Orhan [13] are obtained from Proposition 2 as follows.*

Corollary 7. *Let $B = (b_{jk})$ be a normal matrix and $A = (a_{rj})$ be any row finite matrix. Assume that for a bounded Bx, there exists a bounded sequence $\mathcal{A}x$. Then*

$$\mathrm{B} - core\{Ax\} \subseteq \mathrm{B} - core\{Bx\}$$

if and only if AB^{-1} is F-regular, (i),(iii) of Proposition 1 are satisfied and (v) of Corollary 3 holds.

Proof. In Proposition 2, taking $\lambda(T) = \beta = 1$ and $\lim_n \frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk} = \tau_k = 0$ for fixed k uniformly in i and $\lim_n \sum_k \frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk} = \tau = 1$ uniformly in i, we get

$$\mathrm{B} - core\{Ax\} \subseteq \mathrm{B} - core\{Bx\}$$

if and only if (i) and (iii) of Proposition 1 are satisfied,

$$\lim_n \sup \sup_i \sum_k |\frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk}| = 1$$

and T is almost regular with

$$\lim_n \sup \sup_i \sum_k |\frac{1}{n+1} \sum_{r=i}^{i+n} (t_{rk} - t_{r,k+1})| = 0, \tag{36}$$

i.e. $T = AB^{-1}$ is F-regular.

Hence the above corollary.
Also, taking B as the identity matrix in the above case, we have our next corollary.

Corollary 8. *For $\|A\| < \infty$ and $\forall x \in m$,*

$$\mathrm{B} - core\{Ax\} \subseteq \mathrm{B} - core\{x\}$$

if and only if (i),(iii) of Proposition 1 hold, A is F-regular and

$$\lim_n \sup \sup_i \sum_k |\frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk}| = 1.$$

Conclusion

The above results also determine the inequalities between sublinear functionals forming other invariant cores using different classes of conservative matrices.

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