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On Convergence of Spectral Expansion in Eigenfunctions of Dirac Operator

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Abstract. In this paper, we consider one-dimensional Dirac operator on the interval $G = (0,\pi)$. Absolute and uniform convergences of spectral expansion in eigen vector-functions of this operator are studied. Necessary and sufficient condition for uniform convergence of spectral expansion of the vector-function from the class $W_p^1(G)$, $p \geq 1$, is obtained. The rate of uniform convergence on $\bar{G} = [0,\pi]$ is estimated.

Key Words and Phrases: Dirac operator, eigen vector-function, absolute and uniform convergence.

2010 Mathematics Subject Classifications: 34L10, 42A20

1. Introduction and formulation of results

Consider on the interval $G = (0,\pi)$ the one-dimensional Dirac operator

$$Du = B\frac{du}{dx} + P(x)u, \quad u(x) = \left(u^1(x),\, u^2(x)\right)^{\,T},$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $P(x) = diag\,(p(x),\, q(x))$, and $p(x)$, $q(x)$ are real functions belonging to $L_r\,(0,\pi)$, $r > 1$.

Let $L_p^2(G)$, $p \geq 1$, be a space of two-dimensional vector-functions $f(x) = (f_1(x),\, f_2(x))^{\,T}$ with the norm

$$\|f\|_{p,2} = \left(\int_G |f(x)|^p\,dx\right)^{1/p}, \quad \left(\|f\|_{\infty,2} = \sup_G vrai\,|f(x)|\right),$$
$$|f(x)| = \left(|f_1(x)|^2 + |f_2(x)|^2\right)^{1/2}.$$

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Obviously, for $f(x) \in L_p^2(G)$, $g(x) \in L_q^2(G)$, $p^{-1} + q^{-1} = 1$, $p \geq 1$, there exists the "scalar product"

$$(f, g) = \int_G < f,\ g > dx = \int_G \sum_{j=1}^{2} f_j(x)\overline{g_j(x)}dx\,.$$

Following [1], by the vector-function of the operator D corresponding to the real eigenvalue λ, we mean any non-identically zero vector-function $y(x) = \left(y^1(x),\, y^2(x)\right)^T$, which is absolutely continuous on $\bar{G} = [0, \pi]$ and satisfies the equation $Dy = \lambda y$ almost everywhere in G.

Let $\{u_n(x)\}_{n=1}^{\infty}$ be a complete orthonormalized in $L_2^2(G)$ system consisting of eigen vector-functions of the operator D, and $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n \in R$, be a corresponding system of eigenvalues.

For the vector-functions $f(x) \in W_p^1(G)$, $p \geq 1$, we introduce a partial sum of its spectral expansion with respect to the system $\{u_n(x)\}_{n=1}^{\infty}$:

$$\sigma_v(x, f) = \sum_{|\lambda_n| \leq v} f_n u_n(x) = \sum_{|\lambda_n| \leq v} (f, u_n)\, u_n(x), \quad v \geq 1.$$

Denote $R_v(x, f) = \sigma_v(x, f) - f(x)$ and

$$A_n(f) = \langle f, Bu_n \rangle \Big|_0^{\pi} = \left(f_1(x)\overline{u_n^2(x)} - f_2(x)\,\overline{u_n^1(x)}\right) |_0^{\pi},$$

$$f(x) = (f_1(x), f_2(x))^T, \quad u_n(x) = \left(u_n^1(x),\, u_n^2(x)\right)^T.$$

The following theorems are the main results of this paper.

Theorem 1. *Let* $p(x),\ q(x) \in L_r(G),\ r > 1$, *and the vector-function* $f(x)$ *belong to* $W_p^1(G),\ 1 < p \leq \infty$. *Then*

a) for the uniform convergence of the series

$$\sum_{n=1}^{\infty} |f_n|\ |u_n(x)|\ , \quad x \in \bar{G}, \tag{1}$$

it is necessary and sufficient that the series

$$\sum_{|\lambda_n| \geq 1} |\lambda_n|^{-1} |A_n(f)|\ |u_n(x)|\ , \ \ x \in \bar{G}, \tag{2}$$

is uniformly convergent;

b) for the uniform convergence of the spectral expansion

$$\sum_{n=1}^{\infty} f_n u_n(x), \tag{3}$$

on $\bar{G}$, it is necessary and sufficient that the series

$$\sum_{|\lambda_n|\geq 1} \lambda_n^{-1} A_n(f)\, u_n(x)\,, \ \ x\in \bar{G}, \tag{4}$$

is uniformly convergent;

c) if $A_n(f)=0,\ n=1,2,...$, then spectral expansion (3) *of the vector-function $f(x)$ converges absolutely and uniformly on $\bar{G}=[0,\pi]$ and the following estimates are valid:*

$$\|R_v(\cdot,f)\|_{C[0,\pi]} \leq const\, v^{-\delta}\left\{\|Pf\|_{r,2}+\|f\|_{W_p^1(G)}\right\}, \tag{5}$$

$$\|R_v(\cdot,f)\|_{C[0,\pi]} = o\left(v^{-\delta}\right), \quad v\to+\infty, \tag{6}$$

where $\delta=\min\{1/q,1/r',1/2\}$, $p^{-1}+q^{-1}=1,\ r^{-1}+r'^{-1}=1$, const is independent of f, and the symbol "o" is dependent on f.

Theorem 2. *Let $p(x),\ q(x)\in L_r(G),\ r>1,\ f(x)\in W_1^1(G)$ and, for some $n_0\geq 1$, the following numerical series converge:*

$$\sum_{n=n_0}^{\infty} n^{-1}\omega_1(f',n^{-1}). \tag{7}$$

Then all the statements of Theorem 1 remain valid. But this time, instead of (5), (6), *the following estimates are valid:*

$$\begin{aligned}\|R_v(\cdot,f)\|_{C[0,\pi]} \leq const\, \Big\{ v^{-\alpha}\|Pf\|_{r,2}+ \\ +\left[v^{-1}\|f\|_{W_1^1(G)}+\sum_{n=[v]}^{\infty} n^{-1}\omega_1(f',n^{-1})\right](1+\|P\|_1)\Big\}\,,\end{aligned} \tag{8}$$

where

$$\alpha=\min\left\{\frac{1}{2},\frac{1}{r'}\right\},\ r^{-1}+r'^{-1}=1,\ \omega_1(g,\delta)=\sup_{0<h\leq\delta}\int_0^{\pi-h}|g(x+h)-g(x)|\,dx,$$

$$\|P\|_1=\int_0^{\pi}(|p(x)|+|q(x)|)\,dx.$$

Remark 1. *If the system* $\{u_n(x)\}_{n=1}^{\infty}$ *and the vector-function* $f(x) = (f_1(x), f_2(x))^T$ *satisfy one of the self-adjoint conditions*

$$u^1(0) = u^2(\pi) = 0, \tag{9}$$

$$u^1(0) + \omega u^1(\pi) = 0,\ \ \bar{\omega} u^2(0) + \beta_1 u^1(\pi) + u^2(\pi) = 0, \tag{10}$$

$$\beta_2 u^1(0) + u^2(0) + \omega u^1(\pi) = 0\,,\ \ -\omega u^1(0) + \beta_3 u^1(\pi) + u^2(\pi) = 0, \tag{11}$$

then $A_n(f) = 0,\ n = 1, 2, ...,$ *where* $\beta_i,\ i = \overline{1,3},$ *are arbitrary real numbers, and* $\omega \neq 0$ *is an arbitrary complex number.*

2. Proofs of the results

Proof of Theorem 1. We will consider all the sums only for the numbers n, for which $|\lambda_n| \geq 1$, because the finite number of terms of the series (1), (3), corresponding to eigenvalues $|\lambda_n| < 1$, does not affect the convergence of these series. Finiteness of the number of eigenvalues, satisfying the condition $|\lambda_n| < 1$, follows from the inequality

$$\sum_{\tau \leq |\lambda_n| \leq \tau+1} 1 \leq const, \quad \forall \tau \geq 0, \tag{12}$$

proved in [2].

Since the eigenfunction $u_n(x)$ satisfies the equation $Du_n = \lambda_n u$, the following is valid ($\lambda_n \neq 0$) for the Fourier coefficient of the vector-function $f(x) \in W_p^1(G),\ p \geq 1$, with respect to the system $\{u_n(x)\}_{n=1}^{\infty}$:

$$\overline{f_n} = (u_n, f) = \int_0^{\pi} \langle u_n(x), f(x)\rangle\, dx = \frac{1}{\lambda_n}\int_0^{\pi} \langle Du_n(x), f(x)\rangle\, dx =$$

$$= \frac{1}{\lambda_n}\int_0^{\pi} \left\langle Bu_n'(x), f(x)\right\rangle dx + \frac{1}{\lambda_n}\int_0^{\pi} \langle P(x)u_n(x), f(x)\rangle\, dx.$$

Integrating by parts in the first integral, we get

$$(u_n, f) = \int_0^{\pi} \langle u_n(x), f(x)\rangle\, dx =$$

$$= \lambda_n^{-1} \langle Bu_n(x), f(x)\rangle\, |_0^{\pi} - \lambda_n^{-1}\int_0^{\pi} \left\langle Bu_n(x), f'(x)\right\rangle dx +$$

$$+\lambda_n^{-1}\int_0^{\pi} \langle P(x)u_n(x), f(x)\rangle\, dx = \lambda_n^{-1}\overline{A_n(f)} - \lambda_n^{-1}\int_0^{\pi} \left\langle Bu_n(x), f'(x)\right\rangle dx +$$

$$+\lambda_n^{-1}\int_0^{\pi}\langle P(x)u_n(x), f(x)\rangle\, dx.$$

Consequently, when $|\lambda_n| \geq 1$, for the Fourier coefficient f_n of the vector-function $f(x) \in W_p^1(G)$, $p \geq 1$, the following formula is valid:

$$f_n = (f, u_n) = \lambda_n^{-1} A_n(f) - \lambda_n^{-1}\left(B^* f', u_n\right) + \lambda_n^{-1}\left(P^* f, u_n\right), \tag{13}$$

where

$$B^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad P^*(x) = P(x) = \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix}.$$

Prove that for $1 < p \leq 2$ the series

$$\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-1}\left|\left(B^* f', u_n\right)\right| |u_n(x)|, \tag{14}$$

$$\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-1}\left|(Pf, u_n)\right| |u_n(x)| \tag{15}$$

converge uniformly on $\bar{G} = [0, \pi]$.

Due to the orthonormality of the system $\{u_n(x)\}_{n=1}^{\infty}$ in $L_2^2(G)$, the following inequalities hold:

$$\|u_n\|_{\infty,2} \leq C\,\|u_n\|_{2,2} = C, \tag{16}$$

$$\|u_n\|_{q,2} \leq C_1\,\|u_n\|_{\infty,2} \leq C_2, \quad 1 \leq q < \infty, \tag{17}$$

where C, C_1, C_2 are some constants (see [2]). Then for uniform convergence of the series (14) and (15) it suffices to prove the convergence of numerical series

$$\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-1}\left|\left(B^* f', u_n\,\|u_n\|_{q,2}^{-1}\right)\right|, \tag{18}$$

$$\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-1}\left|\left(Pf, u_n\,\|u_n\|_{\gamma',2}^{-1}\right)\right|, \tag{19}$$

where $\gamma = \min\{2, r\}$, $\gamma^{-1} + \gamma'^{-1} = 1$.

The functions $B^* f'(x)$ and $P(x)f(x)$ belong to the spaces $L_p^2(G)$ and $L_\gamma^2(G)$, respectively. The systems $\left\{u_n(x)\,\|u_n\|_{q,2}^{-1}\right\}_{n=1}^{\infty}$ and $\left\{u_n(x)\,\|u_n\|_{\gamma',2}^{-1}\right\}_{n=1}^{\infty}$ are Riesz spaces in $L_p^2(G)$ and $L_\gamma^2(G)$, respectively (see [3]). Therefore, due to the Hölder and Riesz inequalities, from (18) and (19) we obtain

$$\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-1}\left|\left(B^* f', u_n\,\|u_n\|_{q,2}^{-1}\right)\right| \leq$$

$$\leq \left(\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-p}\right)^{1/p} \left(\sum_{|\lambda_n|\geq 1} \left|\left(B^* f', u_n \|u_n\|_{q,2}^{-1}\right)\right|^q\right)^{1/q} \leq$$

$$\leq C_3 \left\|B^* f'\right\|_{p,2} \left(\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-p}\right)^{1/p}; \tag{20}$$

$$\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-1} \left|\left(Pf, u_n \|u_n\|_{\gamma',2}^{-1}\right)\right| \leq$$

$$\leq \left(\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-\gamma}\right)^{1/\gamma} \left(\sum_{|\lambda_n|\geq 1} \left|\left(Pf, u_n \|u_n\|_{\gamma',2}^{-1}\right)\right|^{\gamma'}\right)^{1/\gamma'} \leq \tag{21}$$

$$\leq C_4 \|Pf\|_{\gamma,2} \left(\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-\gamma}\right)^{1/\gamma}.$$

The inequality (12) implies that the series $\sum_{|\lambda_n|\geq 1} |\lambda_n|^{-\theta}$, $\theta > 1$, converges. Consequently, it follows from (20) and (21) that the series (14) and (15) converge uniformly with respect to $x \in \overline{G}$.

Let the series (1) converge uniformly on $\bar{G}$. Then by equality (13) and uniform convergence of the series (14) and (15), it follows that the series (2) also converges uniformly on $\bar{G}$. Vice versa, if the series (2) converges uniformly on $\bar{G}$, then by uniform convergence of the series (14), (15) and from the equality (13) it follows uniform convergence of the series (1). Item a) of Theorem 1 for $1 < p \leq 2$ is proved. For $p > 2$ the validity of this item follows from $W_p^1(G) \subset W_2^1(G),\ p > 2$.

Now we prove item b). It follows from the uniform convergence of the series (14) and (15) on $\bar{G}$ that the series

$$\sum_{|\lambda_n|\geq 1} \lambda_n^{-1} \left(B^* f', u_n\right) u_n(x), \tag{22}$$

$$\sum_{|\lambda_n|\geq 1} \lambda_n^{-1} \left(Pf, u_n\right) u_n(x) \tag{23}$$

converge uniformly on $\bar{G}$.

Let the series (3) converge uniformly on $\bar{G}$. Then the series $\sum\limits_{|\lambda_n|\geq 1} f_n u_n(x)$ also converges uniformly. By the Cauchy criterion, for any $\varepsilon > 0$ there exists a number $N_0(\varepsilon)$ such that for any $N_1, N_2 \geq N_0$ $(N_1 < N_2)$, the following inequalities hold:

$$\left|\sum_{N_1 \leq |\lambda_n| \leq N_2} \lambda_n^{-1} \left(B^* f', u_n\right) u_n(x)\right| < \varepsilon/3, \tag{24}$$

$$\left|\sum_{N_1\le|\lambda_n|\le N_2}\lambda_n^{-1}\left(Pf,u_n\right)\,u_n(x)\right|<\varepsilon/3, \tag{25}$$

$$\left|\sum_{N_1\le|\lambda_n|\le N_2}f_nu_n(x)\right|<\varepsilon/3. \tag{26}$$

By (12) it follows from the equality (13) that

$$\sum_{N_1\le|\lambda_n|\le N_2}\lambda_n^{-1}A_n(f)u_n(x)=\sum_{N_1\le|\lambda_n|\le N_2}f_nu_n(x)+$$

$$+\sum_{N_1\le|\lambda_n|\le N_2}\lambda_n^{-1}\left(B^*f',u_n\right)\,u_n(x)-\sum_{N_1\le|\lambda_n|\le N_2}\lambda_n^{-1}\left(P^*f,u_n\right)\,u_n(x). \tag{27}$$

Consequently, from (24)-(27) we obtain

$$\left|\sum_{N_1\le|\lambda_n|\le N_2}\lambda_n^{-1}A_n(f)u_n(x)\right|<\varepsilon,$$

i.e. the series (4) converges uniformly on $\bar{G}$.

Let the series (4) converge uniformly in $x\in\bar{G}$. Then, by the Cauchy criterion, for any $\varepsilon>0$ there exists a number $N_0(\varepsilon)$ such that for any $N_1,N_2>N_0(\varepsilon)$, $(N_2>N_1)$, the inequalities (24), (25) and

$$\left|\sum_{N_1\le|\lambda_n|\le N_2}\lambda_n^{-1}A_n(f)u_n(x)\right|<\varepsilon/3,\quad x\in\overline{G},$$

hold.

Hence, from the equality (27) it follows that $\left|\sum_{N_1\le|\lambda_n|\le N_2}f_nu_n(x)\right|<\varepsilon$, which shows the uniform convergence of the series (3) on $\bar{G}$. The item b) is proved.

We now prove item c) of Theorem 1. If $A_n(f)=0$, $n=1,2,...$, then uniform convergence of the series (3) follows from item b). By the completeness of the system $\{u_n(x)\}_{n=1}^{\infty}$ in $L_2^2(G)$ and continuity of $f(x)$, this series converges uniformly just to the function $f(x)$, i.e. the following equality is valid:

$$f(x)=\sum_{n=1}^{\infty}f_n\,u_n(x)\,,\quad x\in\overline{G}. \tag{28}$$

Let us prove the validity of the estimate (5). By the equality (28),

$$\|R_v(\cdot,f)\|_{C[0,\pi]} = \|f-\sigma_v(\cdot,f)\|_{C[0,\pi]} =$$

$$= \left\| \sum_{n=1}^{\infty} f_n u_n(\cdot) - \sum_{|\lambda_n|\le v} f_n u_n(\cdot) \right\|_{C[0,\pi]} = \left\| \sum_{|\lambda_n|>v} f_n u_n(\cdot) \right\|_{C[0,\pi]}.$$

Applying the estimates (16), (17), taking into account the equality (13) and the conditions $A_n(f)=0$, $n=1,2,...$, we obtain (see (14), (15), (18), (19), (20), (21))

$$\|R_v(\cdot,f)\|_{C[0,\pi]} \le C_5 \left(\sum_{|\lambda_n|\ge v} \left| \left(B^* f', u_n \|u_n\|_{q,2}^{-1} \right)^q \right| \right)^{1/q} \left(\sum_{|\lambda_n|\ge v} |\lambda_n|^{-p} \right)^{1/p} +$$

$$+C_6 \left(\sum_{|\lambda_n|\ge v} \left| \left(Pf, u_n \|u_n\|_{\gamma',2}^{-1} \right) \right|^{\gamma'} \right)^{1/\gamma'} \left(\sum_{|\lambda_n|\ge v} |\lambda_n|^{-\gamma} \right)^{1/\gamma} \le$$

$$\le C_7 \left\| B^* f' \right\|_{p,2} \left(\sum_{|\lambda_n|\ge v} |\lambda_n|^{-p} \right)^{1/p} + C_8 \|Pf\|_{\gamma,2} \left(\sum_{|\lambda_n|\ge v} |\lambda_n|^{-\gamma} \right)^{1/\gamma}. \qquad (29)$$

Since by (12)

$$\left(\sum_{|\lambda_n|\ge v} |\lambda_n|^{-p} \right)^{1/p} \le \left(\sum_{k=[v]}^{\infty} \sum_{k\le|\lambda_n|\le k+1} |\lambda_n|^{-p} \right)^{1/p} \le$$

$$\le const \left(\sum_{k=[v]}^{\infty} k^{-p} \right)^{1/p} \le C_9 v^{-\frac{1}{q}},$$

$$\left(\sum_{|\lambda_n|\ge v} |\lambda_n|^{-\gamma} \right)^{1/\gamma} \le C_{10} v^{-\frac{1}{\gamma'}},$$

from (29) we obtain the estimate (5).

To prove the estimate (6), it suffices to note that in (29) the sums of the series

$$\sum_{|\lambda_n|\ge v} \left| \left(B^* f', u_n \|u_n\|_{q,2}^{-1} \right) \right|^q$$

and

$$\sum_{|\lambda_n|\geq v} \left|\left(Pf, u_n \left\|u_n\right\|^{-1}_{\gamma',2}\right)\right|^{\gamma'}$$

are $o(1)$ as $v \to +\infty$.

Theorem 1 is completely proved. ◄

Proof of Theorem 2. Let $p(x)$, $q(x) \in L_r(G)$, $r > 1$, $f(x) \in W_1^1(G)$ and the numerical series (7) converge. Since $B^* f' \in L_1^2(G)$, $Pf \in L_r^2(G)$, $r > 1$, it suffices to prove the uniform convergence only for the series (14) (uniform convergence of the series (15) for $Pf \in L_r^2(G)$, $r > 1$, was already established in the proof of Theorem 1). To this end, we estimate Fourier coefficients of the function $B^* f$, i.e. $(B^* f, u_n)$. For that, we apply the following formula (see [2]):

$$u_n(t) = (\cos\lambda_n t \cdot I - \sin\lambda_n t \cdot B)\, u_n(0)+$$

$$+\int_0^t (\sin\lambda_n(t-\xi)\cdot I + \cos\lambda_n(t-\xi)\cdot B)P(\xi)u_n(\xi)d\xi.$$

As a result, we have

$$(u_n, B^* f) = \int_0^\pi \langle u_n(x), B^* f(x)\rangle\, dx =$$

$$= \int_0^\pi \langle(\cos\lambda_n t\cdot I - \sin\lambda_n t\cdot B)\, u_n(0),\ B^* f(t)\rangle\, dt+$$

$$+\int_0^\pi \left\langle \int_0^t (\sin\lambda_n\,(t-\xi)\cdot I + \cos\lambda_n(t-\xi)\cdot B)\, P(\xi)u_n(\xi)d\xi,\ B^* f(t)\right\rangle dt = T_1 + T_2.$$

By the estimate (16) and the equality $B^* f = (-f_2, f_1)^T$, we obtain

$$|T_1| \leq |u_n(0)| \left\{\left|\int_0^\pi \cos\lambda_n t\overline{f_2(t)}dt\right| + \left|\int_0^\pi \sin\lambda_n t\overline{f_2(t)}dt\right| +\right.$$

$$\left.+\left|\int_0^\pi \cos\lambda_n t\overline{f_1(t)}dt\right| + \left|\int_0^\pi \sin\lambda_n t\overline{f_1(t)}dt\right|\right\} \leq$$

$$\leq C\left\{\left|\int_0^\pi \cos\lambda_n t\overline{f_2(t)}dt\right| + \left|\int_0^\pi \sin\lambda_n t\overline{f_2(t)}dt\right| +\right.$$

$$\left.+\left|\int_0^\pi \cos\lambda_n t\overline{f_1(t)}dt\right| + \left|\int_0^\pi \sin\lambda_n t\overline{f_1(t)}dt\right|\right\}.$$

Here, taking into account (see [4], lemma 2)

$$\left|\int_0^\pi \cos\lambda_n t\overline{g(t)}dt\right|,\ \left|\int_0^\pi \sin\lambda_n t\overline{g(t)}dt\right| \leq$$

$$\leq const\left\{\omega_1\left(g, |\lambda_n|^{-1}\right) + |\lambda_n|^{-1}\|g\|_1\right\}, \quad |\lambda_n| \geq 4,$$

where $g(x) \in L_1(G)$, we obtain

$$|T_1| \leq const\left\{\omega_1\left(B^* f, |\lambda_n|^{-1}\right) + |\lambda_n|^{-1}\|B^* f\|_{1,2}\right\}, \quad |\lambda_n| \geq 4.$$

We estimate the integral T_2. Obviously,

$$|T_2| = \left|\int_0^\pi \left\langle \int_0^t \left(\sin\lambda_n(t-\xi)\cdot I + \cos\lambda_n(t-\xi)\cdot B\right) P(\xi) u_n(\xi) d\xi, \, B^* f(t)\right\rangle dt\right| \leq$$

$$\leq \left|\int_0^\pi p(\xi) u_n^1(\xi)\left(\int_\xi^\pi \overline{f_2(t)}\sin\lambda_n(t-\xi)dt\right)d\xi\right| +$$

$$+\left|\int_0^\pi q(\xi) u_n^2(\xi)\left(\int_\xi^\pi \overline{f_1(t)}\sin\lambda_n(t-\xi)dt\right)d\xi\right| +$$

$$+\left|\int_0^\pi q(\xi) u_n^2(\xi)\left(\int_\xi^\pi \overline{f_2(t)}\cos\lambda_n(t-\xi)dt\right)d\xi\right| +$$

$$+\left|\int_0^\pi p(\xi) u_n^1(\xi)\left(\int_\xi^\pi \overline{f_1(t)}\cos\lambda_n(t-\xi)dt\right)d\xi\right|,$$

where $u_n(\xi) = \left(u_n^1(\xi), u_n^2(\xi)\right)^T$.

Applying for the internal integrals the inequality

$$\left|\int_\xi^\pi g(t)\left\{\begin{array}{c}\sin\lambda_n(t-\xi)\\ \cos\lambda_n(t-\xi)\end{array}\right\}dt\right| \leq const\left\{\omega_1\left(g_\xi\, |\lambda_n|^{-1}\right) + |\lambda_n|^{-1}\|g\|_1\right\},$$

$$|\lambda_n| \geq 4, \;\; g(t) \in L_1(G),$$

and the estimate $\omega_1\left(g_\xi, \delta\right) \leq const\left\{\omega_1(g,\delta) + \delta\|g\|_1\right\}$ (see [5], inequality (19)), where

$$g_\xi(z) = \left\{\begin{array}{ll} g(z+\xi), & 0 \leq z \leq \pi-\xi, \\ 0 & \pi-\xi < z \leq \pi, \end{array}\right.$$

we obtain

$$|T_2| \leq const\,\|u_n\|_{\infty,2}\int_0^\pi \{|p(x)| + |q(x)|\}\,dx \times$$

$$\times\left\{\omega_1\left(B^* f, |\lambda_n|^{-1}\right) + |\lambda_n|^{-1}\|B^* f\|_{1,2}\right\}, \quad |\lambda_n| \geq 4\,.$$

Since $\|u_n\|_{\infty,2} \leq const$, we have

$$|T_2| \leq const\,\|P\|_1\left\{\omega_1\left(B^* f, |\lambda_n|^{-1}\right) + |\lambda_n|^{-1}\|B^* f\|_{1,2}\right\}.$$

Consequently, for $|\lambda_n| \geq 4$ the following estimate holds:

$$|(B^* f, u_n)| \leq const\left\{\omega_1\left(B^* f, |\lambda_n|^{-1}\right) + |\lambda_n|^{-1}\|B^* f\|_{1,2}\right\} \times$$
$$\times\left(1+\|P\|_1\right),\ |\lambda_n| \geq 4. \tag{30}$$

We now prove the uniform convergence of the series (14) on $\overline{G}$. By the estimates (16), (30) and the equality $\omega_1\left(B^* f, |\lambda_n|^{-1}\right) = \omega_1\left(f, |\lambda_n|^{-1}\right)$, we obtain

$$\sum_{|\lambda_n|\geq\mu} |\lambda_n|^{-1}\left|(B^* f, u_n)\ u_n(x)\right| \leq \sum_{|\lambda_n|\geq\mu} |\lambda_n|\ \left|^{-1}(B^* f, u_n)\right|\ \|u_n\|_{\infty,2} \leq$$
$$\leq const \sum_{|\lambda_n|\geq\mu}\left\{|\lambda_n|^{-1}\omega_1\left(B^* f, |\lambda_n|^{-1}\right) + |\lambda_n|^{-2}\|B^* f\|_{1,2}\right\}\left(1+\|P\|_1\right) \leq$$
$$\leq const\left(1+\|P\|_1\right)\left(\sum_{|\lambda_n|\geq\mu}|\lambda_n|^{-1}\omega_1\left(f, |\lambda_n|^{-1}\right) + \|B^* f\|_{1,2}\sum_{|\lambda_n|\geq\mu}|\lambda_n|^{-2}\right),$$

where $\mu \geq 4$.

We estimate each of the series on the right-hand side of this inequality. By the estimate (12) and the convergence of the series (7), we obtain

$$\sum_{|\lambda_n|\geq\mu}|\lambda_n|^{-1}\omega_1\left(f, |\lambda_n|^{-1}\right) \leq \sum_{k=[\mu]}^{\infty}\ \sum_{k\leq|\lambda_n|\leq k+1}|\lambda_n|^{-1}\omega_1\left(f, |\lambda_n|^{-1}\right) \leq$$
$$\leq \sum_{k=[\mu]}^{\infty} k^{-1}\omega_1\left(f, k^{-1}\right)\left(\sum_{k\leq|\lambda_n|\leq k+1} 1\right) \leq const\sum_{k=[\mu]}^{\infty} k^{-1}\omega_1\left(f, k^{-1}\right) < \infty;$$
$$\sum_{|\lambda_n|\geq\mu}|\lambda_n|^{-2} \leq const\sum_{k=[\mu]}^{\infty} k^{-2} \leq const\mu^{-1}.$$

Consequently, the series (14) converges uniformly on $\overline{G}$ and its remainder does not exceed the value

$$const\left(1+\|P\|_1\right)\left\{\sum_{k=[\mu]}^{\infty} k^{-1}\omega_1\left(f, k^{-1}\right) + \|B^* f\|_{1,2}\,\mu^{-1}\right\}.$$

Thus, under the conditions of Theorem 1, the series (14) and (15) converge uniformly on $\overline{G}$. Based on this and using the equality (13), we complete the proof of Theorem 2 by the same scheme used in the proof of Theorem 1. ◀

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