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Maximal-simultaneous Approximation Properties of Faber Series in Weighted Bergman Space

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Abstract. In this work, maximal-simultaneous approximation properties of generalized Faber series in weighted Bergman space, defined on bounded continuums of the complex plane, are studied. The error of this approximation in dependence of the best approximation number and the parameters of considered canonical domains is estimated.

Key Words and Phrases: quasidisc, generalized Faber series, maximal convergence, simultaneous approximation, weighted Bergman spaces.

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1. Introduction

Let $\mathfrak{M}$ be a bounded continuum with more than one point in the complex plane $\mathbb{C}$, $\mathfrak{M}^c{:=}\mathbb{C}\backslash\mathfrak{M}$ be its connected complement, and $D := \{w : |w| < 1\}$. By $w = \varphi(z)$ we denote the Riemann conformal mapping of $\mathfrak{M}^c$ onto $\overline{D}^c := \mathbb{C}\backslash\overline{D}$ with the normalization $\varphi(\infty) = \infty$, $\varphi^{'}(\infty) > 0$. Let also $z = \psi(w)$ be the inverse mapping of φ.

For an arbitrary fixed number $R > 1$ we set

$$L_R := \{z : |\varphi(z)| = R\}, \quad \mathfrak{M}_R := intL_R := \{z : z \in \mathfrak{M}^c \text{ and } |\varphi(z)| < R\} \cup \mathfrak{M}.$$

Let g be an analytic function in $\mathfrak{M}^c$, and $g(\infty) > 0$. Then, the generalized Faber polynomials $F_k(z, g)$, $k = 0, 1, 2, ...$, for $\mathfrak{M}$ can be defined as the coefficients of the series expansion

$$\frac{wg[\psi(w)]\,\psi^{'}(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{F_k(z, g)}{w^k}, \quad z \in \mathfrak{M}, \quad |w| > 1. \tag{1}$$

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It is also known that this series converges uniformly and absolutely on the compact subsets of $\overline{D}^c$x $\mathfrak{M}$.

If we differentiate the series (1) $m+1$ times, $m \in \mathbb{N}$:=$\{0,1,2,3,...\}$, with respect to the variable z, then we have the series representation

$$\frac{(m+1)!wg\left[\psi(w)\right]\psi'(w)}{\left[\psi(w)-z\right]^{m+2}}=\sum_{k=m+1}^{\infty}\frac{F_k^{(m+1)}\left(z,g\right)}{w^k}, \tag{2}$$

which also converges uniformly and absolutely on the compact subsets of $\overline{D}^c$x $\mathfrak{M}$.

Let $G\subset\mathbb{C}$ be a simply connected bounded domain and ω be a weight function defined on G. Then we can define the Bergman space $A^p\left(G,\omega\right)$, $1\leq p<\infty$, of analytic functions f in G, equipped with the norm

$$\|f\|_{A^p(G,\omega)}:=\left(\iint_G\left|f\left(z\right)\right|^p\omega\left(z\right)d\sigma(z)\right)^{1/p}<\infty,$$

where $d\sigma(z)=dxdy$ is the 2-dimensional Lebesgue measure on G. When $\omega\left(z\right)=1$, we denote $A^p\left(G\right):=A^p\left(G,1\right)$ and $A\left(G\right):=A^1\left(G\right)$. Note that $A^p\left(G,\omega\right)$, $1\leq p<\infty$, is a Banach space.

As is known, Faber polynomials and their different generalizations in the approximation theory have been used for construction of approximation aggregates (see, for example, [25, 26, 11, 3, 12, 13, 14, 15, 16, 2, 1, 4, 18, 20, 17, 21, 22, 27]). These polynomials can be also used for solution of different boundary value and basicity problems on domains of the complex plane (see, for example, [7, 8, 24]). Note that fundamental properties of these polynomials have been studied in detail in the monographs [25, 26, 11].

Now let G be a bounded simply connected domain with quasiconformal boundary L. Without loss of generality, we assume that $0\in G$. Since L is quasiconformal, by definition there exists a quasiconformal homeomorphism of $\mathbb{C}$ onto itself that maps a circle onto L. Moreower, there exists (see, for example, [3, pp. 107-109]) a canonical quasiconformal reflection $y=y(\zeta)$ across the boundary L, which is differentiable almost everywhere on $\mathbb{C}$, except possibly at the points of L, and for any small fixed $\delta>0$ satisfies the relations

$$\begin{aligned}|y_\zeta|+\left|y_{\overline{\zeta}}\right| &\leq c_1,\ \delta<|\zeta|<1/\delta;\quad if\quad \zeta\notin L,\\ |y_\zeta|+\left|y_{\overline{\zeta}}\right| &< c_2\left|\zeta\right|^{-2};\ if\quad |\zeta|\geq 1/\delta\ \text{ or }\ |\zeta|\leq\delta,\end{aligned} \tag{3}$$

for some positive constants $c_i=c_i(\delta)$, $i=1,2$.

If f is analytic and bounded in G, then the integral representation:

$$f(z) = -\frac{1}{\pi} \iint\limits_{\overline{G}^c} \frac{(f \circ y)(\zeta)\, y_{\overline{\zeta}}(\zeta)}{(\zeta - z)^2} d\sigma(\zeta), \quad z \in G, \tag{4}$$

holds, proved by V. I. Belyi in [6] (see also [3, pp. 103-113]). This formula plays an important role in proving direct theorems of approximation theory in the uniform norm in domains with a quasiconformal boundary. Considering only canonical reflections, Batchaev [5] proved that this representation is true also in the space $A(G)$.

The integral representation (4) is also very useful for investigation of approximation problems in the weighted and nonweighted Bergman spaces (see, for example, [5, 9, 12, 14]). In particular, in [12] maximal convergence (not simultaneous) properties of partial sums $S_n(f) := \sum\limits_{k=0}^{n} a_k(f) F_k'(z)$ of the series $\sum\limits_{k=0}^{\infty} a_k(f) F_k'(z)$, produced by the integral representation (4), were investigated and corresponding approximation errors were estimated. In other words, for a given $f \in A^2(\mathfrak{M}_R)$, $R > 1$, the error of approximation $|f - S_n(f)|$ in the Bergman space $A^2(\mathfrak{M})$, in dependence of the best approximation number

$$E_n(f, \mathfrak{M}_R) := \inf_{p \in \Pi_n} \|f - p\|_{A^2(\mathfrak{M}_R)},$$

where Π_n is the class of algebraic polynomials of degree at most n and the parameters n, $\mathfrak{M}$ and R was estimated. Similar results for generalized Faber series were obtained in [14].

In the classical Smirnov classes of analytic functions, maximal approximation problems in the uniform norm were investigated in the monogaphs [28, 25, 11, 26]. Let us emphasize that in all of these studies, only maximal convergence problems were considered. On the other hand, some works treated simultaneous approximation problems in the real line and in the complex plane (see, for example, [26, 10, 3, 23]), i.e., convergence of the derivatives of series constructed using the given function to the derivatives of function. But, as far as we know, there are no studies investigating both problems, maximal and simultaneous approximation problems, at the same time in the weighted Bergman spaces.

2. Description of results

In this work, we first define a new set of weight functions ω, and then the corresponding weighted Bergman space $A^2(\mathfrak{M}_R, \omega)$.

Let G be a simply connected bounded domain in $\mathbb{C}$.

Definition 1. *Let g be an analytic function in $\overline{G}^c$, $g(\infty) > 0$, and for some fixed constant $R_0 \in (1, \infty)$*

$$\iint\limits_{G_{R_0} \setminus G} |g(z)|^2 \, d\sigma(z) \cdot \iint\limits_{G_{R_0} \setminus G} |g(z)|^{-2} \, d\sigma(z) < \infty. \tag{5}$$

We define a weight function ω as follows:

$$\omega(z) := |(g \circ y)(z)|^{-2}, \; z \in G.$$

By $W^2(G)$ we denote the set all of weight functions ω defined above.

Lemma 1. *If $f \in A^2(\mathfrak{M}_R, \omega)$, $\omega \in W^2(\mathfrak{M}_R)$, $R > 1$, then $f \in A(\mathfrak{M}_R)$.*

Proof. Let $y = y_R(z)$ be a canonical quasiconformal reflection across the level line $L_R = \partial \mathfrak{M}_R$, $R < R_0$. Using (3) and (5), for any sufficiently small fixed $\delta > 0$ we get

$$\begin{aligned}
&\iint\limits_{\mathfrak{M}_R} |(g \circ y_R)(z)|^2 \, d\sigma(z) \\
= &\iint\limits_{\mathfrak{M}_R^c} |g(z)|^2 \left(\left|(y_R)_{\overline{z}}\right|^2 - \left|(y_R)_z\right|^2 \right) d\sigma(z) \leq \iint\limits_{\mathfrak{M}_R^c} |g(z)|^2 \left|(y_R)_{\overline{z}}\right|^2 d\sigma(z) \\
= &\iint\limits_{\mathfrak{M}_{R_0} \setminus \mathfrak{M}_R} |g(z)|^2 \left|(y_R)_{\overline{z}}\right|^2 d\sigma(z) + \iint\limits_{\mathfrak{M}_{R_0}^c} |g(z)|^2 \left|(y_R)_{\overline{z}}\right|^2 d\sigma(z) \\
\leq \; & c_3 \iint\limits_{\mathfrak{M}_{R_0} \setminus \mathfrak{M}_R} |g(z)|^2 \, d\sigma(z) + c_4 \iint\limits_{\mathfrak{M}_{R_0}^c} \left|(y_R)_{\overline{z}}\right|^2 d\sigma(z) < \infty,
\end{aligned}$$

where $c_4 = \max\left\{|g(z)| : z \in \mathfrak{M}_{R_0}^c\right\}$. Hence, by Hölder's inequality,

$$\begin{aligned}
&\left(\iint\limits_{\mathfrak{M}_R} |f(z)| \, d\sigma(z) \right)^2 \\
\leq \; &\left(\iint\limits_{\mathfrak{M}_R} |f(z)|^2 \omega(z) \, d\sigma(z) \right)^2 \left(\iint\limits_{\mathfrak{M}_R} |(g \circ y_R)(z)|^2 \, d\sigma(z) \right)^2 < \infty.
\end{aligned}$$

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Let $f \in A^2(\mathfrak{M}_R, \omega)$, $\omega \in W^2(\mathfrak{M}_R)$, $R > 1$. Since the level line L_R is a quasiconformal curve and $y_R(z)$ is a canonical quasiconformal reflection across L_R, by Lemma 1 we have $f \in A(\mathfrak{M}_R)$ and then

$$f(z) = -\frac{1}{\pi} \iint\limits_{\mathfrak{M}_R^c} \frac{(f \circ y_R)(\zeta)\,(y_R)_{\bar{\zeta}}(\zeta)}{(\zeta - z)^2} d\sigma(\zeta), \; z \in \mathfrak{M}_R, \tag{6}$$

which, by substituting $\zeta = \psi(w)$, can be rewritten as

$$\begin{aligned} & f(z) \\ = \; & \frac{-1}{\pi} \iint\limits_{|w|>R} (f \circ y_R)(\psi(w))\,(y_R)_{\bar{\zeta}}[\psi(w)]\,\overline{\psi'(w)} \frac{\psi'(w)}{[\psi(w) - z]^2} d\sigma(w) \\ = \; & \frac{-1}{\pi} \iint\limits_{|w|>R} \frac{(f \circ y_R)(\psi(w))\,(y_R)_{\bar{\zeta}}[\psi(w)]\,\overline{\psi'(w)}}{g[\psi(w)]} \frac{g[\psi(w)]\,\psi'(w)}{[\psi(w) - z]^2} d\sigma(w). \end{aligned} \tag{7}$$

Hence, for the m-th, $m \in \mathbb{N}$, order derivatives of (6) and (7) we have

$$\begin{aligned} & f^{(m)}(z) \\ = \; & -\frac{(m+1)!}{\pi} \iint\limits_{\mathfrak{M}_R^c} \frac{(f \circ y_R)(\zeta)\,(y_R)_{\bar{\zeta}}(\zeta)}{(\zeta - z)^{m+2}} d\sigma(\zeta) \qquad (8) \\ = \; & \frac{-1}{\pi} \iint\limits_{|w|>R} \frac{(f \circ y_R)(\psi(w))(y_R)_{\bar{\zeta}}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)]} \frac{(m+1)! g[\psi(w)]\psi'(w)}{[\psi(w) - z]^{m+2}} d\sigma(w). \end{aligned}$$

Now, considering the expansion (2) in (8), we have the series representation

$$f^{(m)}(z) \sim \sum_{k=m+1}^{\infty} a_k(f)\, F_k^{(m+1)}(z, g), \quad z \in \mathfrak{M}_R, \; m \in \mathbb{N}, \tag{9}$$

for $\forall f \in A^2(\mathfrak{M}_R, \omega)$ with the coefficients

$$a_k(f) := -\frac{1}{\pi} \iint\limits_{|w|>R} \frac{(f \circ y_R)(\psi(w))\,(y_R)_{\bar{\zeta}}[\psi(w)]\,\overline{\psi'(w)}}{g[\psi(w)]\, w^{k+1}} d\sigma(w), \; k = m+1, \; m+2, \ldots .$$

As follows from Theorem 1 proved in [14], if $f \in A^2(\mathfrak{M}_R, \omega)$, $\omega \in W^2(\mathfrak{M}_R)$, then in the case of $m = 0$ the generalized Faber series (9) converges uniformly to f on any compact subset of $\mathfrak{M}_R$. Hence, by the Weierstrass theorem on the uniform convergence of derivative series, we have

$$f^{(m)}(z) = \sum_{k=m+1}^{\infty} a_k(f) F_k^{(m+1)}(z,g), \ z \in \mathfrak{M}, \ m \in \mathbb{N}$$

uniformly on any compact subset of $\mathfrak{M}_R$. Now we denote

$$\begin{aligned} & R_n(z, f^{(m)}, g) \\ : & = f^{(m)}(z) - \sum_{k=m+1}^{n} a_k(f) F_k^{(m+1)}(z,g) \\ = & \sum_{k=n+1}^{\infty} a_k(f) F_k^{(m+1)}(z,g), \ n \geq m, \ z \in \mathfrak{M}. \end{aligned}$$

In this work, for a given $f \in A^2(\mathfrak{M}_R, \omega)$ and a fixed $m \in \mathbb{N}$, maximal-simultaneous approximation properties of partial sums of the derivative Faber series

$$\sum_{k=n+1}^{\infty} a_k(f) F_k^{(m+1)}(z,g), \quad n \geq m,$$

in the space $A^2(\mathfrak{M})$ are studied. Namely, the error of approximation $\left|R_n(z, f^{(m)}, g)\right|$ in the Bergman spaces $A^2(\mathfrak{M})$, in dependence of the best approximation number $E_n(f, \mathfrak{M}_R, \omega) := \inf_{p \in \Pi_n} \|f - p\|_{A^2(\mathfrak{M}_R, \omega)}$, and the parameters n, m, $\mathfrak{M}$ and R is estimated.

As can be seen, we intend to investigate, unlike previous studies, both maximal and simulataneous approximation problems at the same time.

Now we state our main results.

Theorem 1. *Let $\mathfrak{M} \subset \mathbb{C}$ be a bounded continuum with more than one point and with connected complement, and $F_k(z,g)$ be its k-th generalized Faber polynomial. If $f \in A^2(\mathfrak{M}_R, \omega)$, where $\omega \in W^2(\mathfrak{M}_R)$, $R > 1$, then for given $m \in \mathbb{N}$ and $r \in (1, R)$ there exists a constant $c(R, \mathfrak{M}, m, r, \omega) > 0$ such that for every $n \geq m+1$ the inequality*

$$\left\| R_n(z, f^{(m)}, \omega) \right\|_{A^2(\mathfrak{M})} \leq c(\mathfrak{M}, R, m, r, \omega) \frac{E_n(f, \mathfrak{M}_R, \omega)\,(n+1)!}{(n+1-m)!} \left(\frac{r}{R}\right)^{n+1}$$

holds.

In the case, of $\omega \equiv 1$we have the estimate

$$\left\|R_n(z, f^{(m)})\right\|_{A^2(\mathfrak{M})} \leq c(\mathfrak{M},R, m, r)\frac{E_n(f, \mathfrak{M}_R)\,(n+1)!}{(n+1-m)!}\left(\frac{r}{R}\right)^{n+1},$$

which improves the estimate

$$\left\|R_n(z, f^{(m)})\right\|_{A^2(\mathfrak{M})} \leq c(\mathfrak{M},R, m, r)\frac{E_n(f, \mathfrak{M}_R)\,(n+1)!}{\sqrt{n+1}\,(n-m)!}\left(\frac{r}{R}\right)^{n+1},$$

proved in [19].

In the case of $m = 0$, we have

Corollary 1. *Under the conditions of Theorem 1, there exists a constant* $c(R, \mathfrak{M}, r, \omega) > 0$ *such that for every* $n \in \mathbb{N}$ *the inequality*

$$\|R_n(z, f, \omega)\|_{A^2(\mathfrak{M})} \leq c(\mathfrak{M},R, r,\omega) E_n(f, \mathfrak{M}_R, \omega)\left(\frac{r}{R}\right)^{n+1}$$

holds.

Corollary 1 was proved in [14, Theorem 3].

If $m = 0$ and $\omega = 1$, then we have

Corollary 2. *Under the conditions of Theorem 1, there exists a constant* $c(R, \mathfrak{M}) > 0$ *such that for every* $n \in \mathbb{N}$ *the inequality*

$$\|R_n(z, f)\|_{A^2(\mathfrak{M})} \leq c(\mathfrak{M},R)\frac{E_n(f, \mathfrak{M}_R)}{R^{n+1}}$$

holds.

This corollary was proved in [12, Theorem 4].

Theorem 2. *For the given numbers* $R > 1$ *and* $m \in \mathbb{N}$*, under the conditions of Theorem 1 the inequality*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{E_n(f^{(m)}, \mathfrak{M}, \omega)/E_n(f, \mathfrak{M}_R, \omega)} \leq 1/R$$

holds.

As can be seen, the upper limit of the quantity $\sqrt[n]{E_n(f^{(m)}, \mathfrak{M}, \omega)/E_n(f, \mathfrak{M}_R, \omega)}$ can be estimated independently of m.

3. Auxiliary results

We will use the following area theorem, due to Lebedev and Milin, which can be found in [26, p. 170].

Theorem A. *Let an analytic function $w = Q(z)$ on a bounded continuum $\mathfrak{M}$ with connected complement be given. Suppose that the expansion of the composite function $w = Q[\psi(t)]$ in a ring $1 < |t| < \rho$ has the form*

$$Q[\psi(t)] = \sum_{j=0}^{\infty} a_j t^j + \sum_{j=0}^{\infty} b_j / t^j, \quad 1 < |t| < \rho.$$

Then the area of the Riemann surface, onto which the function $w = Q(z)$ maps the continuum $\mathfrak{M}$, is given by the formula

$$S = \pi \left(\sum_{j=1}^{\infty} j |a_j|^2 - \sum_{j=1}^{\infty} j |b_j|^2 \right) \geq 0,$$

which implies that

$$\sum_{j=1}^{\infty} j |b_j|^2 \leq \sum_{j=1}^{\infty} j |a_j|^2 .$$

The following lemma was proved in [14, Lemma 2].

Lemma 2. *Let $f \in A^2(\mathfrak{M}_R)$, $R > 1$, and y_R be a quasiconformal reflection across the level line L_R of the continuum $\mathfrak{M}$. Then*

$$\iint_{\mathfrak{M}_R^c} \left| \frac{(f \circ y_R)(\zeta)\,(y_R)_{\bar{\zeta}}(\zeta)}{g(\zeta)} \right|^2 d\sigma(\zeta) \leq \frac{\|f\|^2_{A^2(\mathfrak{M}_R,\omega)}}{1 - k_R^2},$$

where $k_R = (K_R - 1)/(K_R + 1)$ and K_R is a quasiconformality coefficient of the level line L_R.

Lemma 3. *Let $F_k(\cdot, g)$, $k = 0, 1, 2, ...,$ be the generalized Faber polynomials for the continuum $\mathfrak{M}$, and $1 < r < R$, $R > 1$. Then for given $m \in \mathbb{N}$ there exists a constant $c(r, \mathfrak{M}, m, R)$ such that*

$$\sum_{k=n+1}^{\infty} \frac{\left\| F_k^{(m+1)}(\cdot, g) \right\|^2_{A^2(\mathfrak{M})}}{k R^{2k}} \leq \frac{c(\mathfrak{M}, R, m, r)\,[(n+1)!]^2}{[(n+1-m)!]^2} \left(\frac{r}{R}\right)^{2n}, \; n \geq m.$$

Proof. It is well known that if $F_k(z,g)$, $k=0,1,2,...$, is the generalized Faber polynomial of order k for the continuum $\mathfrak{M}$, then

$$F_k(z,g) = [\varphi(z)]^k g(z) + E_k(z,g), \quad z \in \mathfrak{M}^c, \tag{10}$$

where $E_k(z,g)$ is analytic in $\mathfrak{M}^c$, $E_k(\infty,g)=0$, and in some neighborhood of ∞ its power series representation contains negative powers only. In particular, for the first and second derivatives of (10) we have

$$\begin{aligned}
F_k^{'}(z,g) &= k[\varphi(z)]\varphi^{'}(z)g(z) + g^{'}(z)[\varphi(z)]^k + E_k^{'}(z,g) \\
&=: k[\varphi(z)]^{k-1}A_1(z,g) + g^{'}(z)[\varphi(z)]^k + E_k^{'}(z,g) \\
&=: k[\varphi(z)]^{k-1}\Lambda_1(z,g), \\
F_k^{''}(z) &= k(k-1)[\varphi(z)]^{k-2}\varphi^{'}(z)A_1(z,g) + k[\varphi(z)]^{k-1}A_1^{'}(z,g) \\
&\quad + k[\varphi(z)]^{k-1}\varphi^{'}(z)g^{'}(z) + g^{''}(z)[\varphi(z)]^k + E_k^{''}(z,g) \\
&=: k(k-1)[\varphi(z)]^{k-2}A_2(z,g) + g^{''}(z)[\varphi(z)]^k + E_k^{''}(z,g) \\
&=: k(k-1)[\varphi(z)]^{k-2}\Lambda_2(z,g),
\end{aligned}$$

respectively. Using the substitution $z=\psi(w)$ in these equalities, we have

$$\begin{aligned}
F_k^{'}[\psi(w),g] &= kw^{k-1}\Lambda_1[\psi(w),g], \\
F_k^{''}[\psi(w)] &= k(k-1)w^{k-2}\Lambda_2(w,g),
\end{aligned}$$

where $\Lambda_i(w,g)$, $i=1,2$, are analytic functions in $\overline{D}^c$ and hence $F_k^{'}[\psi(w),g]$ and $F_k^{''}[\psi(w),g]$ have a pole of order at most $k-1$ and $k-2$, respectively, in ∞. Generalizing this operation for higher derivatives, we see that the function $F_k^{(m)}[\psi(w),g]$, $k=m,(m+1),...$ can be written as

$$F_k^{(m)}[\psi(w),g] = k(k-1)(k-2)\cdots(k-m+1)w^{k-m}\Lambda_m(w,g), \tag{11}$$

where $\Lambda_m(w,g)$ is analytic in $\overline{D}^c$, with pole of order at most $k-m$ at ∞. Therefore, the function $F_k^{(m)}[\psi(w),g]$ has the series representation

$$F_k^{(m)}[\psi(w)] = \frac{k!}{(k-m)!}\left(\sum_{j=0}^{k-m}\alpha_j^{(m)}w^j + \sum_{j=1}^{\infty}b_j^{(m)}/w^j\right) \tag{12}$$

with the coefficients $b_j^{(m)}$, $j=1,2,...$, and $\alpha_j^{(m)}$, $j=0,1,2,...,k-m$, where the coefficients $\alpha_j^{(m)}$ can be defined as

$$\alpha_j^{(m)} = \frac{1}{2\pi i}\int\limits_{|w|=r}\frac{w^{k-m}\Lambda_m(w,g)\,dw}{w^{j+1}}, \qquad j=0,1,2,...,k-m, \tag{13}$$

for some $r>1$ and estimated by the inequalities

$$\left|\alpha_j^{(m)}\right| \le r^{k-m-j} c_5(r,\mathfrak{M},m,g), \quad j=0,1,2,...k-m. \tag{14}$$

Now denote $Q(z) := F_k^{(m)}(z,g)$. From Theorem A it follows that the area of the Riemann surface, onto which the function $F_k^{(m)}(z,g)$ maps the continuum $\mathfrak{M}$, can be estimated by the formula

$$S = \pi\left(\frac{k!}{(k-m)!}\right)^2 \left(\sum_{j=1}^{k-m} j\left|\alpha_j^{(m)}\right|^2 - \sum_{j=1}^{\infty} j\left|b_j^{(m)}\right|^2\right) \ge 0,$$

and hence by (14)

$$\begin{aligned} S &\le \pi\left(\frac{k!}{(k-m)!}\right)^2 c_5^2(r,\mathfrak{M},m,g)\sum_{j=1}^{k-m} j r^{2(k-m-j)} \\ &= \pi\left(\frac{k!}{(k-m)!}\right)^2 c_5^2(r,\mathfrak{M},m,g) r^{2(k-m)} \sum_{j=1}^{k-m} j r^{-2j}. \end{aligned}$$

Estimating the last sum by

$$\sum_{j=1}^{k-m} j\left(1/r^2\right)^j = (k-m)\sum_{j=1}^{k-m}\left(1/r^2\right)^j \le c_6(r)(k-m)/r^2,$$

we have

$$\begin{aligned} S &\le \pi\left(\frac{k!}{(k-m)!}\right)^2 M^2(r,\mathfrak{M},m,g)c(r)(k-m)r^{2(k-m-1)} \\ &\le c_7(r,\mathfrak{M},m,g)\left(\frac{k!}{(k-m)!}\right)^2 (k-m)\, r^{2(k-m-1)}. \end{aligned}$$

On the other hand, the area of the Riemann surface, onto which the function $w = F_k^{(m)}(z,g)$ maps the continuum $\mathfrak{M}$, can be also calculated with the help of the formula

$$S = \iint_{\mathfrak{M}} \left|F_k^{(m+1)}(\cdot,g)\right|^2 dxdy = \left\|F_k^{(m+1)}(\cdot,g)\right\|^2_{A^2(\mathfrak{M})}.$$

Hence, after simple computations we have

$$\sum_{k=n+1}^{\infty} \frac{\left\|F_k^{(m+1)}(\cdot,g)\right\|^2_{A^2(\mathfrak{M})}}{kR^{2k}}$$

$$
\begin{aligned}
&\leq c_7(r,\mathfrak{M},m,g)\sum_{k=n+1}^{\infty}\frac{\left(\frac{k!}{(k-m)!}\right)^2(k-m)\,r^{2(k-m-1)}}{kR^{2k}}\\
&\leq c_7(r,\mathfrak{M},m,g)\frac{1}{r^{2(m+1)}}\sum_{k=n+1}^{\infty}\left(\frac{k!}{(k-m)!}\right)^2\left(\frac{r}{R}\right)^{2k}\\
&\leq \frac{c(r,R,\mathfrak{M},m,g)\,[(n+1)!]^2}{[(n+1-m)!]^2}\left(\frac{r}{R}\right)^{2n}.
\end{aligned}
$$

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Remark 1. *In particular, when* $m=0$, $g=1$ *and* $\mathfrak{M}:=\overline{D}$, *we have*

$$
\sum_{k=n+1}^{\infty}\frac{\left\|F_k'\right\|^2_{A^2(\mathfrak{M})}}{kR^{2k}}=\frac{\pi}{R^2-1}\frac{1}{R^{2n}},
$$

which shows that the inequalities proved in Lemma 3 are precise in the sense that the degree n *in the factor* $1/R^{2n}$ *cannot be increased even in the case of* $\mathfrak{M}:=\overline{D}$.

4. Proofs of main results

Proof of Theorem 1 Let $f\in A^2(\mathfrak{M}_R,\omega)$, $\omega\in W^2(\mathfrak{M}_R)$, $R>1$ and P_n^* be its best approximation polynomial in the norm $\|\cdot\|_{A^2(\mathfrak{M}_R,\omega)}$, i.e.,

$$
\|f-P_n^*\|_{A^2(\mathfrak{M}_R,\omega)}=E_n(f,\mathfrak{M}_R,\omega).
$$

Then for every $z\in\mathfrak{M}$ and $n\geq m+1$ we have

$$
\begin{aligned}
&\left|R_n(z,f^{(m)},\omega)\right|\\
=&\left|f^{(m)}(z)-\sum_{k=m+1}^{n}a_k(f)\,F_k^{(m+1)}(z,g)\right|=\left|\sum_{k=n+1}^{\infty}a_k(f)\,F_k^{(m+1)}(z,g)\right|\\
=&\frac{1}{\pi}\left|\iint\limits_{|w|>R}\frac{(f-P_n^*)\circ y_R\,[\psi(w)]\ \overline{\psi'(w)}\,(y_R)_{\bar{\zeta}}\,[\psi(w)]}{g\,[\psi(w)]}\sum_{k=n+1}^{\infty}\frac{F_k^{(m+1)}(z,g)}{w^{k+1}}d\sigma(w)\right|.
\end{aligned}
$$

By Hölder's inequality

$$
\left|R_n(z,f^{(m)},\omega)\right|^2
$$

$$\leq \iint\limits_{|w|>R} \left| \frac{(f - P_n^*) \circ y_R \left[\psi(w)\right] \ \overline{\psi'(w)} \left(y_R\right)_{\overline{\zeta}} \left[\psi(w)\right]}{g\left[\psi(w)\right]} \right|^2 d\sigma\left(w\right)$$

$$\cdot \iint\limits_{|w|>R} \left| \sum_{k=n+1}^{\infty} \frac{F_k^{(m+1)}\left(z,g\right)}{w^{k+1}} \right|^2 d\sigma\left(w\right) =: I_1 \cdot I_2.$$

By Lemma 2

$$\begin{aligned} I_1 &= \iint\limits_{|w|>R} \left| \frac{(f - P_n^*) \circ y_R \left[\psi(w)\right] \ \overline{\psi'(w)} \left(y_R\right)_{\overline{\zeta}} \left[\psi(w)\right]}{g\left[\psi(w)\right]} \right|^2 d\sigma\left(w\right) \\ &= \iint\limits_{\mathfrak{M}_R^c} \left| \frac{\left[(f - P_n^*) \circ y_R\right](\zeta) \left(y_R\right)_{\overline{\zeta}}(\zeta)}{g\left(\zeta\right)} \right|^2 d\sigma\left(\zeta\right) \\ &\leq \frac{\|f - P_n^*\|^2_{A^2(\mathfrak{M}_R,\omega)}}{1 - k_R^2}. \end{aligned}$$

For I_2 we have

$$I_2 = \iint\limits_{|w|>R} \left| \sum_{k=n+1}^{\infty} \frac{F_k^{(m+1)}\left(z,g\right)}{w^{k+1}} \right|^2 d\sigma\left(w\right) = \sum_{k=n+1}^{\infty} \frac{\left|F_k^{(m+1)}\left(z,g\right)\right|^2}{kR^{2k}}.$$

Then

$$\left|R_n(z, f^{(m)}, \omega)\right|^2 \leq \frac{\|f - P_n^*\|^2_{A^2(\mathfrak{M}_R,\omega)}}{\left(1 - k_R^2\right)^2} \sum_{k=n+1}^{\infty} \frac{\left|F_k^{(m+1)}\left(z,g\right)\right|^2}{kR^{2k}}.$$

Now, by integrating both sides of this inequality over $\mathfrak{M}$ and applying Lemma 3 we have

$$\begin{aligned} &\left\|R_n(z, f^{(m)}, \omega)\right\|^2_{A^2(\mathfrak{M})} \\ \leq \quad &\frac{\|f - P_n^*\|^2_{A^2(\mathfrak{M}_R,\omega)}}{1 - k_R^2} \sum_{k=n+1}^{\infty} \frac{\left\|F_k^{(m+1)}\left(\cdot,g\right)\right\|^2_{A^2(\mathfrak{M})}}{kR^{2k}} \end{aligned}$$

$$\leq \quad \frac{c_8(\mathfrak{M},R,r,m,g)E_n^2(f,\mathfrak{M}_R,\omega)\,[(n+1)!]^2}{[(n+1-m)!]^2}\left(\frac{r}{R}\right)^{2n},$$

which implies that

$$\left\|R_n(z,f^{(m)},\omega)\right\|_{A^2(\mathfrak{M})} \leq \frac{c(\mathfrak{M},R,r,m,g)E_n(f,\mathfrak{M}_R,\omega)\,(n+1)!}{(n+1-m)!}\left(\frac{r}{R}\right)^{n+1}.$$

◄

Proof of Theorem 2. Since $E_n(f^{(m)},\mathfrak{M},\omega) \leq \left\|R_n(z,f^{(m)},\omega)\right\|_{A^2(\mathfrak{M})}$, by Theorem 1 we have

$$E_n(f^{(m)},\mathfrak{M},\omega) \leq \frac{c(R,\mathfrak{M},r,m,g)E_n(f,\mathfrak{M}_R,\omega)\,(n+1)!}{(n+1-m)!}\left(\frac{r}{R}\right)^{n+1},$$

or

$$\begin{aligned} E_n(f^{(m)},\mathfrak{M},\omega)/E_n(f,\mathfrak{M}_R,\omega) \quad &\leq \quad \frac{c(R,\mathfrak{M},r,m,g)\,(n+1)!}{(n+1-m)!}\left(\frac{r}{R}\right)^{n+1} \\ &\leq \quad c(R,\mathfrak{M},r,m,g)m\,(n+1)\left(\frac{r}{R}\right)^{n+1}. \end{aligned}$$

Taking here the $(n+1)$-th order root of both sides and passing to the limit as $n\to\infty$, we get

$$\overline{\lim_{n\to\infty}}\sqrt[n+1]{E_n(f^{(m)},\mathfrak{M},\omega)/E_n(f,\mathfrak{M}_R,\omega)} \leq \frac{r}{R}.$$

Since $r>1$ is arbitrary, now passing to the limit as $r\to 1$ we obtain the desired inequality

$$\overline{\lim_{n\to\infty}}\sqrt[n+1]{E_n(f^{(m)},\mathfrak{M},\omega)/E_n(f,\mathfrak{M}_R,\omega)} \leq \frac{1}{R}.$$

◄

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