Azerbaijan Journal of Mathematics
V. 15, No 1, 2025, January
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2025.1.169

Modified Inertial Method for Solving Bilevel Split Quasimonotone Variational Inequality and Fixed Point Problems

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Abstract. The bilevel split variational inequality problem (BSVIP), which includes the VIP of quasimonotone mapping and the fixed point problem of demimetric mapping as lower-level problems and the upper-level problem of a strongly monotone operator in real Hilbert spaces, is solved in this paper using a modified algorithm that combines the inertial method and the contraction projection method. We establish a strong convergence under some appropriate parameter assumptions. Finally, a numerical experiment is given to demonstrate the effectiveness of the suggested approach.

Key Words and Phrases: fixed point, quasimonotone operator, variational inequality problem, demimetric mapping, strong convergence.

2010 Mathematics Subject Classifications: 47H06, 47H09, 47J05, 47J25

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1. Introduction

Let H represent a real Hilbert space. The symbols $\langle ., . \rangle$ and $\|.\|$ are used to denote its inner product and norm, respectively. Let $A : H \to H$ be any operator, and C be a nonempty, closed and convex subset of H. For any given mapping $S : H \to H$, let its set of fixed points be denoted by $F(S) = \{Sx = x,\ x \in H\}$. The classical variational inequality problem (VIP) of Fichera [10, 11], is stated as follows: find a point $z \in C$ such that

$$\langle Az, x - z \rangle \geq 0, \quad \text{for all } x \in C. \tag{1}$$

We denote the solution set of the VIP (1) by VI(C,A). Numerous mathematical and applied sciences use the VIP (1) as a fundamental concept; its theoretical and

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algorithmic foundations, as well as its applications, have been extensively studied in the literature and are still the subject of active research. The extragradient method, developed by Korpelevich [16] in 1976, is one of the most widely used techniques for resolving the VIP (1). The method is formulated as follows: For any point $x_0 \in C$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{2}$$

where $A : C \to \mathbb{R}^n$ is monotone and L-Lipschitz continuous with $L > 0$ and $\lambda \in (0, 1/L)$, P_C is a metric projection from H onto C. If the solution set VI(C,A) of (2) is nonempty, then the iterative sequence $\{x_n\}$ generated by the algorithm (2) converges weakly to a point in VI(C,A). Each iteration of the extragradient method (2) requires the computation of two projections onto the closed convex set C. If C is a general closed convex set, this may have a significant impact on the algorithm's effectiveness. It is also important to note that the mapping in the extragradient method requires knowledge of the Lipschitz constant. Lipschitz constants are, regrettably, frequently unknown or challenging to accurately estimate. Many scholars have paid close attention to Korpelevich's extragradient method, (2), and have greatly improved it in various ways (see, for example, [1, 2, 12, 13, 16, 17, 18, 24, 25, 27] and the references therein). Many authors have recently suggested and examined a number of iterative approaches to solving the VIP (1). In order to update the step size in each iteration, in 2021, Tan et al. [23] studied the inertial modified extragradient projection and contraction method with the hybrid steepest descent method with Armijo-type line search as follows:

$$\begin{cases} x_0, x_1 \in H \\ w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \lambda_n A w_n) \\ z_n = P_{T_n}(w_n - \rho\lambda_n\eta_n A y_n) \\ T_n = \{x \in H : \langle w_n - \lambda_n A w_n - y_n, x - y_n\rangle \leq 0\} \\ d_n = (w_n - y_n) - \lambda_n(A w_n - A y_n) \\ \eta_n := \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2} \\ x_{n+1} = z_n - \alpha_n \sigma S z_n, \ \forall n \geq 1, \end{cases} \tag{3}$$

where λ_n is chosen to be largest $\lambda \in \{\delta, \delta\xi, \delta\xi^2, \cdots\}$, $\delta, \xi \in (0, 1)$, A is Lipschitz continuous and pseudomonotone, S is Lipschitz and α-strongly monotone and Lipschitz continuous, $\{\alpha_n\}$ is a control sequence in $(0, 1)$ with some condition. They demonstrated that the sequence produced by (3) exhibits strong convergence under appropriate parameter conditions. We observe that the Armijo-type line search criteria are used by the algorithm (3) proposed by Thong and Hieu [25] to update the step size of each iteration. It is well known that a line search approach would necessitate numerous additional computations and further decrease the computational effectiveness of the employed method.

Next, let us also assume that C and Q are nonempty, closed, and convex subsets of their respective Hilbert spaces, H_1 and H_2. Let $A : C \to H_1$ and $G_2 : Q \to H_2$ be any nonlinear mappings, $G_1 : H_1 \to H_1$ be a $\beta-$strongly monotone and $L-$Lipschitz continuous operator on C, and $B : H_1 \to H_2$ be a bounded and linear operator. After that, the "bilevel split variational inequality problem" (BSVIP) was introduced by Ani et al. [3] as follows:

$$\text{Find} \quad z \in \Gamma \quad \text{such that} \quad \langle G_1 z, x - z\rangle \geq 0, \quad \text{for all } x \in \Gamma, \tag{4}$$

where $\Gamma := \{z \in VI(C, A) : Bz \in VI(Q, G_2)\}$ is the solution set of the following split variational inequality problem (SVIP) introduced by Censor et al. [12]:

$$\text{Find } z \in C \text{ that solves } \langle A(z), x - z\rangle \geq 0, \ \forall x \in C \tag{5}$$

such that

$$x^* = Bz \in Q \text{ solves } \langle G_2(x^*), y - x^*\rangle \geq 0, \quad \forall y \in Q, \tag{6}$$

and $VI(C, A)$ and $VI(Q, G_2)$ denote the solution sets of the variational inequalities (5) and (6), respectively. Censor et al. [12] proposed and studied the following method for solving the SVIP (5)-(6): For $x_1 \in H_1$,

$$x_{n+1} = P_C(I - \lambda A)(x_n + \tau B^*(P_Q(I - \lambda G_2) - I)Bx_n), \quad n \geq 1, \tag{7}$$

where A, G_2 are β_1, β_2-inverse strongly monotone and λ, τ satisfy some conditions. They proved weak convergence of the sequence generated by (7) to a solution of problem (5)-(6). In order to study traffic equilibrium control issues and partial differential equations, a powerful methodology known as the SVIP (5)-(6) which incorporates the classical variational inequality problem (VIP) (1) has been used; for more information, see [8, 9]. The split feasibility problem (SFP), developed and studied by Censor and Elfving [6], is a special case of the SVIP that arises when $A = G_2 = 0$. It has been studied and applied in a variety of scientific fields, including phase retrieval, medical image reconstruction, signal processing, and radiation therapy treatment planning (for more information, see [5, 7, 26]. Additionally, it is well known that the fixed point problem and problem (6) are equivalent in the sense that:

$$z \in Q \text{ if and only if } z = P_C(z - \lambda G_2 z), \tag{8}$$

where $\lambda > 0$ and P_C is the metric projection of H_1 onto C. Suppose $S(z) := P_Q(I - \lambda G_2)z, \quad \lambda > 0$. Then, from (8), we get $VI(Q, G_2) = F(S)$. In this regard, BSVIP was defined by Ugwunnadi et al. [28] as follows: Suppose that $A : H_1 \to H_1$ is pseudomonotone and L-Lipschitz continuous, $F : H_1 \to H_1$ is $\beta-$strongly monotone and $L-$Lipschitz continuous, $B : H_1 \to H_2$ is a bounded linear operator with $B \neq 0$ and $S : H_2 \to H_2$ is a κ-generalized demimetric mapping with $\kappa > 0$. Then

$$\text{find } z^* \in \Gamma \text{ such that } \langle F(z^*), z - z^* \rangle \geq 0, \quad \forall z \in \Gamma, \tag{9}$$

where $\Gamma := \{z^* \in VI(C, A) : Bz^* \in F(S)\}$. They proposed a modified projection and contraction method and showed that the sequence it produces strongly converges to a distinct BSVIP (9) solution under the assumption of an operator norm on the given step size.

Question: How can BSVIP and related results be solved using the modified algorithm (3) in such a way that strong convergence is achieved without the use of operator norms or Armijo-type line search criteria?

Inspired and motivated by the literature findings, in this paper, we provide a positive response to the aforementioned query and investigate strong convergence for the solution of the bilevel split variational inequality problem (BSVIP) in real Hilbert spaces using Lipschitz-continuous and quasimonotone mapping, demimetric mapping, as lower-level problems and the upper-level problem of a strongly monotone operator. Using the inertial modified extragradient projection and contraction method, we demonstrate that the proposed algorithm strongly converges to a particular point in the solution set of BSIVP under a number of suitable conditions imposed on the parameters. Finally, we demonstrate the effectiveness of our findings through a number of numerical experiments.

2. Preliminaries

Throughout this section, the symbols " $\to$ " and " $\rightharpoonup$ " represent the strong and weak convergences, respectively.

Let C be a closed convex subset of a real Hilbert space H. The metric projection from H onto C is the mapping $P_C : H \to C$ such that for each $x \in H$, there exists a unique point $z = P_C(x)$ with

$$\|x - z\| = \inf_{y \in C} \|x - y\|.$$

Lemma 1. *Let $x \in H$ and $z \in C$ be any point. Then we have*

(i) $z = P_C(x)$ if and only if the following relation holds:

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \tag{10}$$

(ii) For all $x, y \in H$, we have

$$\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2.$$

(iii) For $x \in H$ and $y \in C$

$$\|y - P_C(x)\|^2 + \|x - P_C(x)\|^2 \leq \|x - y\|^2.$$

The mapping T is called

(1) L-Lipschitz continuous with $L > 0$ if for all $x, y \in C$,

$$\|Tx - Ty\| \leq L\|x - y\|.$$

If $L = 1$, then T is called a nonexpansive mapping.

(2) quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C, y \in F(T)$,

(4) τ-demicontractive if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + \tau\|x - Tx\|^2, \; for\ all\ x \in C, y \in F(T).$$

(6) τ-demimetric [20] if $F(T) \neq \emptyset$ and there exists $\tau \in (-\infty, 1)$ such that for any $x \in C$ and $y \in F(T)$, we have

$$\langle x - y, x - Tx\rangle \geq \frac{1-\tau}{2}\|x - Tx\|^2.$$

Lemma 2 ([14])**.** *Let $T : C \to H$ be a nonexpansive mapping. Then T is demiclosed on C in the sense that if $\{x_n\}$ converges weakly to $x \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $x \in F(T)$.*

Lemma 3. *[22] Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in \mathbb{R}$, the following hold:*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(2) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$.

Let us review some important nonlinear mappings (or operators) and their relationship in convex analysis. For any $x, y \in H$, the operator $A : H \to H$ is said to be:

(a) $\eta-$strongly monotone, if there exists $\eta > 0$ such that

$$\langle Ax - Ay, \; x - y\rangle \geq \eta||x - y||^2;$$

(b) monotone, if

$$\langle Ax - Ay,\ x - y\rangle \geq 0.$$

(c) pseudomonotone, if

$$\langle Ax,\ y - x\rangle \geq 0 \Rightarrow \langle Ay,\ y - x\rangle \geq 0;$$

(d) quasimonotone, if

$$\langle Ax,\ x - y\rangle > 0 \Rightarrow \langle Ay,\ x - y\rangle \geq 0.$$

It is obvious that (a) ⇒(b)⇒(c)⇒ (d). But the converses are not generally true.

Lemma 4. *[15, 30]* Let C be a nonempty, closed and convex subset of a Hilbert space H and $F : H \to H$ be an L-Lipschitzian and quasimonotone operator. Let $y \in C$. If for some $x^* \in C$, we have $\langle F(y), x^* - y\rangle \geq 0$, then at least one of the following must hold:

$$\langle F(x^*), x^* - y\rangle \geq 0 \ \text{ or } \ \langle F(y), z^* - y\rangle \leq 0\ \forall z^* \in C.$$

Lemma 5. ([21]) *Let H be a Hilbert space and C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 0)$ and T be a $\tau-$demimetric mapping of C into H such that $F(T) \neq \emptyset$. Then, $F(T)$ is closed and convex.*

Lemma 6. [3] *Let H be a real Hilbert space and $F : H \to H$ be a $\beta-$strongly monotone and $L-$Lipschitz continuous mapping on H. If $\alpha \in (0,1)$, $\eta \in [0, 1-\alpha]$ and $\mu \in \left(0, \frac{2\beta}{L^2}\right)$, then for all $x, y \in H$, we have*

$$\|[(1-\eta)x - \alpha\mu F(x)] - [(1-\eta)y - \alpha\mu F(y)]\| \leq (1 - \eta - \alpha\delta)\|x - y\|,$$

where $\delta = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0, 1]$.

Lemma 7. *[19] Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with condition*

$$\sum_{n=1}^{\infty} \alpha_n = \infty$$

and $\{b_n\}$ be a sequence of real numbers. Assume that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Assumption 1. *Let H_1 and H_2 be real Hilbert spaces and C and Q be nonempty closed and convex subsets of H_1 and H_2, respectively. Suppose the following conditions are satisfied:*

(C1) $A : H_1 \to H_1$ *is a quasimonotone and L_1-Lipschitz continuous with $L_1 > 0$. Also, A is sequentially weakly continuous, i.e., each sequence $\{x_n\} \subset C$ converging weakly to x implies $\{A(x_n)\}$ converging weakly to $\{A(x)\}$.*

(C2) $B : H_1 \to H_2$ *is a bounded linear operator with $B \neq 0$ and $S : H_2 \to H_2$ is a $\nu-$demimetric mapping and demiclosed at zero.*

(C3) $G : H_1 \to H_1$ *is a $\beta-$strongly monotone and L_2-Lipschitz continuous operator on H_1 with $L_2 > 0$ such that $\delta = 1 - \sqrt{1 - \gamma(2\beta - \gamma L_2^2)}$, where $\gamma \in \left(0, \frac{2\beta}{L_2^2}\right)$.*

(C4) *$\{\mu_n\}$ is a positive sequence with $\mu_n = \circ(\alpha_n)$, $\{\beta_n\} \subset (a, 1 - \alpha_n)$ for some $a > 0$ and $\{\alpha_n\} \subset (\alpha, 1)$, where $\alpha > 0$, $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$.*

(C5) *Denote the set of common solutions by $\Gamma := \{z \in VI(C, A) : Bz \in F(S))\}$, where z is a unique solution of $VI(\Gamma, G)$.*

Algorithm 1. Initialization: *Choose $\theta > 0$, $\lambda > 0$, $\mu \in (0, 1)$, $\rho \in (0, 2)$. Let $x_0, x_1 \in H$ be arbitrary.*

Iterative Steps: *Calculate x_{n+1} as follows:*

Step 1. *Given the iterates x_{n-1} and x_n for each $n \geq 1$, $\theta > 0$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$, where*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\mu_n}{\|x_n - x_{n-1}\|}\}, & if \quad x_n \neq x_{n-1}, \\ \theta, \quad otherwise. \end{cases} \tag{11}$$

Step 2. Compute

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = (y_n - \lambda_n B^*(I - S)By_n), \\ w_n = P_C(u_n - \tau_n A u_n), \end{cases} \tag{12}$$

where the steps size λ_n and τ_n are chosen as follows:

$$0 < \epsilon \leq \lambda_n \leq \frac{(1 - \tau)\|By_n - SBy_n\|^2}{\|B^*(By_n - SBy_n)\|^2} - \epsilon, \tag{13}$$

if $By_n \neq SBy_n$*; otherwise* $\lambda_n = \lambda$ *and*

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\mu\|u_n - w_n\|}{\|Au_n - Aw_n\|}, \tau_n\right\}, if Au_n \neq Aw_n. \\ \tau_n, \quad otherwise \end{cases} \tag{14}$$

Step 3. Compute

$$\begin{cases} v_n = P_{T_n}(u_n - \rho\tau_n\eta_n(Aw_n)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)v_n - \alpha_n\sigma G v_n, \ n \in \mathbb{N}, \end{cases} \tag{15}$$

where $T_n := \{z \in H_1 : \langle u_n - Au_n - w_n, z - w_n\rangle \leq 0\}$ *and* $\eta_n := \frac{\langle u_n - w_n, d_n\rangle}{\|d_n\|^2}$,
$d_n = u_n - w_n - \tau_n(Au_n - Aw_n)$.
Set $n := n+1$ *and return to* **Step 1**.

Lemma 8. *Let* C *and* Q *be nonempty closed and convex subsets of real Hilbert spaces* H_1 *and* H_2*, respectively. Suppose conditions (C1) to (C5) are satisfied. Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by Algorithm 1. Then* $\{x_n\}_{n=1}^{\infty}$ *is bounded.*

Proof. Let $p \in \Gamma$. By (y_n) in (12) of Step 2, we get

$$\begin{aligned} \|y_n - p\| &= \|x_n - p + \theta_n(x_n - x_{n-1})\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - p\| + \alpha_n.\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|. \end{aligned}$$

But from (11) in Algorithm 1, with $x_n \neq x_{n-1}$, $\theta_n \leq \bar{\theta_n} \leq \frac{\mu_n}{\|x_n - x_{n-1}\|}$ for all $n \geq 1$, that is $\theta_n \leq \frac{\mu_n}{\|x_n - x_{n-1}\|}$. Then from (C4), using the fact that $\mu_n = \circ(\alpha_n)$, we get

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{\mu_n}{\alpha_n} \to 0 \quad as \quad n \to \infty. \tag{16}$$

Hence, the sequence $\left\{\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\right\}$ is bounded and so there exists a constant $M > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M$ for all $n \neq 1$. Thus,

$$\|y_n - p\| \leq \|x_n - p\| + \alpha_n M. \tag{17}$$

By definition of (u_n) in (12) and step size λ_n in (13), we get

$$\begin{aligned} \|u_n - p\|^2 &= \|y_n - \lambda_n B^*(I-S)Ay_n - p\|^2 \\ &= \|y_n - p\|^2 - 2\lambda_n\langle y_n - p, B^*((I-S))By_n\rangle \\ &\quad + \|\lambda_n B^*(I-S)By_n\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|y_n - p\|^2 - 2\lambda_n\langle By_n - Bp, (I-S)By_n\rangle \\
&\quad + \|\lambda_n B^*(I-S)By_n\|^2 \\
&\leq \|y_n - p\|^2 - \lambda_n(1-\tau)\|By_n - SBy_n\|^2 \\
&\quad + \lambda_n^2\|B^*(I-S)By_n\|^2 \\
&= \|y_n - p\|^2 - \lambda_n\Big[(1-\tau)\|By_n - SBy_n\|^2 \qquad (18) \\
&\quad - \lambda_n\|B^*(By_n - SBy_n)\|^2\Big] \\
&\leq \|y_n - p\|^2.
\end{aligned}$$

This implies

$$\|u_n - p\| \leq \|y_n - p\| \leq \|x_n - p\| + \alpha_n M. \qquad (19)$$

Next, since $p \in \Gamma \subset C$, from (12) we have $w_n \subset C$. Then

$$\langle Ap, w_n - p\rangle \geq 0, \qquad (20)$$

and with A being a mapping on C, by Lemma 4, we have

$$\langle Aw_n, w_n - p\rangle \geq 0. \qquad (21)$$

Then

$$\begin{aligned}
\langle Aw_n, v_n - p\rangle &= \langle Aw_n, v_n - w_n\rangle + \langle Aw_n, w_n - p\rangle \\
&\geq \langle Aw_n, v_n - w_n\rangle. \qquad (22)
\end{aligned}$$

By definition of T_n in Algorithm 1, we get $v_n \in T_n$. Thus

$$\langle u_n - \tau_n Au_n - w_n, v_n - w_n\rangle \leq 0,$$

which implies

$$\begin{aligned}
\langle d_n, v_n - w_n\rangle &= \langle u_n - w_n - \tau_n(Au_n - Aw_n), v_n - w_n\rangle \\
&= \langle u_n - \tau_n Au_n - w_n, v_n - w_n\rangle + \tau_n\langle Aw_n, v_n - w_n\rangle \\
&\leq \tau_n\langle Aw_n, v_n - w_n\rangle. \qquad (23)
\end{aligned}$$

Combining (22) and (23), we get

$$\begin{aligned}
\langle d_n, v_n - u_n\rangle + \langle d_n, u_n - w_n\rangle &= \langle d_n, v_n - w_n\rangle \\
&\leq \tau_n\langle Aw_n, v_n - p\rangle. \qquad (24)
\end{aligned}$$

Using (15), (24) and the fact that then Projection mapping is firmly nonexpansive, by Lemma 1(ii), we get

$$2\|v_n - p\|^2 \leq 2\langle u_n - p - \rho\tau_n\eta_n Aw_n, v_n - p\rangle$$

$$
\begin{aligned}
&= \|v_n - p\|^2 + \|u_n - p - \rho\tau_n\eta_n Aw_n\|^2 - \|v_n - u_n + \rho\tau_n\eta_n Aw_n\|^2 \\
&= \|v_n - p\|^2 + \|u_n - p\|^2 - \|v_n - y_n\|^2 - 2\rho\tau_n\eta_n\langle Aw_n, u_n - p\rangle \\
&\quad -2\rho\tau_n\eta_n\langle Aw_n, v_n - u_n\rangle \\
&= \|v_n - p\|^2 + \|u_n - p\|^2 - \|v_n - u_n\|^2 - 2\rho\tau_n\eta_n\langle Aw_n, v_n - p\rangle \\
&\leq \|v_n - p\|^2 + \|u_n - p\|^2 - \|v_n - u_n\|^2 - 2\rho\eta_n\langle d_n, v_n - y_n\rangle \\
&\quad -2\rho\eta_n\langle d_n, u_n - w_n\rangle \\
&\leq \|v_n - p\|^2 + \|u_n - p\|^2 - \|v_n - u_n\|^2 + 2\rho\eta_n\langle d_n, u_n - v_n\rangle - 2\rho\eta_n^2\|d_n\|^2 \\
&= \|v_n - p\|^2 + \|u_n - p\|^2 - \|v_n - u_n\|^2 + \rho^2\eta_n^2\|d_n\|^2 - 2\rho\eta_n^2\|d_n\|^2 \\
&\quad +\|v_n - u_n\|^2 - \|u_n - v_n - \rho\eta_n d_n\|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|v_n - p\|^2 &\leq \|u_n - p\|^2 + \rho^2\eta_n^2\|d_n\|^2 - 2\rho\eta_n^2\|d_n\|^2 \\
&\quad -\|u_n - v_n - \rho\eta_n d_n\|^2. \qquad (25)
\end{aligned}
$$

And

$$
\begin{aligned}
\langle d_n, u_n - w_n\rangle &= \|u_n - w_n\|^2 - \tau_n\langle Au_n - Aw_n, u_n - w_n\rangle \\
&\geq \|u_n - w_n\|^2 - \tau_n\|Au_n - Aw_n\|\|u_n - w_n\| \\
&\geq \|u_n - w_n\|^2 - \frac{\tau_n\mu}{\tau_{n+1}}\|u_n - w_n\|^2 \\
&= \left(1 - \frac{\tau_n\mu}{\tau_{n+1}}\right)\|u_n - w_n\|^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
\|d_n\| &\leq \|y_n - w_n\| + \tau_n\|Ay_n - Aw_n\| \\
&\leq \|y_n - w_n\| + \frac{\tau_n}{\tau_{n+1}}\mu\|Ay_n - Aw_n\| \\
&= \left(1 + \frac{\tau_n}{\tau_{n+1}}\mu\right)\|y_n - w_n\|,
\end{aligned}
$$

we have

$$
\begin{aligned}
\eta_n^2\|d_n\|^2 &= \langle d_n, u_n - w_n\rangle \cdot \frac{\langle d_n, u_n - w_n\rangle}{\|d_n\|^2} \\
&\geq \left(\frac{\tau_{n+1} - \tau_n\mu}{\tau_{n+1} + \tau_n\mu}\right)^2\|u_n - w_n\|^2. \qquad (26)
\end{aligned}
$$

Combining (25) and (26), we get

$$
\|v_n - p\|^2 \leq \|u_n - p\|^2 - \|u_n - v_n - \rho\eta_n d_n\|^2
$$

$$-\rho(2-\rho)\Big(\frac{\tau_{n+1}-\tau_n\mu}{\tau_{n+1}+\tau_n\mu}\Big)^2\|u_n-w_n\|^2. \quad (27)$$

Therefore

$$\|v_n-p\| \le \|u_n-p\|. \quad (28)$$

It follows from (19) that

$$\|v_n-p\| \le \|x_n-p\|+\alpha_n M. \quad (29)$$

Hence, using definition of (x_{n+1}), Lemma 6 and (29), we get

$$\begin{aligned}
\|x_{n+1}-p\| &= \|[(1-\beta_n)v_n-\alpha_n\sigma G(v_n)]-[(1-\beta_n)p-\alpha_n\sigma G(p)]\\
&\quad +\beta_n x_n - p]-\alpha_n\sigma G(p)\|\\
&\le \|[(1-\beta_n)v_n-\alpha_n\sigma G(v_n)]-[(1-\beta_n)p-\alpha_n\sigma G(p)]\|\\
&\quad +\beta_n\|x_n-p\|+\alpha_n\sigma\|G(p)\|\\
&\le (1-\beta_n-\alpha_n\delta)\|v_n-p\|+\beta_n\|x_n-p\|+\beta_n\sigma\|G(p)\|\\
&\le (1-\beta_n-\alpha_n\delta)\|v_n-p\|+\beta_n\|x_n-p\|+\beta_n\sigma\|G(p)\|\\
&\le (1-\beta_n-\alpha_n\delta)[\|x_n-p\|+\alpha_n M]+\beta_n\|x_n-p\|+\beta_n\sigma\|G(p)\|\\
&\le (1-\alpha_n\delta)\|x_n-p\|+\alpha_n[\sigma\|G(p)\|+M]\\
&\le (1-\alpha_n\delta)\|x_n-p\|+\frac{\alpha_n\delta[\sigma\|G(p)\|+M]}{\delta}\\
&\le \max\{\|x_n-p\|,\delta^{-1}[\sigma\|G(p)\|+M]\}.
\end{aligned}$$

Thus, by induction for all $n \ge 1$

$$\|x_n-p\| \le \max\{\|x_1-p\|,\delta^{-1}[\sigma\|G(p)\|+M]\}.$$

Hence $\{x_n\}$ is bounded. It follows that $\{y_n\},\{u_n\}$ and $\{v_n\}$ are all also bounded. ◄

Lemma 9. (See Lemma 3.6 in [29]) *Let $\{w_n\}$ and $\{u_n\}$ be sequences generated by Algorithm 1 with conditions (C1)-(C4) in Assumption 1. Suppose there exist the subsequences $\{w_{n_s}\}$ of $\{w_n\}$ and $\{u_{n_s}\}$ of $\{u_n\}$ such that $\{u_{n_s}\}$ converges weakly to $z \in H_1$ and $\lim_{s\to\infty}\|w_{n_s}-u_{n_s}\|=0$. Then $z \in VI(C,A)$.*

Lemma 10. *Let $\{x_n\}$ be a sequence generated by Algorithm 1 such that Assumption 1 holds. Then for any $p \in \Gamma$, the following inequality holds:*

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\alpha_n\delta)\|x_n-p\|^2\\
&\quad +\alpha_n\delta[\delta^{-1}\{\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|M_1+2\sigma\langle G(p),p-x_{n+1}\rangle\}]. \quad (30)
\end{aligned}$$

Proof. Let $p \in \Gamma$. Then, by Lemma 6 and (C3), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|[(1-\beta_n)v_n - \alpha_n \sigma G(v_n) + \beta_n x_n - p\|^2 \\
&= \|[(1-\beta_n)v_n - \alpha_n \sigma G(v_n)] - [(1-\beta_n)p - \alpha_n \sigma G(p)] \\
&\quad + \beta_n x_n - p) - \alpha_n \sigma G(p)\|^2 \\
&\leq \|[(1-\beta_n)v_n - \alpha_n \sigma G(v_n)] - [(1-\beta_n)p - \alpha_n \sigma G(p)] \\
&\quad + \beta_n x_n - p)\|^2 + 2\alpha_n \sigma \langle G(p), p - x_{n+1}\rangle \\
&\leq \{\|[(1-\beta_n)v_n - \alpha_n \sigma G(v_n)] - [(1-\beta_n)p - \alpha_n \sigma G(p)]\| \\
&\quad + \beta_n \|x_n - p\|\}^2 + 2\alpha_n \sigma \langle G(p), p - x_{n+1}\rangle \\
&\leq \{(1-\beta_n - \alpha_n \delta)\|v_n - p\| + \beta_n \|x_n - p\|\}^2 \\
&\quad + 2\alpha_n \sigma \langle G(p), p - x_{n+1}\rangle \\
&\leq (1-\beta_n - \alpha_n \delta)\|v_n - p\|^2 + \beta_n \|x_n - p\|^2 \qquad (31) \\
&\quad + 2\alpha_n \sigma \langle G(p), p - x_{n+1}\rangle.
\end{aligned}$$

We deduce from stepsize λ_n in (13) that

$$\begin{aligned}
&\epsilon^2 \|A^*(yu_n - SBy_n)\|^2 < \lambda_n \epsilon \|B^*(By_n - SBy_n)\|^2 \\
&\leq \lambda_n[(1-\tau)\|By_n - SBy_n\|^2 - \lambda_n \|B^*(1-S)By_n\|^2]. \qquad (32)
\end{aligned}$$

Thus, combining (19), (30) and (32), we get

$$\begin{aligned}
\|v_n - p\|^2 &\leq \|y_n - p\|^2 - \epsilon^2 \|A^*(By_n - SBy_n)\|^2 - \|u_n - v_n - \rho \eta_n d_n\|^2 \\
&\quad - \rho(2-\rho)\Big(\frac{\tau_{n+1} - \tau_n \mu}{\tau_{n+1} + \tau_n \mu}\Big)^2 \|u_n - w_n\|^2. \qquad (33)
\end{aligned}$$

And from definition of (y_n) in (12), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1}\rangle + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n \|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\|[2\|x_n - p\| + \theta_n \|x_n - x_{n-1}\|] \\
&\leq \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| M_1, \qquad (34)
\end{aligned}$$

for some constant $M_1 > 0$. Combining (32), (33) and (34), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1 - \alpha_n \delta)\|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| M_1 + 2\alpha_n \sigma \langle G(p), p - x_{n+1}\rangle \\
&\quad - (1 - \beta_n - \alpha_n \delta)\Big[\epsilon^2 \|B^*(By_n - SBy_n)\|^2
\end{aligned}$$

$$
\begin{aligned}
&-\|u_n - v_n - \rho\eta_n d_n\|^2 - \rho(2-\rho)\Big(\frac{\tau_{n+1}-\tau_n\mu}{\tau_{n+1}+\tau_n\mu}\Big)^2\|u_n - w_n\|^2\Big] \quad (35)\\
\leq\ & (1-\alpha_n\delta)\|x_n - p\|^2 + \alpha_n\delta\Big(\delta^{-1}[\theta_n/\alpha_n\|x_n - x_{n-1}\|M_1\\
&+2\sigma\langle G(p), p - x_{n+1}\rangle]\Big).
\end{aligned}
$$

This completes the proof. ◄

Theorem 1. *Let Assumption 1 hold. Then, the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to an element $z \in \Gamma$, which is also a unique solution of the variational inequality problem*

$$
\langle Gz, p - z\rangle \geq 0 \qquad \text{for all } p \in \Gamma. \tag{36}
$$

Proof. Let $p \in \Gamma$. Then by Lemmas 6 and 10, we only need to show that

$$
\limsup_{s\to\infty}\langle G(p), p - x_{n_s+1}\rangle \leq 0
$$

for every subsequence $\{\|x_{n_s} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying

$$
\liminf_{s\to\infty}(\|x_{n_s+1} - p\| - \|x_{n_s} - p\|) \geq 0.
$$

Now, let $\{\|x_{n_s} - p\|\}$ be a subsequence of $\{\|x_n - p\|\}$ such that

$$
\liminf_{s\to\infty}(\|x_{n_s+1} - p\| - \|x_{n_s} - p\|) \geq 0.
$$

Then, since $\{\|x_{n_s+1} - p\| + \|x_{n_s} - p\|\}$ is bounded, there exists greatest lower bound $K \geq 0$ such that

$$
\begin{aligned}
\liminf_{s\to\infty}(\|x_{n_s+1} - p\|^2 - \|x_{n_s} - p\|^2) &= \liminf_{s\to\infty}\{(\|x_{n_s+1} - p\| + \|x_{n_s} - p\|)\\
&\quad\times(\|x_{n_s+1} - p\| - \|x_{n_s} - p\|)\}\\
&\geq K\liminf_{s\to\infty}\{(\|x_{n_s+1} - p\| - \|x_{n_s} - p\|)\}\\
&\geq 0. \qquad (37)
\end{aligned}
$$

From (35) and (37), we obtain

$$
\begin{aligned}
0\ <\ & \epsilon^2\|B^*(By_n - SBy_n)\|^2 + \|u_n - v_n - \rho\eta_n d_n\|^2\\
&+\rho(2-\rho)\Big(\frac{\tau_{n+1}-\tau_n\mu}{\tau_{n+1}+\tau_n\mu}\Big)^2\|u_n - w_n\|^2\\
\leq\ & -(\|x_{n+1} - p\|^2 - \|x_n - p\|^2)
\end{aligned}
$$

$$+\alpha_n\delta\Big(\delta^{-1}[\theta_n/\alpha_n\|x_n - x_{n-1}\|M_1$$
$$+2\sigma\langle G(p), p - x_{n+1}\rangle] - \|x_n - p\|^2\Big).$$

By taking limsup as $s \to \infty$ and using (37), we obtain

$$\begin{aligned} 0 &< \limsup_{s\to\infty}\Big(\epsilon^2\|B^*(By_{n_s} - SBy_{n_s})\|^2 + \|u_{n_s} - v_{n_s} - \rho\eta_n d_{n_s}\|^2 \\ &\quad +\rho(2-\rho)\Big(\frac{\tau_{n_s+1} - \tau_{n_s}\mu}{\tau_{n_s+1} + \tau_{n_s}\mu}\Big)^2\|u_{n_s} - w_{n_s}\|^2\Big) \\ &\leq -\liminf_{s\to\infty}(\|x_{n_s+1} - p\|^2 - \|x_{n_s} - p\|^2) \\ &\leq 0. \end{aligned} \tag{38}$$

This demonstrates that

$$\lim_{s\to\infty}\Big(\epsilon^2\|B^*(By_{n_s} - SBy_{n_s})\|^2 + \|u_{n_s} - v_{n_s} - \rho\eta_n d_{n_s}\|^2$$
$$+\rho(2-\rho)\Big(\frac{\tau_{n_s+1} - \tau_{n_s}\mu}{\tau_{n_s+1} + \tau_{n_s}\mu}\Big)^2\|u_{n_s} - w_{n_s}\|^2\Big) = 0.$$

Thus,

$$\begin{cases} \lim_{s\to\infty}\|B^*(By_{n_s} - SBy_{n_s})\| = 0, \\ \lim_{s\to\infty}\|u_{n_s} - v_{n_s} - \rho\eta_n d_{n_s}\| = 0, \\ \lim_{s\to\infty}\|u_{n_s} - w_{n_s}\| = 0. \end{cases} \tag{39}$$

Since S is a $\tau-$demimetric mapping, B is a linear operator and by boundedness of $\{y_n\}$ there exists $M > 0$, for $By_n \neq Sy_n$ we have

$$\begin{aligned} 0 < \frac{1-\tau}{2}\|By_n - S(By_n)\|^2 &\leq \langle By_n - Bp, By_nSBy_n\rangle \\ &= \langle y_n - p, B^*(By_nSBy_n)\rangle \\ &\leq \|y_n - p\|\|B^*(By_n - S(By_n))\| \\ &\leq M\|B^*(By_n - S(By_n))\|. \end{aligned}$$

It follows from (39) that

$$0 < \frac{1-\tau}{2}\|By_{n_s} - S(By_{n_s})\|^2 \leq M\|B^*(By_{n_s} - S(By_{n_s}))\| \to 0 \text{ as } s \to \infty.$$

Hence

$$\lim_{s\to\infty}\|By_{n_s} - S(By_{n_s})\| = 0. \tag{40}$$

From (13), we know that $0 < \epsilon \leq \lambda_n$ for all $n \geq 1$. Then with $u_n = y_n - \lambda_n B^*(I - S)By_n$ and (39), we get

$$\lim_{s\to\infty} \|u_{n_s} - y_{n_s}\| = 0. \tag{41}$$

And combining (39) with (41), we get

$$\lim_{s\to\infty} \|w_{n_s} - y_{n_s}\| = 0. \tag{42}$$

Furthermore, from (12) we get

$$\|y_{n_s} - x_{n_s}\| = \frac{\theta_{n_s}}{\alpha_{n_s}} \|x_{n_s} - x_{n_s-1}\| \to 0 \tag{43}$$

as $s \to \infty$, and it follows from (41) that

$$\|w_{n_s} - x_{n_s}\| \leq \|w_{n_s} - y_{n_s}\| + \|y_{n_s} - x_{n_s}\| \to 0 \tag{44}$$

as $s \to \infty$. Thus, combining (39) and (44), we get

$$\lim_{s\to\infty} \|u_{n_s} - x_{n_s}\| = 0. \tag{45}$$

And, we know that

$$\begin{aligned} \|u_n - v_n\| &\leq \|u_n - v_n - \rho\eta_n d_n\| + \rho\eta_n \|d_n\| \\ &\leq \|u_n - v_n - \rho\eta_n d_n\| + \rho \cdot \frac{\langle u_n - w_n, d_n \rangle}{\|d_n\|} \\ &\leq \|u_n - v_n - \rho\eta_n d_n\| + \rho \|u_n - w_n\|. \end{aligned}$$

Thus,

$$\|u_{n_s} - v_{n_s}\| \leq \|u_{n_s} - v_{n_s} - \rho\eta_{n_s} d_{n_s}\| + \|u_{n_s} - w_{n_s}\|.$$

With this and (39), we get

$$\lim_{s\to\infty} \|u_{n_s} - v_{n_s}\| = 0. \tag{46}$$

And with $\|v_n - x_n\| \leq \|v_n - u_n\| + \|u_n - x_n\|$, it follows from (45) and (46) that

$$\lim_{s\to\infty} \|v_{n_s} - x_{n_s}\| = 0. \tag{47}$$

From (15) we have

$$\|x_{n+1} - x_n\| \leq (1 - \beta_n)\|v_n - x_n\| + \alpha_n \|\sigma G v_n\|$$

It follows from (C4) and (47) that

$$\lim_{s\to\infty}\|x_{n_s+1}-x_{n_s}\|=0. \tag{48}$$

Furthermore, since $\{x_{n_s}\}$ is bounded, there exists a subsequence $\{x_{n_{s_t}}\}$ of $\{x_{n_t}\}$ such that $\{x_{n_{s_t}}\}$ converges weakly to z in H_1 as $t\to\infty$. From (45), we know that $\{u_{n_{k_s}}\}$ converges weakly to $z\in H_1$, thus with (39), we conclude by Lemma 9 that $z\in VI(C,A)$. Also, from (43), we have $y_{n_{s_t}}\rightharpoonup z$. Since B is bounded and linear, we get $By_{n_{s_t}}\rightharpoonup Bz\in H_2$ and from (40) we know that $\lim_{s\to\infty}\|By_{n_s}-SBy_{n_s}\|=0$. With S been demicclosed at zero, we have $Bz=SBz$, that is $z\in B^{-1}F(S)$. Therefore, $z\in\Gamma=VI(C,A)\cap B^{-1}F(S)$. Following that, we demonstrate that z is a unique solution to the variational inequality problem (36). With respect to (15), we have

$$G(v_n)=\frac{1}{\sigma\alpha_n}\Big(x_n-x_{n+1}+(1-\beta_n)(v_n-x_n)\Big).$$

For any $p\in\Gamma$, since $\alpha_n>\alpha>0$, $\{\beta_n\}\subset(a,1-\alpha_n)$ for some $a>0$ and $\{v_n\}$ is bounded, we have

$$\begin{aligned}\langle G(v_n),v_n-p\rangle &= \frac{1}{\sigma\alpha_n}\Big(\langle x_n-x_{n+1},v_n-p\rangle\\ &\quad+(1-\beta_n)\langle v_n-x_n,v_n-p\rangle\Big)\\ &\le \frac{M}{\sigma\alpha}\Big(\|x_n-x_{n+1}\|+(1-a)\|v_n-x_n\|\Big).\end{aligned}$$

Therefore,

$$\langle G(v_{n_{s_t}}),v_{n_{s_t}}-p\rangle\le\frac{M}{\sigma\alpha}\Big(\|x_{n_{s_t}}-x_{n_{s_t}+1}\|+(1-a)\|v_{n_{s_t}}-x_{n_{s_t}}\|\Big).$$

Taking limit as $t\to\infty$ on both sides of the above inequality, and knowing that $x_{n_{s_t}}\rightharpoonup z$, by using (47) and (48), we get $v_{n_{s_t}}\rightharpoonup z$, which implies

$$\langle G(z),z-p\rangle\le 0,\qquad\forall\, p\in\Gamma.$$

Thus, $z\in\Gamma$ is a solution of the variational inequality problem (36). Since G is $\beta-$strongly monotone, z is a unique solution of (36).
Next, we show that $\limsup_{s\to\infty}\langle F(z),z-x_{n_s}\rangle\le 0$. Without loss of generality, for any $p\in\Gamma$, there exists a subsequence $\{x_{n_{s_t}}\}$ of $\{x_s\}$ which converges weakly to p. Then

$$\limsup_{s\to\infty}\langle G(z),z-x_{n_s}\rangle = \lim_{t\to\infty}\langle G(z),z-x_{n_{s_t}}\rangle$$

$$= \langle G(z), z-p\rangle \leq 0. \quad (49)$$

We also know that

$$\langle G(z), z - x_{n+1}\rangle = \langle G(z), z - x_n\rangle + \langle G(z), x_n - x_{n+1}\rangle. \quad (50)$$

From (48), (49) and (50), we get

$$\limsup_{s\to\infty}\langle G(z), z - x_{n_s+1}\rangle \leq 0. \quad (51)$$

Finally, we show that $x_n \to z \in \Gamma$. From Lemma 10, we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1-\alpha_n\delta)\|x_n - z\|^2 \\ &\quad +\alpha_n\delta[\delta^{-1}\{\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|M_1 + 2\sigma\langle G(z), z - x_{n+1}\rangle\}]. \end{aligned} \quad (52)$$

Therefore, by (51), $\limsup_{s\to\infty}\Psi_{n_s} \leq 0$, where $\Psi_{n_s} := \delta[\delta^{-1}\{\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|M_1 + 2\sigma\langle G(z), z - x_{n+1}\rangle\}]$, under the assumption that $\liminf_{s\to\infty}\Big(\|x_{n_s+1} - z\|^2 - \|x_{n_s - z\|^2}\Big) \geq 0$. Then from (52) and Lemma 7, we obtain $\lim_{n\to\infty}\|x_n - z\| = 0$, hence $x_n \to z \in \Gamma$. This completes the proof. ◄

4. Numerical Example

In this section we present a numerical illustration of our Algorithm 1 and then compare it with Algorithm 3.1 of Tan et al. [23].

Let $H = \mathbb{R}^5$ and $C = \{x \in \mathbb{R}^5 : 1 \leq x_i \leq 3,\ i = 1, 2\cdots, 5\}$. Consider the quadratic fractional programming problem (see [4])

$$\min_{x\in C} f(x) = \frac{x^T Bx + a^T x + a_0}{b^T x + b_0},$$

where

$$B = \begin{pmatrix} 5 & -1 & 2 & 0 & 2 \\ -1 & 6 & -1 & 3 & 0 \\ 2 & -1 & 3 & 0 & 1 \\ 0 & 3 & 0 & 5 & 0 \\ 2 & 0 & 1 & 0 & 4 \end{pmatrix},\quad a = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 1 \end{pmatrix},\quad b = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},\quad a_0 = -2,\quad b_0 = 20.$$

It is easy to see that

$$\nabla f(x) = \frac{(b^T x + b_0)(2Bx + a^T) - b(x^T Bx + a^T x + a_0)}{(b^T x + b_0)^2}.$$

Let $M = \nabla f$. Then M is Lipschitz continuous on C with the constant $L = \max\{\|M(x)\| : x \in C\}$. We compute the value of L using Matlab to obtain $L \approx 149$. The mapping M is said to be pseudomonotone since f is pseudo-convex. Now, let the mapping $S : \mathbb{R}^5 \to \mathbb{R}^5$ be given in the form $S(x) = Bx + q$, where $B \in \mathbb{R}^{5\times 5}$ is a positive definite and symmetric matrix and $q \in \mathbb{R}^5$ with their entries in $(0, 2)$. It is obvious that S is Lipschitz continuous with a constant $L_S = \max\{eig(P)\}$ and α-strongly monotone with the coefficient $\alpha = \min\{eig(P)\}$, where $eig(P)$ denotes all the eigenvalues of P. For implementation of both algorithms, we choose $\theta = \frac{1}{3}$, $\mu = 0.5$, $\rho = 1.5$, $\sigma = 0.03$, $\epsilon_n = \frac{1}{(n+1)^2}$ and $\alpha_n = \frac{0.1}{n+3}$. In particular, we let $\beta_n = \frac{1}{n+1}$ in Algorithm 1 and we choose $\delta = 0.003$ and $\xi = 0.9$ in Algorithm 3.1 of Tan et al. [23]. We let the stopping criterion be given as $\|x_{n+1} - x_n\| \leq \epsilon$, where $\epsilon = 10^{-4}$. Our implementation of the methods is completed by selecting various initial values of x_0 and x_1 as follows:

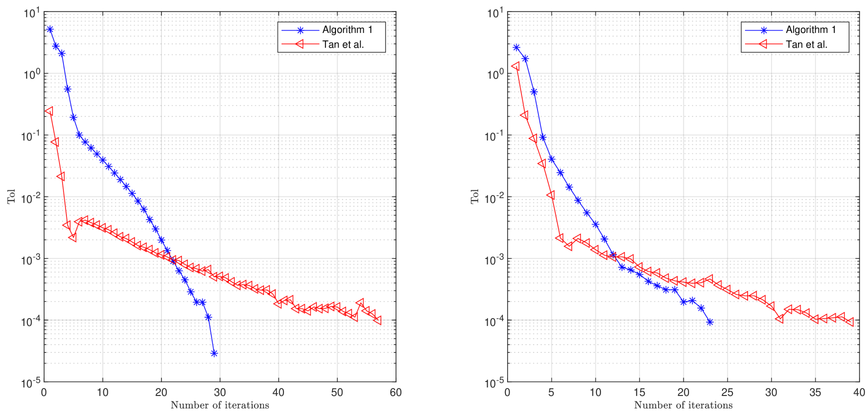
(i) $x_0 = (0, 0, 0, 0, 0)$ and $x_1 = (1, 1, 1, 1, 1)$;

(ii) $x_0 = 1.5 \times rand(5, 1)$ and $x_1 = 2 \times rand(5, 1)$;

(iii) $x_0 = 2.5 \times rand(5, 1)$ and $x_1 = 2 \times rand(5, 1)$;

(iv) $x_0 = 5 \times rand(5, 1)$ and $x_1 = 4 \times rand(5, 1)$.

The result of this experiment is presented in Figure 1 below.



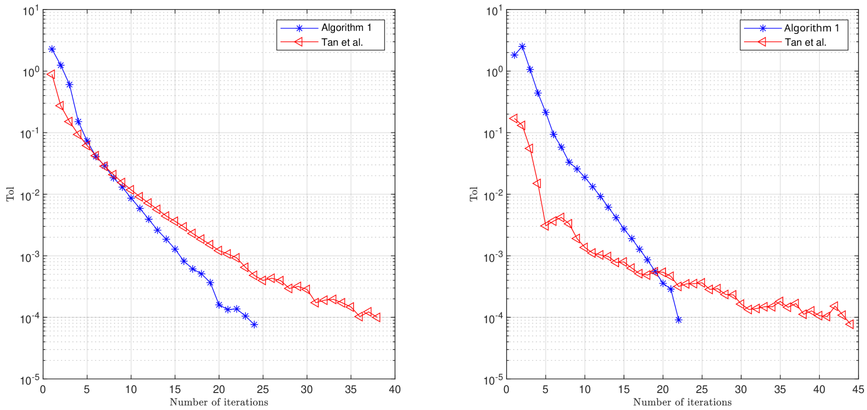


Figure 1: Top left: Case (i); Top right: Case (ii); Bottom left : Case (iii); Bottom right: Case (iv).

Acknowledgement

Authors are grateful to Department of Mathematics and Applied Mathematics, Sefako Makgato Health Science University South Africa for supporting this research work.

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Received 19 January 2024
Accepted 14 April 2024