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On Quasilinear Parabolic Equations of Higher Order

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Abstract. This paper deals with quasilinear parabolic equations of any order $2b \geq 2$ with a main quasilinear elliptic operator under general linear boundary conditions. The interpolation method for obtaining a priori estimates for strong solutions to quasilinear parabolic equations with unbounded singularities on the right-hand side, provided the availability of the first a priori estimate in the space of summable functions, is described. For elliptic equations, this approach was developed by S.I. Pokhozhaev [11, 12].

Key Words and Phrases: quasilinear equation, Sobolev space, interpolation method.

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1. Introduction

Sufficient conditions for the existence of strong solution to the initial boundary value problem for the quasilinear parabolic equations having higher order in spectral variables and first order in time variable are found. For these problems, an estimate is established for the power growth rate of the subordinate linear operator with respect to the corresponding derivative. Obtaining this estimate is based on the theory of a priori estimates to solve the considered quasilinear problems. This paper provides a theorem on a priori estimates for the solutions in the norm of the Sobolev space $W_p^{2b,1}(Q_T)$, by means of the norm

$$\|u\|_{k,\infty} = \sum_{|\gamma| \leq k} \sup_{Q_T} |D^\gamma u(x,t)|.$$

with some $k,\ 0 \leq k \leq 2b-1$.

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The existence of these a priori estimates for $\|u\|_{W_p^{2b,1}(Q_T)}$ is due to the corresponding growth rates of nonlinear operators with respect to lower order derivatives. Counter examples given in this paper show that the corresponding equalities for growth rates are unimprovable without additional assumptions. The main attention is paid to the question of the attainability of maximum degree of growth of nonlinear component of equations and to restrictions on its summability index. As is known, for second order elliptic [11] and parabolic [9] type equations limiting degrees are attainable. But, as the examples constructed in [12] show, this is not true for higher order equations. In the parabolic case, Wahl [16, 17] proved that subject to a coerciveness type condition for higher order equations and systems, the limiting degree of growth is acceptable.

The proof of the solvability of boundary value problems is carried out on the basis of the given theorems on a priori estimates using the Leray-Schauder method [7, p.235-236].

Quasilinear elliptic and parabolic equations were considered by M.I.Vishik [15], Yu. A. Dubinskiy [6], F. Browder [5], V. Wahl [18], S.I. Pokhozhaev [11, 12], G.G. Laptev [8, 9], etc. In [5, 6, 15, 18] it was supposed that the nonlinear operator has a divergent structure, while in [8, 9, 11, 12], and in the present paper there is no assumption on a divergent structure of the subordinate nonlinear operator. This enables us to consider more general nonlinear subordinate operators.

Let $a \in R_+ \equiv \{a \in R \,:\, a \geq 0\}$. We will use the following notations: $x = (x_1, \ldots, x_n)$ is a point in the space R^n ; $\Omega \subset R^n$ is a bounded domain with the boundary $\partial\Omega$ of class c^{2b}, $b \geq 1$; $Q_t = \Omega \times (a, a+t)$ is a cylindrical domain in the space R^{n+1}, $t \in R_+$; $\partial Q_t = \partial\Omega \times (a, a+t)$, is a lateral surface of the cylinder Q_t; $Q_T = \Omega \times (0, T)$ is a cylinder with the given height of $T > 0$.

Throughout this paper the functions are considered to be real-valued. We will use the following function spaces ([2], [3, p.126], [4] and [14, p.118]): the space of summable functions $L_p(Q_T)$, $p \geq 1$, with the norm

$$\|u\|_{p;Q_t} = \left(\int\limits_a^{a+t} \int\limits_{\Omega} |u(x,t)|^p \, dx \, dt \right)^{1/p};$$

anisotropic Sobolev spaces $W_p^{2b,1}(Q_t)$ with the norm

$$\|u\|_{W_p^{2b,1}(Q_t)} = \|u\|_{p;Q_t} + \sum_{i=1}^{n} \left\| \frac{\partial^{2b} u}{\partial x_i^2 b} \right\|_{p;Q_t} + \left\| \frac{\partial u}{\partial t} \right\|_{p;Q_t}.$$

By $D^i u$ we denote the vectors from the derivatives $D^\alpha u$ of the function $u(x,t)$, $\alpha = (\alpha_1, \ldots, \alpha_n, 0)$, $|\alpha| = i$. These vectors have n_i components, where n_i is the number of various multiindices $\alpha = (\alpha_1, \ldots, \alpha_n, 0)$, $|\alpha| = i$.

Each paragraph uses its own system of notations. Therefore, in various paragraphs, c_1, c_2, c_3, etc. can denote different constants.

Let us consider the boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} + Lu = f\left(x,t,u,Du,\ldots,D^{2b-1}u\right), \ (x,t) \in Q_T, \\ B_i u\Big|_{\partial Q_T} = 0 \ \ (i = 0,1,\ldots,b-1), \ x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) = 0, \ \ x \in \Omega \end{cases} \tag{1}$$

in real anisotropic Sobolev space $W_p^{2b,1}(Q_T)$ with $p > 1$ provided that there exists a priori estimate $\|u\|_{k,\infty}$ and $(2b-k)\,p > n+2b$ with some k, $0 \le k \le 2b-1$.

For the boundary value problem (1) we assume that the following conditions are fulfilled.

A.1) Let the function $f(x,t,\xi_0,\xi_1,\ldots,\xi_{2b-1})$ with $\xi_l = \left\{\xi_\gamma \Big| \gamma \text{ is a multi-index}, |\gamma| = l\right\}$ be defined on $Q_T \times R \times R^n \times \cdots \times R^{n_{2b-1}}$ with the values in R and be a Caratheodory function, i.e. measurable with respect to (x,t) for all $(\xi_0,\ldots,\xi_{2b-1}) \in R \times \cdots \times R^{n_{2b-1}}$ and continuous with respect to $(\xi_0,\xi_1,\ldots,\xi_{2b-1})$ almost for all $(x,t) \in Q_T$.

A. 2) Let

$$|f(x,t,\xi_0,\xi_1,\ldots,\xi_{2b-1})| \le$$

$$\le b(x,t,\xi_0,\xi_1,\ldots,\xi_k) + \sum_{l=k+1}^{2b-1} b_l(x,t,\xi_0,\ldots,\xi_k) \cdot |\xi_l|^{\mu_l}$$

almost for all $(x,t) \in Q_T$ and for all $\xi_0 \in R$, $\xi_1 \in R^n, \ldots, \xi_k \in R^{n_k}, \ldots, \xi_{2b-1} \in R^{n_{2b-1}}$ for some k, $0 \le k \le 2b-1$, with such nonnegative Caratheodory functions b, $b_{k+1},\ldots,b_{2b-1}$ ($b_l \equiv 0$ for $k = 2b-1$) that for any $r \ge 0$ the function

$$\hat{b}_r(x,t) \equiv \sup\left\{b(x,t,\xi_0,\ldots,\xi_k) \,\Big|\, |\xi_0| + \cdots + |\xi_k| \le r\right\}$$

belongs to $L_p(Q_T)$, $p > 1$ and $(2b-k)\,p > n+2b$; the function

$$\hat{b}_{l,r}(x,t) \equiv \sup\left\{b_l(x,t,\xi_0,\ldots,\xi_k) \,\Big|\, |\xi_0| + \cdots + |\xi_k| \le r\right\}$$

belongs to $L_{q_l}(Q_T)$, $(l = k+1,\ldots,2b-1)$ with $q_l \ge p$, if $l < 2b - \frac{n+2b}{p}$, $q_l > p$, if $l = 2b - \frac{n+2b}{p}$, $q_l \ge \frac{n+2b}{2b-l}$, if $l > 2b - \frac{n+2b}{p}$. Here $|\xi_l| = \sum\limits_{|\gamma|=l} |\xi_\gamma|$.

A. 3) Let

$$\mu_l = \frac{2b-k}{l-k} - \frac{n+2b}{l-k} \cdot \frac{1}{q_l} \ \ (l = k+1,\ldots,2b-1). \tag{2}$$

From A.2)-A.3) it follows that $\mu_l > 1$ for any l.

A. 4) Let Lu be a quasilinear elliptic operator of order $2b \geq 2$ of the form

$$Lu = \sum_{|\alpha|=2b} a_\alpha \left(x, t, u, Du, \ldots, D^k u\right) D^\alpha u,$$

where the coefficients a_α $(|\alpha| = 2b)$ of the operator L are real and continuous functions on $\overline{Q_T} \times R \times R^n \times \cdots \times R^{n_k}$.

Let the operator $L_v u$

$$L_v u = \sum_{|\alpha|=2b} a_\alpha \left(x, t, v, \ldots, D^k v\right) D^\alpha u,$$

linear with respect to $u(x,t)$, be a linear elliptic operator, for any function $v(x,t) \in C^{k,0}\left(\overline{Q_T}\right)$ and B_i $(i = 0, 1, \ldots, b-1)$ be linear boundary differential operators of orders $b_i \leq 2b - 1$, respectively, with real coefficients such that the linear (with respect to $u(x,t)$) boundary value problem

$$\begin{cases} L_v u + \frac{\partial u}{\partial t} = g(x,t), \quad (x,t) \in Q_T, \\ B_i u\Big|_{\partial Q_T} = 0 \quad (i = 0, 1, \ldots, b-1), \quad (x,t) \in \partial Q_T, \\ u\Big|_{t=0} = 0, \quad x \in \Omega \end{cases} \tag{3}$$

is coercive in the space $W_p^{2b,1}(Q_T)$, i.e. a priori estimate

$$\|u\|_{W_p^{2b,1}(Q_T)} \leq C \left(\|g\|_{p;Q_T} + \|u\|_{p;Q_T}\right) \tag{4}$$

with a positive constant C, independent of $g \in L_p(Q_T)$ and on the solution $u \in W_p^{2b,1}(Q_T)$ of the linear problem (3), holds. This time, the constants C can depend on the moduli of continuity of the coefficients a_α $(|\alpha| = 2b)$ and the moduli of continuity of the functions $v, Dv, \ldots, D^k v$.

Note that sufficient conditions on the coefficient of the operators L and B_i $(i = 0, 1, \ldots, b-1)$, providing coerciveness of linear (with respect to $u(x,t)$) boundary value problem (3), were given by S. Agmon, A. Duglis, L. Nirenberg [1], V.A. Solonnikov [14, p.112] and [4].

This paper is based on the following two points: using interpolation inequality for estimating $\left\|D^j u\right\|^{\mu_l}$, $|j| = l$, $2b > l > k$ through $\|u\|_{W_p^{2b,1}(Q_t)}$ and $\|u\|_{k,\infty;Q_t}$ and application of the theorem on solvability of linear parabolic problems in a small cylinder Q_t [14, p.112-129]. By means of special constructions it was shown that the coefficients for $\|u\|_{W_p^{2b,1}(Q_t)}$ are independent of the cylinder height when this height is low. This allows using low height cylinders to establish the achievability of limiting degrees μ_l that are not achievable in the elliptic case.

Lemma 1. *Let $u \in W_p^{2b,1}(Q_\tau)$, $Q_\tau = \Omega \times (t_0, t+\tau) \subset Q_T$, $u(x,t_0) = 0$ and conditions A.1) - A.3) be fulfilled. Then*

$$\left\|D^j u\right\|_{S_l;Q_\tau} \leq C_1 \cdot \|u\|^{\theta_l}_{W_p^{2b,1}(Q_\tau)} \cdot \|u\|^{1-\theta_l}_{k,\infty} + C_2 \cdot \|u\|_{k,\infty} \tag{5}$$

with positive constants C_1 and C_2, independent of the function $u(x,t)$ from $W_p^{2b,1}(Q_\tau)$, t_0 and τ, where

$$\frac{1}{S_l} = \frac{l-k}{n+2b} + \theta_l \left(\frac{1}{p} - \frac{2b-k}{n+2b}\right), \quad \frac{l-k}{2b-k} \leq \theta_l < 1, \tag{6}$$

$|j| = l$, $2b > l > k$.

Proof. By the conditions, the function $u(x,t)$ is originally given in a cylinder with low-height τ. Extend it to the whole cylinder Q_T by assuming $u = 0$ on the segment $[0, t_0]$ and then mapping it in an even manner with respect to the point $t_0 + \tau$. More precisely,

$$\tilde{u}(x,t) = \begin{cases} 0, \quad t \leq t_0, \\ u(x,t), \ \ t_0 < t < t_0 + \tau, \\ u(x, 2(t_0+\tau) - t), \ \ t_0 + \tau \leq t < t_0 + 2\tau, \\ 0, \quad t \geq t_0 + 2\tau. \end{cases}$$

Note that according to the trace theorem [14, p.116] for each $t \in [t_0, t_0+\tau]$ the trace $u(x,t)$ is defined as a function continuous in $\overline{\Omega}$. In particular, the functions $u(x,t_0)$ and $u(x,t_0+\tau)$ are well defined. Obviously, $\tilde{u} \in W_p^{2b,1}(Q_T)$, and

$$\begin{aligned} \|\tilde{u}\|_{k,\infty;Q_T} &\leq 2 \cdot \|u\|_{k,\infty;Q_\tau}, \\ \|\tilde{u}\|_{W_p^{2b,1}(Q_T)} &\leq 2\|u\|_{W_p^{2b,1}(Q_\tau)}, \\ \left\|D^j \tilde{u}\right\|_{S_l;Q_T} &\geq \left\|D^j u\right\|_{S_l;Q_T}. \end{aligned} \tag{7}$$

From the Gagliardo-Nirenberg interpolation inequality [10] for the derivative $\left\|D^j \tilde{u}\right\|$, $|j| = l$, $2b > l > k$, we have

$$\left\|D^j \tilde{u}\right\|_{S_l;Q_T} \leq C_1 \cdot \|\tilde{u}\|^{\theta_l}_{W_p^{2b,1}(Q_T)} \cdot \|\tilde{u}\|^{1-\theta_l}_{k,\infty;Q_T} + C_2 \cdot \|\tilde{u}\|_{k,\infty;Q_T}.$$

Direct calculation shows that $\theta_l = 1/\mu_l$. Taking into account inequality (7), we obtain the statement of Lemma 1. ◄

Lemma 2. *Let conditions A. 1) - A. 2) be fulfilled. Then the operator $F(u) \equiv f\left(x, t, u, \ldots, D^{2b-1}u\right)$ acts completely continuously from $W_p^{2b,1}(Q_\tau)$ to $L_p(Q_\tau)$.*

Proof. Let us estimate $\|F(u)\|_{p;Q_\tau}$ by means of conditions A.1)-A.3). From condition A.2) it follows that

$$\|F(u)\|_{p;Q_\tau} \leq \left\|\hat{b_r}\right\|_{p;Q_\tau} + C_3 \cdot \sum_{l=k+1}^{2b-1} \sum_{|j|=l} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot \left\|D^j u\right\|_{S_l;Q_\tau}^{\mu_l}$$

with

$$S_l = \frac{p \cdot q_l}{q_l - p} \cdot \mu_l \quad (l = k+1, \dots, 2b-1), \;\; r = \|u\|_{k,\infty}$$

and a positive constant C_3 independent of the function $u(x,t)$ from $W_p^{2b,1}(Q_\tau)$, t_0 and τ . Then, based on inequalities (2) and interpolation inequalities (5) with $|j| = l = k+1, \dots, 2b-1$ and the corresponding S_l and μ_l, defined by formula (6), we obtain

$$\|F(u)\|_{p;Q_\tau} \leq \left\|\hat{b_r}\right\|_{p;Q_\tau} + C_3 \cdot \sum_{l=k+1}^{2b-1} \sum_{|j|=l} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot$$

$$\cdot \left[C_1 \cdot \|u\|_{W_p^{2b,1}(Q_\tau)}^{\theta_l} \cdot \|u\|_{k,\infty}^{1-\theta_l} + C_2 \|u\|_{k,\infty}\right]^{\mu_l} \leq$$

$$\leq \left\|\hat{b_r}\right\|_{p;Q_\tau} + C_3 \cdot \sum_{l=k+1}^{2b-1} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot 2^{\mu_l - 1} \cdot C_1^{\mu_l} \cdot \|u\|_{W_p^{2b,1}(Q_\tau)} \cdot \|u\|_{k,\infty}^{\mu_l - 1} +$$

$$+ C_3 \cdot \sum_{l=k+1}^{2b-1} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot 2^{\mu_l - 1} \cdot \|u\|_{k,\infty}^{\mu_l} \cdot C_2^{\mu_l} =$$

$$= \left\|\hat{b_r}\right\|_{p;Q_\tau} + \Phi_1\left(\|u\|_{k,\infty}\right) \cdot \|u\|_{W_p^{2b,1}(Q_\tau)} + \Phi_2\left(\|u\|_{k,\infty}\right), \tag{8}$$

where

$$\Phi_1\left(\|u\|_{k,\infty}\right) = C_3 \cdot \sum_{l=k+1}^{2b-1} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot C_1^{\mu_l} \cdot 2^{\mu_l - 1} \cdot \|u\|_{k,\infty}^{\mu_l - 1},$$

$$\Phi_2\left(\|u\|_{k,\infty}\right) = C_3 \cdot \sum_{l=k+1}^{2b-1} \left\|\hat{b_{l,r}}\right\|_{q_l;Q_\tau} \cdot C_2^{\mu_l} \cdot 2^{\mu_l - 1} \cdot \|u\|_{k,\infty}^{\mu_l},$$

i.e. Φ_1, $\Phi_2 : R_+ \to R_+$ are the increasing functions determined by the known data. This proves the boundedness of the operator $F(u)$.

Since $p(2b-k) > n + 2b$ with some k, $0 \leq k \leq 2b-1$. By virtue of the Sobolev embedding theorem [13, p.74-95], it follows that the embedding operator

$I : W_p^{2b,1}(Q_\tau) \to L_{S_l}(Q_\tau)$ is completely continuous. By virtue of the estimate (8), the operator $F : L_{S_l}(Q_\tau) \to L_p(Q_\tau)$ is bounded, and by general properties of the superposition operator, continuous. Then the operator $F : W_p^{2b,1}(Q_\tau) \to L_p(Q_\tau)$ is completely continuous as a composition of completely continuous and continuous operators.

Lemma 2 is proved. ◄

Theorem 1. *Let conditions A.1) - A.4) be fulfilled. Then there exists an increasing function $\Phi : R_+ \to R_+$ such that for any possible solution $u \in W_p^{2b,1}(Q_T)$ of problem (1) the following a priori estimate holds:*

$$\|u\|_{W_p^{2b,1}(Q_T)} \leq \Phi\left(\|u\|_{k,\infty}\right). \tag{9}$$

The function Φ depends only on the known data contained in the theorem conditions (including the quantities $\left\|\hat{b_r}\right\|_{p,Q_T}, \left\|b_{k+\hat{1},r}\right\|_{q_{k+1}}, \ldots, \left\|b_{2b-\hat{1},r}\right\|_{q_{2b-1}}$ with $r = \|u\|_{k,\infty}$).

Proof. Let us consider in the cylinder $Q_\tau = \Omega \times (t_0, t_0 + \tau)$ the following linear problem:

$$\begin{cases} L_v u + \frac{\partial u}{\partial t} = g(x,t), \ (x,t) \in Q_\tau, \\ B_i u\Big|_{\partial Q_T} = 0 \ \ (i = 0, 1, \ldots, b-1), \ x \in \partial\Omega, \ t \in (t_0, t_0+\tau), \\ u(x, t_0) = 0, \ x \in \Omega. \end{cases} \tag{10}$$

The linear boundary value problem (10) (with respect to $u(x,t)$) is coercive in the space $W_p^{2b,1}(Q_\tau)$, i.e. the a priori estimate

$$\|u\|_{W_p^{2b,1}(Q_\tau)} \leq C_4\left(\|g\|_{p;Q_\tau} + \|u\|_{p;Q_\tau}\right) \tag{11}$$

with positive constant C_4, independent of $g \in L_p(Q_\tau)$, the solution $u \in W_p^{2b,1}(Q_\tau)$ and t_0, τ, holds. On the other hand, by [13, p.74] we have

$$\|u\|_{p;Q_\tau} \leq C_5 \cdot \|u\|_{k,\infty;Q_\tau} \tag{12}$$

with a constant $C_5 > 0$, independent of the function $u(x,t)$ and t_0, τ.

We now use the coerciveness inequality (11) with

$$g(x,t) = f\left(x, t, u(x,t), \ldots, D^{2b-1}u(x,t)\right) \equiv F(u)(x,t)$$

for the problem

$$\begin{cases} Lu + \frac{\partial u}{\partial t} = f\left(x, t, u, Du, \ldots, D^{2b-1}u\right), & (x,t) \in Q_\tau, \\ B_i u\Big|_{\partial Q_\tau} = 0 \quad (i = 0, 1, \ldots, b-1), & x \in \partial\Omega, \ t \in (t_0, t_0 + \tau), \\ u(x, t_0) = 0, \ x \in \Omega. \end{cases} \tag{13}$$

From estimate (12) and inequality (8) we obtain

$$\|u\|_{W_p^{2b,1}(Q_\tau)} \le C_4 \cdot \left[\left\|\hat{b_r}\right\|_{p,Q_T} + \Phi_1\left(\|u\|_{k,\infty}\right) \times \right.$$

$$\left. \times \|u\|_{W_p^{2b,1}(Q_\tau)} + \Phi_2\left(\|u\|_{k,\infty}\right) + C_5 \cdot \|u\|_{k,\infty} \right]. \tag{14}$$

We choose τ so that the inequality $C_4 \cdot \Phi_1 \le \frac{1}{2}$ or

$$C_3 C_4 \cdot \sum_{l=k+1}^{2b-1} C_1^{\mu_l} \cdot 2^{\mu_l - 1} \cdot \|u\|_{k,\infty}^{\mu_l - 1} \cdot \left\|\hat{b_{l,r}}\right\|_{q_l; Q_\tau} \le \frac{1}{2} \tag{15}$$

is fulfilled.

We divide the domain Q_T into the cylinders $Q^0 = \Omega \times (0, \tau)$, $Q^1 = \Omega \times (\tau, 2\tau), \ldots, Q^k = \Omega \times (k\tau, (k+1)\tau), \ldots, Q^K = \Omega \times (K\tau, T)$ of height τ. Since τ is fixed, the number of these cylinders is finite.

Let $u_0(x,t) \in W_p^{2b,1}(Q_T)$ be the solution of (1). Let us consider the problem (1) in the cylinder Q^0 as a linear problem with the right-hand side $f_0 = f\left(x, t, u_0, \ldots, D^{2b-1}u_0\right)$.

According to the inequalities (11), (12) and (14),

$$\|u_0\|_{W_p^{2b,1}(Q^0)} \le C_4 \left(\|f_0\|_{p;Q^0} + \|u_0\|_{p;Q^0} \right) \le$$

$$\le C_4 \left(\left\|\hat{b_r}\right\|_{p;Q^0} + \Phi_1\left(\|u_0\|_{k,\infty}\right) \cdot \|u_0\|_{W_p^{2b,1}(Q^0)} + \Phi_2\left(\|u_0\|_{k,\infty}\right) + C_5 \|u_0\|_{k,\infty} \right). \tag{16}$$

By virtue of (15), the coefficient for $\|u_0\|_{W_p^{2b,1}(Q^0)}$ does not exceed $\frac{1}{2}$, so, inequality (16) takes the form

$$\|u_0\|_{W_p^{2b,1}(Q^0)} \le \frac{1}{2} \|u_0\|_{W_p^{2b,1}(Q^0)} + \frac{1}{2} \cdot C^0,$$

where the constant C^0 is determined from (16). Hence,

$$\|u_0\|_{W_p^{2b,1}(Q^0)} \le C^{(0)}.$$

Let us consider the cylinders Q^{k-1} and $Q^k, 1 \le k \le K$, and suppose that the estimate

$$\|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)} \le C^{(k-1)}$$

has already been obtained.

Let us make sure that it yields the following estimate:

$$\|u_0\|_{W_p^{2b,1}\left(Q^k\right)} \le C^{(k)}.$$

Denote $V(x,t) = u_0(x, 2k\tau - t)$ and assume

$$\tilde{u}(x,t) = u_0(x,t) - V(x,t), \quad t \in (k\tau, (k+1)\tau).$$

Obviously, $\tilde{u} \in W_p^{2b,1}\left(Q^k\right)$ and $\tilde{u}(x, k\tau) = 0$.

In the cylinder Q^k we consider

$$\begin{cases} L\tilde{u} + \frac{\partial \tilde{u}}{\partial t} = f\left(x, t, \tilde{u} + V, \ldots, D^{2b-1}(\tilde{u} + \tilde{v})\right) - \left(LV + \frac{\partial V}{\partial t}\right), \quad (x,t) \in Q^k, \\ \tilde{u}(x, k\tau) = 0, \quad x \in \Omega, \\ B_i \tilde{u}\Big|_{\partial Q^k} = 0 \quad (i = 0, 1, \ldots, b-1), \ (x,t) \in \partial Q^k. \end{cases} \tag{17}$$

The function $\tilde{u}(x,t)$ is the solution of problem (17), since, by assumption, the function $u_0(x,t)$ is the solution of problem (1).

Using the estimate

$$\|\tilde{u}\|_{k,\infty;Q^k} \le 2 \cdot \|u_0\|_{k,\infty;Q_T} \tag{18}$$

and the equality

$$\|V\|_{W_p^{2b,1}\left(Q^k\right)} = \|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)}, \tag{19}$$

similar to (16) we obtain

$$\|\tilde{u}\|_{W_p^{2b,1}\left(Q^k\right)} \le C_4 \cdot$$

$$\cdot \left(\left\| f\left(x, t, \tilde{u} + V, \ldots, D^{2b-1}(\tilde{u} + V)\right) - \left(\frac{\partial V}{\partial t} + LV\right) \right\|_{p;Q^k} + \|\tilde{u}\|_{p;Q^k} \right) \le$$

$$\le C_4 \left(\left\| \hat{b}_r \right\|_{p;Q^k} + C_3 \cdot \sum_{l=k+1}^{2b-1} \sum_{|j|=l} \left\| \hat{b}_{l;r} \right\|_{q_l;Q^k} \cdot \left\| D^j(\tilde{u} + V) \right\|_{S_l;Q^k}^{\mu_l} + \right.$$

$$\left. + C_6 \cdot \|V\|_{W_p^{2b,1}\left(Q^k\right)} + C_5 \cdot \|\tilde{u}\|_{k,\infty;Q^k} \right) \le$$

$$\le C_4 \left(\left\| \hat{b}_r \right\|_{p;Q^k} + \Phi_1\left(\|\tilde{u} + V\|_{k,\infty;Q^k}\right) \cdot \|\tilde{u} + V\|_{W_p^{2b,1}\left(Q^k\right)} + \right.$$

$$+\Phi_2\left(\|\tilde{u}+V\|_{k,\infty;Q^k}\right)+C_6\cdot\|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)}+$$

$$+C_5\cdot\|u_0-V\|_{k,\infty;Q^k}\Big)\le C_4\left(\left\|\hat{b}_r\right\|_{p;Q^k}+\right.$$

$$+\Phi_1\left(\|u_0\|_{k,\infty;Q^k}\right)\cdot\|u_0\|_{W_p^{2b,1}\left(Q^k\right)}+\Phi_2\left(\|u_0\|_{k,\infty;Q^k}\right)+$$

$$\left.+C_6\|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)}+2C_5\|u_0\|_{k,\infty;Q^k}\right), \tag{20}$$

where $C_6>0$ is a constant independent of the function $u(x,t)$ and t,τ. Since the coefficients and $\|u_0\|_{W_p^{2b,1}\left(Q^k\right)}$ do not exceed $\frac{1}{2}$, inequality (20) takes the form

$$\|\tilde{u}\|_{W_p^{2b,1}\left(Q^k\right)}\le\frac{1}{2}\|u_0\|_{W_p^{2b,1}\left(Q^k\right)}+C_7\|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)}+C_8, \tag{21}$$

where C_7 and C_8 are determined from (20).

Note that $u_0=\tilde{u}+V$, so

$$\|u_0\|_{W_p^{2b,1}\left(Q^k\right)}\le\|\tilde{u}\|_{W_p^{2b,1}\left(Q^k\right)}+\|V\|_{W_p^{2b,1}\left(Q^k\right)}.$$

Using (21) and (19), we obtain

$$\|u_0\|_{W_p^{2b,1}\left(Q^k\right)}\le\frac{1}{2}\|u_0\|_{W_p^{2b,1}\left(Q^k\right)}+$$

$$+C_7\cdot\|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)}+C_8+\|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)}.$$

Hence,

$$\|u_0\|_{W_p^{2b,1}\left(Q^k\right)}\le 2\left(C_7+1\right)\|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)}+2C_8. \tag{22}$$

According to the assumption, $\|u_0\|_{W_p^{2b,1}\left(Q^{k-1}\right)}\le C^{(k-1)}$. So from (22) it follows that

$$\|u_0\|_{W_p^{2b,1}\left(Q^k\right)}\le C^{(k)}.$$

Application of induction completes the proof of Theorem 1. ◄

2. Unimpovability of growth rates under the conditions of Theorem 1

The growth exponents $\mu_{k+1}, \dots, \mu_{2b-1}$, determined by the conditions A.3), are unimprovable, i.e. none of the equalities can be replaced by the corresponding inequality (without additional assumptions). In this section we give an example of a boundary value problem (1) for which all the conditions of Theorem 1 except condition A. 3), i.e. equality (2), are fulfilled. For this counterexample the corresponding inequality is fulfilled, and it is shown that the statement of Theorem 1 is not true.

The scheme for constructing a counterexample. Let us consider the function $u(x,t)$, dependent on the parameter ε, $0 < \varepsilon \le 1$, defined by means of the relation

$$u(x,t) = \varepsilon^k v(y,\tau), \quad y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^{2b}}, \tag{23}$$

where $v \in C^{\infty,1}\left(R^{n+1}\right)$ and

$$\sup_{R^{n+1}} |v(y,\tau)| < \infty, \dots, \sup_{R^{n+1}} |D^\gamma v(y,\tau)| < \infty \tag{24}$$

for all multiindices γ with $|\gamma| \le k$, $0 \le k \le 2b-1$. Such a choice of the function $u(x,t)$ provides the boundedness of the norm

$$\|u\|_{k,\infty} = \sum_{|\gamma| \le k} \sup_{Q_T} |D^\gamma u(x,t)| \le C, \tag{25}$$

where the constant C is independent of $\varepsilon \in (O,1]$.

On the other hand, we have

$$\|u\|_{W_p^{2b,1}(Q_T)} = \left[\sum_{|\alpha| \le 2b} \int_{Q_T} |D^\alpha u|^p \, dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^p dxdt\right]^{1/p} \ge$$

$$\ge \left[\sum_{|\alpha| = 2b} \int_{Q_T} |D^\alpha u|^p \, dxdt + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^p dxdt\right]^{1/p} =$$

$$= \varepsilon^{k-2b+\frac{n+2b}{p}} \cdot \left[\sum_{|\alpha| = 2b} \int_{Q_T^\varepsilon} |D^\alpha v|^p \, dyd\tau + \int_{Q_T^\varepsilon} \left|\frac{\partial v}{\partial \tau}\right|^p dyd\tau\right]^{1/p},$$

where $Q_T^{\varepsilon} = \left\{ (y,\tau) \middle| \varepsilon y \in \Omega,\ \tau \cdot \varepsilon^{2b} \in (0,T) \right\}$. Hence, for a domain Q_T such that

$$\Omega \subseteq \Omega_{\varepsilon} = \left\{ y \in R^n \middle| \varepsilon y \in \Omega \right\} \text{ for } \varepsilon \in (0,1], \tag{26}$$

we obtain

$$\|u\|_{W_p^{2b,1}(Q_T)} \le \varepsilon^{k-2b+\frac{n+2b}{p}} \cdot \left[\sum_{|\alpha|=2b} \int\limits_{Q_T} |D^{\alpha} v|^p \, dyd\tau + \int\limits_{Q_T} \left| \frac{\partial v}{\partial \tau} \right|^p dyd\tau \right]^{1/p}.$$

Then for the function $v(y,\tau)$ with

$$\sum_{|\alpha|=2b} \int\limits_{Q_T} |D^{\alpha} v|^p \, dyd\tau + \int\limits_{Q_T} \left| \frac{\partial v}{\partial \tau} \right|^p dyd\tau > 0 \tag{27}$$

and

$$(2b-k)\, p > n + 2b \tag{28}$$

we obtain

$$\|u\|_{W_p^{2b,1}(Q_T)} \to +\infty \text{ for } \varepsilon \to 0 \ (\varepsilon > 0). \tag{29}$$

Note that the inequality (28) yields the embedding

$$W_p^{2b,1}(Q_T) \subset C^{k,1}(Q_T), \ 0 \le k \le 2b-1.$$

Now for the function $u(x,t)$ we draw up an equation of the form

$$\Delta^b u(x,t) - \frac{\partial u}{\partial t} = b_l(x,t) \cdot |L_l u(x,t)|^{\mu_l}, \tag{30}$$

where L_l is a linear homogeneous differential operator of order l with constant coefficients. Then,

$$b_l(x,t) = \frac{\Delta^b u(x,t)}{|L_l u(x,t)|^{\mu_l}} - \frac{\partial u(x,t)/\partial t}{|L_l u(x,t)|^{\mu_l}}$$

and

$$\|b_l\|_{q_l;Q_T} = \left(\int\limits_{Q_T} |b_l|^{q_l} \, dxdt \right)^{1/q_l} \le \varepsilon^{k-2b-(k-l)\mu_l+\frac{n+2b}{q_l}} \cdot$$

$$\cdot \left(\int\limits_0^T \int\limits_{\Omega_{\varepsilon}} \frac{|\Delta^b v|^{q_l} \, dyd\tau}{|L_l v|^{q_l \mu_l}} + \int\limits_0^T \int\limits_{\Omega_{\varepsilon}} \frac{|\partial v/\partial \tau|^{q_l} \, dyd\tau}{|L_l v|^{q_l \mu_l}} \right)^{1/q_l}.$$

Hence for

$$\mu_l > \frac{2b-k}{l-k} - \frac{n+2b}{l-k} \cdot \frac{1}{q_l}, \tag{31}$$

we obtain

$$\|b_l\|_{q_l;Q_T} \le \left(\int\limits_{R^{n+1}} \frac{\left|\Delta^b v\right|^{q_l} dyd\tau}{|L_l v|^{q_l \mu_l}} + \int\limits_{R^{n+1}} \frac{|\partial v/\partial \tau|^{q_l} dyd\tau}{|L_l v|^{q_l \mu_l}} \right)^{1/q_l}. \tag{32}$$

Now we add to the equation (30) the initial boundary conditions

$$\begin{cases} u\Big|_{\partial Q_T} = 0, \ \frac{\partial u}{\partial v}\Big|_{\partial Q_T} = 0, \dots, \frac{\partial^{b-1} u}{\partial v^{b-1}}\Big|_{\partial Q_T} = 0, \ x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) = 0, \ x \in \Omega, \end{cases} \tag{33}$$

where $\frac{\partial u}{\partial v}, \dots, \frac{\partial^{b-1} u}{\partial v^{b-1}}$ are the derivatives of the function $u(x,t)$ of corresponding order on ∂Q_T in the direction of the outer unit normal ν of ∂Q_T.

Thus, the construction of a counterexample in the domain Q_T, satisfying the condition (26) for an integer $k \ge 0$ and the inequality (28), is reduced to constructing the function $v \in C^\infty\left(R^{n+1}\right)$, satisfying the inequalities (24), (27) and the inequality

$$\left(\int\limits_{R^{n+1}} \frac{\left|\Delta^b v\right|^{q_l} dyd\tau}{|L_l v|^{q_l \mu_l}} + \int\limits_{R^{n+1}} \frac{|\partial v/\partial \tau|^{q_l} dyd\tau}{|L_l v|^{q_l \mu_l}} \right)^{1/q_l} < \infty \tag{34}$$

and generating boundary conditions (33) for the function $u(x,t)$.

Now we implement this scheme in the case $b = 1, \ k = 0, \ l = 1$ and $n \ge 1$. Let us consider an initial boundary value problem for a second order equation

$$\Delta u - \frac{\partial u}{\partial t} = f(x,t,u,Du), \ (x,t) \in Q_T,$$

in a real Sobolev space $W_p^{2,1}(Q_T)$ with $p > 1$ provided that there exists a priori estimate

$$\|u\|_{0,\infty} = \sup_{Q_T} |u(x,t)|$$

and $2p > n+2$, where $f(x,t,\xi_0,\xi_1)$ is a Caratheodary function defined on $Q_T \times R \times R^n$ with the values in R.

Let

$$|f(x,t,\xi_0,\xi_1)| \le b(x,t,\xi_0) + b_1(x,t,\xi_0)|\xi_1|^{\mu_1} \tag{35}$$

for almost all $(x,t) \in Q_T$ and all $\xi_0 \in R$, $\xi_1 \in R^n$, with the nonnegative Caratheodory functions $b(x,t,\xi_0)$, $b_1(x,t,\xi_0)$ such that for any $r \geq 0$ the function

$$\hat{b_r}(x,t) \equiv \sup\left\{ b(x,t,\xi_0) \,\middle|\, |\xi_0| \leq r \right\}$$

belongs to $L_p(Q_T)$, with $p > 1$ and $2p > n+2$, and the function

$$\hat{b}_{1,r}(x,t) \equiv \sup\left\{ b_1(x,t,\xi_0) \,\middle|\, |\xi_0| \leq r \right\}$$

belongs to $L_q(Q_T)$ with $q > p$.

Let

$$\mu_1 = 2 - \frac{n+2}{q}. \tag{36}$$

Then by Theorem 1 we have an estimate

$$\|u\|_{W_p^{2,1}(Q_T)} \leq \Phi\left(\|u\|_{0,\infty}\right). \tag{37}$$

Thus, the exponent $\mu_1 = 2 - \frac{n+2}{q}$ in the condition (35) can not be replaced without additional assumptions for $\mu_1 > 2 - \frac{n+2}{q}$.

Let us consider the function

$$u(x,t) = v(y,\tau), \quad y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^2}, \quad \varepsilon \in (0,1], \tag{38}$$

where $v \in C^\infty\left(R^{n+1}\right)$ and

$$\sup_{R^{n+1}} = |v(y,\tau)| < \infty. \tag{39}$$

Such a choice of the function $u(x,t)$ provides the boundedness of the norm

$$\|u\|_{0,\infty} = \sup_{Q_T} |u(x,t)| \leq C, \tag{40}$$

where C is independent of ε.

On the other hand, we have

$$\|u\|_{W_p^{2,1}(Q_T)} \geq \varepsilon^{-2+\frac{n+2}{p}} \cdot \left[\sum_{|\alpha|=2} \int\limits_{Q_T^\varepsilon} |D^\alpha v|^p \, dy d\tau + \int\limits_{Q_T^\varepsilon} \left| \frac{\partial v}{\partial \tau} \right|^p dy d\tau \right]^{1/p},$$

where $Q_T^\varepsilon = \left\{ (y,\tau) \,\middle|\, \varepsilon y \in \Omega,\, \tau \cdot \varepsilon^2 \in (0,T) \right\}$ and Q_T is such that

$$\Omega \subseteq \Omega_\varepsilon = \left\{ y \in R^n \,\middle|\, \varepsilon y \in \Omega \ \text{ for } \varepsilon \in (0,1] \right\}. \tag{41}$$

Then for the function $v(y,\tau)$ with

$$\sum_{|\alpha|=2}\int_{Q_T}|D^\alpha v|^p\,dyd\tau+\int_{Q_T}\left|\frac{\partial v}{\partial\tau}\right|^p dyd\tau>0 \tag{42}$$

and

$$2p>n+2b \tag{43}$$

we obtain

$$\|u\|_{W_p^{2,1}(Q_T)}\to+\infty\ \text{ for }\varepsilon\to 0\ \ (\varepsilon>0)\,. \tag{44}$$

Note that from the inequality (43) we have the embedding

$$W_p^{2,1}(Q_T)\subset C^{0,1}(Q_T)\,.$$

Consider the equation of the form

$$\Delta u(x,t)-\frac{\partial u}{\partial t}=b_l(x,t)\cdot|L_l u(x,t)|^{\mu_l}\,, \tag{45}$$

where L_l is a linear homogeneous differential operator of order $l=1$ with constant coefficients. Then,

$$b_1(x,t)=\frac{\Delta u(x,t)}{|L_1 u|^{\mu_1}}-\frac{u_t}{|L_1 u|^{\mu_1}}$$

and

$$\|b_1\|_{q;Q_T}\le\varepsilon^{-2+\mu_1+\frac{n+2}{q}}\cdot$$

$$\cdot\left(\int_0^T\int_{\Omega_\varepsilon}\frac{|\Delta v|^q\,dyd\tau}{|L_1 v|^{q\mu_1}}+\int_0^T\int_{\Omega_\varepsilon}\frac{|v_\tau|^q\,dyd\tau}{|L_1 v|^{\mu_1 q}}\right)^{1/q}.$$

Hence for $\mu_1>2-\frac{n+2}{q}$ we have

$$\|b_1\|_{q;Q_T}\le\left(\int_{R^{n+1}}\frac{|\Delta v|^q\,dyd\tau}{|L_1 v|^{q\mu_1}}+\int_{R^{n+1}}\frac{|v_\tau|^q\,dyd\tau}{|L_1 v|^{q\mu_1}}\right)^{1/q}. \tag{46}$$

Now we add to the equation (45) the initial boundary conditions

$$\begin{cases} u(x,t)\Big|_{\partial Q_T}=0,\ \ x\in\partial\Omega,\ \ t\in(0,T)\,,\\ u(x,0)=0,\ \ x\in\Omega.\end{cases} \tag{47}$$

Construction of a counterexample satisfying the condition (41) and the inequality (43) is reduced to constructing the function $v \in C^{\infty}\left(R^{n+1}\right)$, satisfying the inequalities (39), (42) and

$$\left(\int\limits_{R^{n+1}} \frac{|\Delta v|^q \, dy d\tau}{|L_1 v|^{q\mu_1}} + \int\limits_{R^{n+1}} \frac{|v_\tau|^q \, dy d\tau}{|L_1 v|^{\mu_1 q}}\right)^{1/q} < \infty, \tag{48}$$

and generating boundary conditions (47) for the function $u(x,t)$.

Thus, for the problem (45), (47), with $b = 1$, $k = 0$, $l = 1$, $n \geq 1$, and p and q such that

$$2p > n+2, \quad q > p$$

for $\mu_1 > 2 - \frac{n+2}{q}$, the statement of Theorem 1 is not valid. In fact, if this statement was valid for problem (45), (47), then from the uniform boundedness of the quantity $\|u\|_{0,\infty}$ for all $\varepsilon \in (0,1)$ there should follow the uniform boundedness of the norm $\|u\|_{W_p^{2,1}(Q_T)}$, for all $\varepsilon \in (0,1]$, which is not valid, since

$$W_p^{2,1}(Q_T) \to +\infty, \quad \text{as } \varepsilon \to 0 \ (\varepsilon > 0).$$

As $\mu_1 > 2 - \frac{n+2}{q}$, it follows that under the conditions of Theorem 1, the equality $\mu_1 = 2 - \frac{n+2}{p}$ can not be replaced by inequality (without additional assumption).

3. Solvability of boundary value problems for quasilinear parabolic equations of arbitrary orders with a priori estimate for $\|u\|_{k,\infty}$

Let us consider the boundary value problem (1)in the real Sobolev space $W_p^{2,1}(Q_T)$ with $p > 1$ provided that there exists a priori estimate $\|u\|_{k,\infty}$ independent of $\tau \in [0,1]$, for solving parametric family of problems

$$\begin{cases} Lu + \frac{\partial u}{\partial t} = \tau \cdot f\left(x,t,u,Du,\ldots,D^{2b-1}u\right), \ (x,t) \in Q_T, \\ B_i u\Big|_{\partial Q_T} = 0 \ \ (i = 0,1,\ldots,b-1), \ x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) = 0, \ \ x \in \Omega \end{cases} \tag{49}$$

for some k, $0 \leq k \leq 2b-1$ and $p(2b-k) > n+2b$.

For the operator L and the boundary operators B_i $(i = 0,1,\ldots,b-1)$, we suppose that the following condition is fulfilled.

A. 5) Let L be a quasilinear elliptic operator of order $2b \geq 2$ with real and continuous coefficients, and B_i $(i = 0,1,\ldots,b-1)$ be linear boundary differential operators of orders $b_i \leq 2b-1$, respectively, with real coefficients such that the linear boundary value problem (3) for any $g \in L_p(Q_T)$ is uniquely solvable in the space $W_p^{2b,1}(Q_T)$, and the estimate

$$\|u\|_{W_p^{2b,1}(Q_T)} \leq C \cdot \|g\|_{p;Q_T}$$

with a positive constant C, independent of $g \in L_p(Q_T)$ and the solution $u \in W_p^{2b,1}(Q_T)$, holds.

Theorem 2. *Let conditions A.1) - A.3), A.5) be fulfilled, and for parametric family of problems* (49) *for* $\tau \in [0,1]$ *there exist a priori estimate* $\|u\|_{k,\infty}$*, independent of* τ *with some* k, $0 \leq k \leq 2b-1$. *Then there exists a solution of problem* (1) *in the space* $W_p^{2b,1}(Q_T)$ *with* $p > 1$ *and* $(2b-k)\,p > n+2b$.

Proof. Let us consider a parametric family of problems (49) for $\tau \in [0,1]$. For this family of problems, the conditions of Theorem 1 are fulfilled, by virtue of which there exists such a constant C_1, independent of τ, that for any possible solution $W_p^{2b,1}(Q_T)$ of problem (49) the inequality

$$\|u\|_{W_p^{2b,1}(Q_T)} \leq C_1, \ \ \forall\, \tau \in [0,1]\,, \tag{50}$$

is fulfilled.

From condition A. 5) it follows that the linear boundary value problem (3) is uniquely solvable for any function $g(x,t)$ from $L_p(Q_T)$, so that $u = Ag$, where A is a linear bounded operator corresponding to problem (3) and acting from the space $L_p(Q_T)$, to the space

$$W_p^{2b,1}(Q_T;0) = \{u \in W_p^{2b,1}(Q_T) \ \ |B_i u|_{\partial Q_T} = 0 \ \ (i = 0,1,\dots,b-1)\,,$$

$$(x,t) \in \partial Q_T,\ u(x,0) = 0,\ x \in \Omega\}.$$

Then, boundary value problem (49) is equivalent to the operator equation

$$u = \tau p(u) \tag{51}$$

in the Banach space $W^{2b,1}(Q_T;0)$, where $p(u) = A \cdot \tilde{f}(u)$ and

$$\tilde{f}(u) = f\left(x,t,u(x,t),\dots,D^{2b-1}u(x,t)\right).$$

The operator $\tilde{f}$ is defined on the space $W_p^{2b,1}(Q_T;0)$ with the values in $L_p(Q_T)$ and is a completely continuous operator from $W_p^{2b,1}(Q_T;0)$ to $L_p(Q_T)$. This fact follows from Sobolev-Kondrashev embedding theorem [13] under the basic condition of Theorem 2. Therefore, the operator p is a completely continuous operator acting in the space $W_p^{2b,1}(Q_T;0)$.

For possible solutions $u(x,t)$ from the space $W_p^{2b,1}(Q_T;0)$, by virtue of inequality (50) the following a priori estimate holds:

$$\|u\|_{W_p^{2b,1}(Q_T;0)} \leq C_2, \ \ \forall \, \tau \in [0,1],$$

where C_2 is a positive constant independent neither of $u(x,t)$ nor of τ. Then, by the virtue of Leray-Schauder principle [7], the equation (51) for $\tau = 1$ and, consequently, the boundary value problem (1) has the solution $u_0 \in W_p^{2b,1}(Q_T)$.

Theorem 2 is proved. ◄

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