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On the Existence and Uniqueness of a Positive Solution to a Boundary Value Problem for a Nonlinear Functional Fractional Order Differential Equation

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Abstract. This article considers a two-point boundary value problem for a nonlinear functional-differential equation of a fractional order with boundary conditions of the Sturm-Liouville type. Sufficient conditions for the existence of a unique positive solution of the problem under consideration are obtained using special topological tools. The existence of a positive solution is proved with the help of the well-known Go-Krasnoselsky theorem on the fixed point of the operator, and the uniqueness is established using the contraction mapping principle. An example is given that illustrates the fulfillment of sufficient conditions for the unique solvability of the problem posed. The results obtained complement the previous results of the author on the existence and uniqueness of positive solutions to boundary value problems for non-linear functional-differential equations.

Key Words and Phrases: positive solution, boundary value problem, cone, Green's function.

2010 Mathematics Subject Classifications: 34K10

1. Introduction

Fractional derivatives are a generalization of integer order derivatives. There are several types of fractional derivatives, for example, the Riemann-Liouville fractional derivative, the Marchaux fractional derivative, the Caputo derivative, etc. Fractional calculus is an excellent tool for modeling various phenomena in applied research. Fractional differential equations are often encountered in various fields of science and technology, such as physics, chemistry, economics, etc. In particular, they are widely used in the study of viscoelasticity, electrochemical control, porous media, electromagnetics, etc. As a result, fractional equations have recently attracted wide attention and acquired great practical importance.

A fairly large number of studies have been dedicated to nonlinear differential equations of fractional order (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]). However, the theory of boundary value problems for non-linear fractional equations, containing a linear operator is still at the initial stage, and many aspects of this direction have not been fully studied.

This paper attempts to fill this gap to some extent. Based on the well-known Go-Krasnoselsky theorem, we established sufficient conditions for the existence of a positive solution to a boundary value problem for a nonlinear functional-differential equation of a fractional order. To prove the uniqueness of the solution, the contraction mapping principle was applied. The results obtained complement our research on this topic.

2. Problem Statement and Main Results

For the convenience of calculations, we will use the following notations: C will denote the space $C[0,1]$, $\mathbb{L}_p$ $(1 < p < \infty)$ will be the space $\mathbb{L}_p(0,1)$ and $\mathbb{W}^2$ will denote the space of real functions defined on $[0,1]$ with an absolutely continuous derivative.

Consider the boundary value problem

$$D^{\alpha}_{0+}x(t) + f\left(t, (Tx)(t)\right) = 0, \quad 0 < t < 1, \tag{1}$$

$$x(0) + x'(0) = 0, \quad x(1) + x'(1) = 0, \tag{2}$$

where $\alpha \in (1,2]$ is a real number, D^{α}_{0+} is a Riemann-Liouville fractional derivative, and $T\colon C \to \mathbb{L}_p$ $(1 < p < \infty)$ is a linear positive continuous operator.

Regarding the function $f(t,u)$, suppose that it is non-negative in the domain $[0,1] \times [0,\infty)$, increases monotonically in u, satisfies the Carathéodory condition and $f(\cdot, 0) \equiv 0$.

Definition 1. *By a positive solution to the problem* (1)–(2) *we mean the function* $x \in \mathbb{W}^2$*, positive in the interval* $(0,1)$*, satisfying almost everywhere equation* (1) *and boundary conditions* (2)*.*

It is easy to verify that the problem (1)–(2) is equivalent to the equation

$$x(t) = \int_0^1 G(t,s) f\left(s, (Tx)(s)\right) ds, \quad 0 \le t \le 1, \tag{3}$$

where

$$G(t,s)=\begin{cases}\dfrac{(1-s)^{\alpha-1}(1-t)+(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & \text{if } 0\le s\le t,\\ \dfrac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & \text{if } t\le s\le 1.\end{cases}$$

It was shown in [15] that the Green's function of the operator D^{α}_{0+} with boundary conditions (2) has the following properties:

1. $G(t,s)>0, \quad t,s\in(0,1)$,
2. $\min_{1/4\le t\le 3/4} G(t,s)\ge \frac{1}{8}M(s)$, $s\in(0,1)$,
3. $\max_{0\le t\le 1} G(t,s)\le M(s)$, $s\in(0,1)$,

where $M(s)=\dfrac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$, $s\in[0,1)$.

Let's assume that for almost all $t\in[0,1]$ the function $f(t,u)$ satisfies the condition

$$f(t,u)\le a_1(t)+bu^{p/q}, \tag{4}$$

where $b>0$ is a constant, $a_1\in\mathbb{L}_q$, $1<q<p<\infty$.

In operator form, the equation (3) can be rewritten as

$$x=GNTx,$$

where $N\colon \mathbb{L}_p\to\mathbb{L}_q$ is a Nemytsky operator, and $G\colon \mathbb{L}_q\to C$ is a linear continuous operator defined by the kernel $G(t,s)$.

The operator A defined by the equality

$$(Ax)(t)=\int_0^1 G(t,s)f\left(s,(Tx)(s)\right)ds, \quad 0\le t\le 1,$$

acts in the space of non-negative continuous functions and is completely continuous [16, p. 161].

Assume that, for almost all $t\in[0,1]$ and arbitrary $u\ge 0$,

$$f(t,u)\ge a_0(t)u^{p/q}, \tag{5}$$

where $a_0\in\mathbb{L}_q$, $a_0\ge 0$, $1<q<p<\infty$.

We define the cone of a nonnegative functions $\tilde{K}$ as follows:

$$\tilde{K}=\left\{x\in C:\ \min_{1/4\le t\le 3/4} x(t)\ge \frac{1}{8}\|x\|_C\right\}.$$

Lemma 1. *The operator A leaves the cone $\tilde{K}$ invariant.*

Proof. By virtue of the above properties of the Green's function, we have

$$\min_{1/4\le t\le 3/4}(Ax)(t) = \min_{1/4\le t\le 3/4}\int_0^1 G(t,s)f\left(s,(Tx)(s)\right)\,ds \ge$$
$$\ge \frac{1}{8}\int_0^1 M(s)f\left(s,(Tx)(s)\right)\,ds.$$

On the other hand

$$\|Ax\|_C = \max_{0\le t\le 1}|(Ax)(t)| \le \int_0^1 M(s)f\left(s,(Tx)(s)\right)\,ds.$$

Finally we have

$$\min_{1/4\le t\le 3/4}(Ax)(t) \ge \frac{1}{8}\|Ax\|_C.$$

◀

Let us introduce the following notations:

$$\Omega_1 = \left\{x\in\tilde{K} : \|x\|_C < r_1\right\},$$
$$\Omega_2 = \left\{x\in\tilde{K} : \|x\|_C < r_2\right\},$$
$$\partial\Omega_1 = \left\{x\in\tilde{K} : \|x\|_C = r_1\right\},$$
$$\partial\Omega_2 = \left\{x\in\tilde{K} : \|x\|_C = r_2\right\},$$
$$\Omega = \overline{\Omega}_2\backslash\Omega_1,$$

where r_1, r_2 are some ordered positive numbers, the selection rule for which we will discuss below.

Theorem 1. *Let's assume that the conditions* (4), (5) *are satisfied and*

1. $\dfrac{p-q}{q}\left(\dfrac{q}{pb\|M\|_{\mathbb{L}_{q'}}\gamma^{\frac{p}{q}}}\right)^{\frac{q}{p-q}} \ge \|a_1\|_{\mathbb{L}_q}\|M\|_{\mathbb{L}_{q'}}$,
   where $\frac{1}{q}+\frac{1}{q'}=1$, γ *is the norm of the operator* T*;*
2. $0 < \int_{1/4}^{3/4} M(s)a_0(s)(T\chi)^{\frac{p}{q}}(s)ds < 64b\frac{p}{q}\|M\|_{\mathbb{L}_{q'}}\gamma^{\frac{p}{q}}$,
   where $\chi(t)$ *is any continuous function on* $[\frac{1}{4},\frac{3}{4}]$ *such that* $0\le\chi(t)\le 1$.

Then the boundary value problem (1)–(2) *has at least one positive solution* $x\in\Omega$.

Proof. To prove the existence of a positive solution to problem (1)–(2), we use the Go-Krasnoselsky theorem on a fixed point of a positive operator [17].

Let us first show the existence of a positive number r_1 such that for $x \in \tilde{K} \cap \partial\Omega_1$

$$\|Ax\|_C \leq \|x\|_C. \tag{6}$$

In view of the above properties of the Green's function and the condition (4), we obtain

$$\|(Ax)(t)\|_C = \max_{0 \leq t \prec 1} \int_0^1 G(t,s) f\left(s, (Tx)(s)\right) ds \leq$$
$$\leq \int_0^1 M(s) a_1(s)\, ds + b \int_0^1 M(s) \left(Tx\right)^{\frac{p}{q}}(s)\, ds \leq$$
$$\leq \|a_1\|_{\mathbb{L}_q} \|M\|_{\mathbb{L}_{q'}} + b\|M\|_{\mathbb{L}_{q'}} \|Tx\|_{\mathbb{L}_p}^{\frac{p}{q}} \leq \|a_1\|_{\mathbb{L}_q} \|M\|_{\mathbb{L}_{q'}} + b\|M\|_{\mathbb{L}_{q'}} \gamma^{\frac{p}{q}} \|x\|_C^{\frac{p}{q}}.$$

Consider the function

$$\varphi(r) = r - \beta_1 r^\delta - \beta_2,$$

where $\beta_1 > 0$, $\beta_2 \geq 0$, $\delta > 1$.

It is easy to verify that for $r > 0$ the function $\varphi(r)$ reaches its maximum value at $r = r_{max} = \left(\frac{1}{\delta\beta_1}\right)^{\frac{1}{\delta-1}}$. In turn, $\varphi(r_{max})$ is non-negative if the inequality

$$\left(1 - \frac{1}{\delta}\right)\left(\frac{1}{\delta\beta_1}\right)^{\frac{1}{\delta-1}} \geq \beta_2 \tag{7}$$

holds.

Let $\delta = \frac{p}{q}$, $\beta_1 = b\|M\|_{\mathbb{L}_{q'}} \gamma^{\frac{p}{q}}$, $\beta_2 = \|a_1\|_{\mathbb{L}_q} \|M\|_{\mathbb{L}_{q'}}$ and $r_1 = r_{max}$. By virtue of (7), the non-negativity of $\varphi(r_1)$ obviously provides satisfaction of condition 1 of this theorem. Thus, the validity of relation (6) is established.

Let us now choose the number $r_2 > 0$ such that for $x \in \tilde{K} \cap \partial\Omega_2$

$$\|Ax\|_C \geq \|x\|_C. \tag{8}$$

By virtue of (5) and the corresponding properties of the Green's function, we have

$$(Ax)(t) = \int_0^1 G(t,s) f\left(s, (Tx)(s)\right) ds \geq \int_{1/4}^{3/4} G(t,s) f\left(s, (Tx)(s)\right) ds \geq$$

$$\geq \frac{1}{8}\int_{1/4}^{3/4} M(s)a_0(s)(Tx)^{\frac{p}{q}}(s)ds \geq \frac{1}{64}\int_{1/4}^{3/4} M(s)a_0(s)(T\chi)^{\frac{p}{q}}(s)ds \cdot \|x\|_C^{\frac{p}{q}}.$$

Putting $r_2 = \left(\frac{64}{\int_{1/4}^{3/4} M(s)a_0(s)(T\chi)^{\frac{p}{q}}(s)ds}\right)^{\frac{q}{p-q}}$, we get the inequality (8).

By virtue of condition 2 of the theorem, $0 < r_1 < r_2$. Therefore, according to the Go–Krasnoselsky theorem, a completely continuous operator A has at least one fixed point in Ω, which in turn is equivalent to the existence of at least one positive solution $x \in \Omega$ of the boundary value problem (1)–(2). ◄

Theorem 2. *Let the conditions of Theorem 1 be satisfied. In addition, suppose that the function $f(t,u)$ is differentiable with respect to u, $f'_u(t,u)$ is monotonically increasing with respect to the second argument, and*

$$\gamma\|\theta\|_{\mathbb{L}_{p'}} < 1, \tag{9}$$

where $\theta(t) \equiv M(t)\left|f'_u\left(t, r_2\left(T\chi\right)(t)\right)\right|$, $\frac{1}{p'} + \frac{1}{p} = 1$.

Then the boundary value problem (1)–(2) has a unique positive solution $x \in \Omega$.

Proof. Due to the monotonicity of the derivative $f'_u(t,u)$ with respect to the second argument, applying Lagrange's finite increment formula, for any $x_1, x_2 \in \Omega$ we obtain

$$|(Ax_1)(t) - A(x_2)(t)| = \left|\int_0^1 G(t,s)f'_u\left(s, \tilde{y}(s)\right)(Ty)(s)\,ds\right| \leq$$

$$\leq \int_0^1 M(s)\left|f'_u\left(s, r_2\left(T\chi\right)(s)\right)\right|\left|(Ty)(s)\right|ds \leq$$

$$\leq \|\theta\|_{\mathbb{L}_{p'}}\|Ty\|_{\mathbb{L}_p} \leq \gamma\|\theta\|_{\mathbb{L}_{p'}}\|y\|_C,$$

where $\tilde{y}(t)$ takes values between $\left(Tx_1\right)(t)$ and $\left(Tx_2\right)(t)$, and $y(t)$ denotes the absolute value of the difference $x_1(t) - x_2(t)$.

Taking into account the condition (9) of the theorem, based on the contraction mapping principle, we conclude that the boundary value problem (1)–(2) has a unique positive solution $x \in \Omega$. ◄

Example 1. *Consider the problem*

$$D_{0+}^{3/2}x(t) + \left(\int_0^1 x(s)ds\right)^2 = 0, \quad 0 < t < 1, \tag{10}$$

$$x(0) + x'(0) = 0, \quad x(1) + x'(1) = 0, \tag{11}$$

where $p/q = 2$, $\gamma = 1$. *We set* $b = 1$, *and as* $a_1(t)$ *we take any non-negative function summable with degree* $q > 1$ *and satisfying the condition* $\|a_1\|_{\mathbb{L}_q} \leq \frac{1}{2\|M\|^2_{\mathbb{L}_{q'}}}$, *which follows from the condition 1 of Theorem 1. Now, if we take, for example,* $\chi(t) = 1$, *then the requirement 2 of Theorem 1, takes the form*

$$0 < \int_{1/4}^{3/4} M(s)a_0(s)ds \leq 128\|M\|_{\mathbb{L}_{q'}}$$

and, obviously, always holds for any non-negative function $a_0(t) \leq 1$, $t \in [0, 1]$.

Thus, based on Theorem 1, we can conclude that the boundary value problem (8)–(9) *has at least one positive solution.*

Let us now find out under what conditions the positive solution of the problem (8)–(9) *is unique. The condition* (9) *of Theorem 2 in this case has the form*

$$\|\theta\|_{\mathbb{L}_{p'}} = \frac{\|M\|_{\mathbb{L}_{p'}}}{\|M\|_{\mathbb{L}_{q'}}} < 1.$$

But since $p' < q'$, *obviously* $\|M\|_{\mathbb{L}_{p'}} < \|M\|_{\mathbb{L}_{q'}}$.

Therefore, the problem (10)–(11) *has a unique positive solution.*

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