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Universal Covariant Representations and Positive Elements

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Abstract. Let (G, P) be a quasi-lattice ordered group. In this paper, we establish the amenability of (G, P) by demonstrating the existence of a positive, faithful, linear mapping from $C^*(G, P)$ into a specific subalgebra within $C^*(G, P)$. Afterward, we employ our findings to derive generalizations of Murphy and Cuntz's theorems.

Key Words and Phrases: quasi-lattice ordered groups, C^*-algebra, true representation, amenable groups, faithful.

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1. Introduction

The theory of crossed products of C^*-algebras by endomorphisms has been experiencing rapid development. This theory constitutes a broader perspective than the theory of crossed products of C^*-algebras by semigroups of automorphisms, which stands as an interesting area within modern operator algebra theory. The importance of this theory has motivated many authors to investigate broader cases of crossed products of C^*-algebras by semigroups of endomorphisms ([3, 5, 6, 7]). The roots of the theory of crossed products by semigroups of endomorphisms can be traced back to the pioneering work of Cuntz in [4].

In this paper, our focus lies in the realm of crossed products by semigroups of endomorphisms, encompassing actions by general semigroups, specifically the positive cone P of a partially ordered discrete abelian group G.

For a quasi-lattice ordered group (G, P), as established in [1, Theorem 3.2], the universal covariant representation (A, U) is a true representation. Furthermore, if we consider a covariant representation (A, V) of the lattice-ordered group (G, P), there exists a $*$-homomorphism denoted as $\phi : C^*(G, P) \to C^*(V)$, with the property that $\phi(U_p) = V_p$.

We aim to provide simplified characterizations of amenability by demonstrating the existence of a positive, faithful, linear map of $C^*(G,P)$ onto a certain subalgebra of $C^*(G,P)$.

In Section 2, we give the background material about quasi-lattice ordered groups and covariant representations. In Section 3, we establish the faithfulness of the universal representation Φ_U on the positive elements of certain subalgebra of $C^*(G,P)$. In Section 4, we apply [1, Theorem 3.6] to show that abelian quasi-lattice ordered groups are amenable. Then we show that the results of Murphy and Cuntz are special cases of our results.

2. Preliminaries and background material

In this section we give some background material that can be found in [1].

Definition 1. *The partially ordered group (G,P) is* quasi-lattice ordered *if every finite subset of G with an upper bound in P has a least upper bound in P [3, Section 2].*

Definition 2. *A covariant isometric representation of the quasi-lattice ordered group (G,P) may be defined as a pair (A,V) consisting of a unital C^*-algebra A and a map V from P to A such that*

(i) $V_e = 1_A$;

(ii) $V_pV_q = V_{pq}$ *for all* $p,q \in P$;

(iii) $V_p^*V_q = \begin{cases} V_{p^{-1}(p\vee q)}V^*_{q^{-1}(p\vee q)}, & \text{when } p,q \text{ have a common upper bound in } P; \\ 0, & \text{otherwise.} \end{cases}$

Remark 1. *The C^*-algebra generated by the set $\{V_p : p \in P\}$ will be denoted by $C^*(V)$.*

Definition 3. *A covariant representation (A,V) of the quasi-lattice ordered group (G,P) is called a true representation if $\prod_{p\in F}(1 - V_pV_p^*) \neq 0$ for all finite subsets F of $P\backslash\{e\}$.*

Definition 4. *A universal covariant representation (A,U) of the quasi-lattice ordered group (G,P) is a covariant representation such that if (B,V) is any other covariant representation of (G,P), then there is a unique $*$-homomorphism $\phi : C^*(U) \to C^*(V)$ such that $\phi(U_p) = V_p$ for all $p \in P$.*

3. Faithfulness on the positive elements

In this section, we establish the faithfulness of the universal representation Φ_U on the positive elements of certain subalgebra of $C^*(G,P)$. If a positive, faithful, linear map of $C^*(G,P)$ can be found, then by [2, Proposition 4.1], (G,P) is amenable.

Lemma 1. *Let A be a C^*-algebra and $\mathcal{K}$ be a family of C^*-subalgebras such that $A = \overline{\bigcup_{K\in\mathcal{K}} K}$. If M is a closed ideal in A, then $M = \overline{\bigcup_{K\in\mathcal{K}} K \cap M}$.*

Proof. To simplify the notation, we write $M_K = M \cap K$ for each $K \in \mathcal{K}$, and $\bigcup M_K$ rather than $\bigcup_{K\in\mathcal{K}} K \cap M$.

First note that $M_K \subset M$ for each $K \in \mathcal{K}$, so $\overline{\bigcup M_K} \subset M$, since M is closed. To show the reverse inclusion contrapositively, consider $a \in A$ such that $a \notin \overline{\bigcup M_K}$. Let

$$\epsilon = \inf_{b\in\bigcup M_K} \|a-b\| > 0.$$

Then, since $A = \overline{\bigcup_{K\in\mathcal{K}} K}$, there is $K \in \mathcal{K}$ and $k \in K$ such that $\|a-k\| \le \epsilon/3$. We claim

$$\inf_{b\in M} \|k-b\| = \inf_{b\in M_K} \|k-b\|.$$

Use this claim and the triangle inequality in the form

$$\|a-b\| \ge \|k-b\| - \|a-k\|$$

to obtain

$$\begin{aligned}\inf_{b\in M} \|a-b\| &\ge \inf_{b\in M} (\|k-b\| - \|a-k\|)\\ &= \inf_{b\in M} (\|k-b\|) - \|a-k\|\\ &= \inf_{b\in M_K} (\|k-b\|) - \|a-k\|.\end{aligned}$$

Now $\inf_{b\in M_K} (\|k-b\|) \ge \inf_{b\in\bigcup M_K} (\|k-b\|)$, since $M_K \subset \bigcup M_K$. Also $\|k-b\| \ge \|a-b\| - \|a-k\|$ by the triangle inequality, so

$$\begin{aligned}\inf_{b\in M} \|a-b\| &\ge \inf_{b\in\bigcup_{K\in\mathcal{K}} M\bigcap K} (\|k-b\|) - \|x-k\|\\ &\ge \inf_{b\in\bigcup_{K\in\mathcal{K}} M\bigcap K} (\|a-b\|) - 2\|x-k\|\end{aligned}$$

$$\geq \epsilon - 2\epsilon/3 > 0.$$

Thus $a \notin M$ and $M \subset \overline{\bigcup M_K}$ giving $M = \overline{\bigcup M_K}$ as required.

To prove the claim, consider the quotient map $\rho : A \to A/M$. This is a $*$-homomorphism and it is well-known that $\rho(K)$ is a C^*-subalgebra of A/M. Also, M_K is an ideal and so K/M_K is a C^*-algebra. Now, the C^*-algebra homomorphism $k + M \mapsto k + M_K$ is injective, since if $k + M_K = 0$ for some $k \in K$, then $k \in M_K \subset M$ and hence $k + M = 0$. Thus $\|k + M\| = \|k + M_K\|$ for all $k \in K$. Therefore by [4, Proposition 5],

$$\inf_{b \in M} \|k - b\| = \|k + M\| = \|k + M_K\| = \inf_{b \in M_K} \|k - b\|$$

as claimed. ◀

The following theorem is an improvement of Laca and Raeburn's argument in [5, Lemma 4.1] as we use our result in [2, Lemma 3.2] to show that the universal representation Φ_U is faithful on the positive elements of certain subalgebras of $C^*(G, P)$.

Theorem 1. *Let (G, P) and $(\mathcal{G}, \mathcal{P})$ be quasi-lattice ordered groups and let (Φ_U, U) denote the universal covariant isometric representation of the quasi-lattice ordered group (G, P). Suppose there is an order preserving homomorphism $\psi : G \to \mathcal{G}$ such that for $p, q \in P$ with a common upper bound in P,*

(i) *$\psi(p) = \psi(q)$ implies $p = q$, and*

(ii) *$\psi(p \vee q) = \psi(p) \vee \psi(q)$.*

Then

$$K := \overline{\text{span}}\{U_p U_q^*, \text{ for } p, q \in P \text{ and } \psi(p) = \psi(q)\}$$

is a C^*-algebra and Φ_U is faithful on the positive elements of K.

Proof. We claim that K is a C^*-algebra with a family of C^*-subalgebras $\{K_F\}_{F \in \mathcal{F}}$ such that $K = \overline{\bigcup_{F \in \mathcal{F}} K_F}$. Let $M = \{a \in C^*(G, P) : \Phi_U(a^* a) = 0\}$. By [2, Lemma 3.2], M is a closed ideal in $C^*(G, P)$, so $K \bigcap M$ is a closed ideal in K. We claim that $K_F \bigcap M = \{0\}$ for each $F \in \mathcal{F}$ and hence by Lemma 1,

$$K \cap M = \overline{\bigcup_{F \in \mathcal{F}} K_F \cap M} = \{0\}$$

and the claim follows. To prove the first claim, let $\mathcal{F}$ be the collection of non-empty finite subsets of $\mathcal{P}$ such that if $s, t \in F$ have a common upper bound in $\mathcal{P}$, then $s \vee t \in F$. For each $F \in \mathcal{F}$, define

$$K_F = \overline{\text{span}}\{U_p U_q^* : \psi(p) = \psi(q) \in F\} \subset K$$

and write K_s instead of $K_{\{s\}}$ for each $s \in \mathcal{P}$. We claim K_F is a C^*-algebra. To see this, consider $p, q, r, s \in P$ such that $\psi(p) = \psi(q) \in F$ and $\psi(r) = \psi(s) \in F$ and notice that

$$U_p U_q^* U_r U_s^* = \begin{cases} U_{pq^{-1}(q\vee r)} U^*_{sr^{-1}(q\vee r)}, & \text{if } q, r \text{ have a common upper bound in } P \\ 0, & \text{otherwise.} \end{cases}$$

Now $\psi(pq^{-1}) = 1$, so

$$\psi(pq^{-1}(q \vee r)) = \psi(q \vee r) = \psi(q) \vee \psi(r) \in F$$

by assumption. Similarly $\psi(sr^{-1}(q\vee r)) = \psi(q)\vee\psi(r) \in F$ and hence $U_p U_q^* U_r U_s^* \in K_F$. Thus by linearity and continuity of the algebraic product, K_F is a C^*-algebra. Note that the finiteness of F has not been used yet, so the same reasoning gives that $K = \overline{\text{span}}\{U_p U_q^* : \psi(p) = \psi(q) \in \mathcal{P}\}$ is a C^*-algebra.

Note that $\overline{\bigcup_{F\in\mathcal{F}} K_F} \subset K$, since $K_F \subset K$ for each $F \in \mathcal{F}$ and K is closed. To show the reverse inclusion, let

$$a \in \text{span}\{U_p U_q^* : \psi(p) = \psi(q) \in \mathcal{P}\}.$$

Since a could be written as a finite sum of elements $U_p U_q^*$, there is a finite set $H \subset \mathcal{P}$ such that

$$a \in \text{span}\{U_p U_q^* : \psi(p) = \psi(q) \in H\}.$$

Define

$$H^\vee = \{s \vee t \vee ... \vee u : s, t, ..., u \in H \text{ have a common upper bound in } \mathcal{P}\}.$$

This set contains at most $2^{|H|}$ elements, so $H^\vee \in \mathcal{F}$. Moreover, $a \in K_{H^\vee} \subset \bigcup_{F\in\mathcal{F}} K_F$, since $H \subset H^\vee$. Hence

$$\text{span}\{U_p U_q^* : \psi(p) = \psi(q) \in \mathcal{P}\} \subset \bigcup_{F\in\mathcal{F}} K_F$$

and thus $K \subset \overline{\bigcup_{F\in\mathcal{F}} K_F}$, and $K = \overline{\bigcup_{F\in\mathcal{F}} K_F}$ as required.

To show that $K_F \cap M = \{0\}$ for each $F \in \mathcal{F}$, consider $F \in \mathcal{F}$ and $a \in K_F \cap M$. Write $a = \lim_{n\to\infty} \sum_{s\in F} a_{n,s}$, where $a_{n,s} \in K_s$ for each integer n and $s \in F$. Now, F is finite, hence it has an element s_0 which is minimal in the quotient order. Let P_{s_0} denote the projection of $\ell^2(P)$ onto the subspace

$$H_{s_0} = \overline{\text{span}}\{\delta_p : \psi(p) = s_0\}.$$

We claim that for each integer n,

$$\|a_{n,s_0}\| = \|\phi(\sum_{s\in F} a_{n,s})P_{s_0}\|,$$

where $\phi : C^*(G,P) \to B(\ell^2(P))$ is the unique $*$-homomorphism such that $\phi(U_p) = T_p$ for all $p \in P$. By the continuity of ϕ and since $M = \ker(\phi)$ ([2, Lemma 3.2]), we have

$$\phi(\sum_{s\in F} a_{n,s}) \to \phi(a) = 0.$$

So now $a_{n,s_0} \to 0$ and $a = \lim_{n\to\infty} \sum_{s\in F\setminus\{s_0\}} a_{n,s}$. This process may be repeated at most $|F|$ times to deduce that $a = 0$.

Finally, to show

$$\|a_{n,s_0}\| = \|\phi(\sum_{s\in F} a_{n,s})\, P_{s_0}\|,$$

consider $p, q, r \in P$ such that $\psi(p) = \psi(q) \in F$ and $\delta_r \in H_{s_0}$. Use the covariance condition to obtain

$$\begin{aligned}\phi(U_pU_q^*)\delta_r &= T_pT_q^*T_r\delta_e\\ &= \begin{cases} T_{pq^{-1}(q\vee r)}\, T^*_{r^{-1}(q\vee r)}\delta_e, & \text{if } q, r \text{ have c.u.b in } P\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Notice that $T_z\delta_e = 0$ for all $z \in P\setminus\{e\}$, so $\phi(U_pU_q^*)\delta_r = 0$ unless $r^{-1}(q\vee r) = e$. But then $q \le q\vee r = r$ and $\psi(q) \le \psi(r)$, since ψ is order preserving. Thus $\psi(q) = \psi(r) = s_0$ by the minimality of s_0 and in fact $q = r$ by hypothesis one. Then

$$\phi(U_pU_q^*)\delta_r = T_pT_q^*T_r\delta_e = \begin{cases} \delta_p, & \text{if } q = r,\\ 0, & \text{otherwise.}\end{cases}$$

This gives $\psi(U_pU_q^*)P_{s_0} = 0$ unless $\psi(p) = \psi(q) = s_0$, so

$$\phi\big(\sum_{s\in F} a_{n,s}\big)\, P_{s_0} = a_{n,s_0}\, P_{s_0}.$$

Moreover, if $\psi(p) = \psi(q) = s_0$, then $\phi(U_pU_q^*)P_{s_0}$ is the rank one operator $(.\,|\delta_q)\delta_p$ on the Hilbert space H_{s_0}. Hence the map $b \to \phi(b)P_{s_0}$ is an injection of K_{s_0} onto the closure of the finite rank operators on H_{s_0}. This map is a $*$-homomorphism, since given $a, b \in K_{s_0}$,

$$\phi(a)P_{s_0}\,(\phi(b)P_{s_0})^* = \phi(a)P_{s_0}\,\phi(b^*)P_{s_0} = \phi(ab^*)P_{s_0}.$$

Thus,

$$\|a_{n,s_0}\| = \|\phi(a_{n,s_0}) = \|\phi(\sum_{s\in F} a_{n,s})\|$$

as required. ◄

Remark 2. *If $\mathcal{G}$ in Theorem 1 is abelian, then by [2, Proposition 4.5] there is a positive, faithful, linear map of $C^*(G,P)$ onto K which commutes with Φ_U. Then by [2, Proposition 4.2], (G,P) is amenable. This fact will be useful in the following section.*

4. Some amenable quasi-lattice ordered groups

In this section, we leverage our findings to establish the amenability of abelian quasi-lattice ordered groups and derive extensions of the theorems of Murphy and Cuntz. Furthermore, we demonstrate that the results of Murphy and Cuntz can be seen as specific instances within our broader results in [1, Theorem 3.2, Theorem 3.6].

Theorem 2. *Abelian quasi-lattice ordered groups are amenable.*

Proof. Let (G,P) be an abelian quasi-lattice ordered group. The identity map $\iota : G \to G$ is a group homomorphism, and hence by [2, Proposition 4.3] there is a continuous, faithful linear map $\Phi : C^*(G,P) \to C^*(G,P)$ such that

$$\begin{aligned}\Phi(U_pU_q^*) &= \begin{cases} U_pU_q^* & \text{if } p = \iota(p) = \iota(q) = q, \\ 0 & \text{otherwise.}\end{cases} \\ &= \Phi_U(U_pU_q^*).\end{aligned}$$

By linearity and continuity of Φ and Φ_U, we have $\Phi = \Phi_U$. Hence (G,P) is amenable. ◄

Recall the following Theorem of Murphy:

Theorem 3. *Let G be a totally ordered abelian group with sub-semigroup $P = \{p \in G : p \geq 0\}$. Then there is a representation (A,U) of P by isometries that has the following properties:*

(i) *Let (B,V) be a representation of P by isometries. Then there is a $*$-homomorphism $\phi_V : C^*(U) \to C^*(V)$ such that $\phi_V(U_p) = V_p$ for each $p \in P$.*

(ii) *If V_p is non-unitary for all $p \in P$, then ϕ_V is an isomorphism.*

Proof. Recall that if a semigroup P induces a total order on a group G, then any representation of P by isometries is covariant. Thus by [1, Theorem 3.2] there is a representation (A, U) which has the required universal property. Suppose that V_p is non-unitary for all $p \in P$. Then by section § 4.1 of [1], (B, V) is true. By Theorem 2 (G, P) is amenable, so [1, Theorem 3.6] implies that (B, V) is isomorphic to the universal object. ◄

Recall the following Theorem of Cuntz:

Theorem 4. *Given $n \in \mathbb{N}$, there is a pairwise orthogonal family of n isometries X which generates a C^*-algebra $C^*(X)$ with the following properties:*

(i) *Let $C^*(Y)$ be the C^*-algebra generated by some pairwise orthogonal family of n isometries Y. There is a $*$-homomorphism $\phi_Y : C^*(X) \to C^*(Y)$ such that $\phi_Y(X) = Y$.*

(ii) *If $\sum_{V \in Y} VV^* < 1$, then ϕ_Y is an isomorphism.*

Proof. Let (G, P) be the free product of a family $\{(G_i, P_i) : i \in I\}$, where $(G_i, P_i) = (\mathbb{Z}, \mathbb{N})$ for all $i \in I$. By [1, Theorem 3.2] there is a universal covariant isometric representation (A, U) of (G, P). Recall that the covariance of (A, U) implies that $C^*(G, P)$ is generated by the pairwise orthogonal family of isometries $X = \{U_{a_i} : i \in I\}$.

Moreover, we know that any pairwise orthogonal family of isometries $Y = \{V_i : i \in I\}$ induces a covariant isometric representation (B, V) of (G, P) such that $V_{a_i} = V_i$. Then, by [1, Theorem 3.2] there is a $*$-homomorphism $\phi_Y : C^*(G, P) \to C^*(V)$ such that $\phi_Y(U_{a_i}) = V_{a_i}$ and hence $\phi_Y(X) = Y$.

Suppose $|I| = n < \infty$. Then, if $\sum_{V \in Y} VV^* < 1$, (A, V) is true by [1, §2]. Hence by [5, Theorem 4.4] and [1, Theorem 3.6], ϕ_Y is an isomorphism. ◄

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