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Duality and Preservation of Continuous K-J-Frames in Krein Spaces

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Abstract. The purpose of this paper is to introduce and study the notion of continuous K-frames in the Krein space setting under the name continuous K-J-frames. We will also present the definition of K-J-dual continuous Bessel family and some results concerning this notion. The preservation of the continuous K-J-frames by usual operations (the composition by a bounded operator, the sum) will also be studied. Finally, we will see how to achieve a transfer of continuous K-frames from a Hilbert space to a Krein space arising from a possibly non-regular Gram operator.

Key Words and Phrases: continuous K-J-frames, K-dual Bessel families, Krein spaces, non-regular Gram operator.

2010 Mathematics Subject Classifications: 42C15, 42C40, 46C20

1. Introduction

Duffin and Schaeffer [11] introduced the concept of discrete frames in Hilbert spaces for the first time in 1952, and they have since evolved into an important tool in several applications in both pure and applied mathematics, such as signal processing, sampling, and wavelet analysis [7].

Within the framework of the generalizations of the notion of frame, we find the K-frames proposed by L. Găvruta in [13] in order to study atomic systems with respect to a bounded linear operator K in Hilbert spaces. We also find continuous frames, which were originally introduced by Ali, Gazeau, and Antoine [2], and independently by Kaiser [10]. They offer interesting mathematical problems that have been studied in many papers with various aspects of the concept [4, 8, 12].

The first papers carrying the frame generalization to Krein spaces were presented by Giribet, Maestripieri, Martinez Peria, and Massey in [14], and Esmeral,

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Ferrer, and Wagner in [17] but with separate approaches, and a more general definition was given by Kamuda and Kuzhel in [1].

Continuous frames in Krein spaces setting were defined and studied in [19] by Diego Carrillo, Kevin Esmeral, Elmar Wagner and discrete K-frames for Krein spaces were exposed by Atmani Mohammed, Kabbaj Samir, and Nourdine Bounader in [18] and by Osmin Ferrer Villar, Jesús Domínguez Acosta, Edilberto Arroyo Ortiz in [20].

The main purpose of our paper is to introduce the notion of continuous K-J-frames for Krein spaces and study it along several axes. Section 3 will be intended to give the definition of a continuous K-J-frames in a Krein space $\mathscr{H}$ and characterize them by their pre-frame and frame operators. In Section 4 we will expose the notion of the K-J-dual continuous Bessel family as well as its role in the reconstruction of the elements from the range of the operator K. We will also see that every continuous K-J-frame admits a K-J-dual continuous Bessel family. Furthermore, when $R(K)$, the range of K, is closed, we can find for every continuous K-J-frame a K-J-dual continuous Bessel family, which can be expressed more explicitly. This kind of Bessel family is called the canonical K-J-dual continuous Bessel family. In section 5, we discuss the preservation of continuous K-J-frames, first by composition by a bounded operator, and after by the sum. In the final section, we will examine the transfer problem of continuous K-frame from a Hilbert space $\mathcal{H}$ to an associated Krein space $\mathcal{H}_W$ which is derived from a potentially non-regular Gram operator W. This problem was introduced and solved for discrete frames [17], for discrete K-frames in [18], and for continuous frames in [19].

2. Preliminaries

2.1. Krein spaces.

In this section, we provide a brief overview of Krein spaces. However, for a comprehensive understanding of Krein spaces, please refer to [3] and [5].

Definition 1. *1. A Krein space $(\mathscr{H}, [\cdot,\cdot])$ with a fundamental decomposition $(\mathscr{H}_+, \mathscr{H}_-)$ is a direct sum of two Hilbert spaces $(\mathscr{H}_+, [\cdot,\cdot])$ and $(\mathscr{H}_-, -[\cdot,\cdot])$.*

2. The application J is defined as follows:

$$J(x_+ + x_-) = x_+ - x_-, \quad x_\pm \in \mathscr{H}_\pm \tag{1}$$

is called the fundamental symmetry associated to the decomposition $(\mathscr{H}_+, \mathscr{H}_-)$.

Remark 1. • *J is involutive.*

- $\mathscr{H}$ *can be equipped with an inner product associated to the operator J called J-inner product defined as follows:*

$$[x_+ + x_-, y_+ + y_-]_J := [x_+ + x_-, J(y_+ + y_-)] =$$
$$= [x_+, y_+] - [x_-, y_-], \forall x_\pm, y_\pm \in \mathscr{H}_\pm. \quad (2)$$

Definition 2. *$\mathscr{H}$ equipped with the inner product $[\cdot,\cdot]_J$ is a Hilbert space, it is the associated Hilbert space to the Krein space $(\mathscr{H}, [\cdot,\cdot])$ and $\mathscr{H}_+ \oplus_J \mathscr{H}_-$ becomes an orthogonal sum of Hilbert spaces.*

Remark 2.

- *The topology defined on $\mathscr{H}$ will be the one induced by the norm $\|\cdot\|_J := \sqrt{[\cdot,\cdot]_J}$.*
- *This topology is independent of the choice of the fixed fundamental symmetry because all the topologies induced by these norms are equivalent [3].*

Proposition 1. *Let $K : (\mathscr{H}_1, [\cdot,\cdot]_1) \to (\mathscr{H}_2, [\cdot,\cdot]_2)$ be a bounded linear operator. Then there exist:*

1. *A unique operator*

$$K^{*_{J_1 J_2}} : (\mathscr{H}_2, [\cdot,\cdot]_{J_2}) \to (\mathscr{H}_1, [\cdot,\cdot]_{J_1})$$

that verifies

$$[Kf, g]_{J_1} = [f, K^{*_{J_1 J_2}} g]_{J_2} \quad \textit{for all } f \in \mathscr{H}_1, \quad f \in \mathscr{H}_2.$$

It is called the J-adjoint of K.

2. *A unique operator*

$$K^{[*]} : (\mathscr{H}_2, [\cdot,\cdot]_2) \to (\mathscr{H}_1, [\cdot,\cdot]_1)$$

such that

$$[Kf, g]_1 = [f, K^{[*]} g]_2 \quad \textit{for all } f \in \mathscr{H}_1, \quad g \in \mathscr{H}_2.$$

It is called the adjoint of K.

Remark 3.

- *If $K \in \mathcal{B}(\mathscr{H})$, we will denote the J-adjoint of K by K^{*_J}.*
- *$K^{[*]}$ is bounded and we have:*

$$K^{[*]} = J_1 K^{*_{J_1 J_2}} J_2.$$

- *K is J-self-adjoint if $K = K^{*_{J_1 J_2}}$, and self-adjoint if $K = K^{[*]}$.*

Definition 3. *Let $(\mathscr{H}, [\cdot,\cdot])$ be a Krein space and $\mathscr{M}$ be a subspace of $\mathscr{H}$. The orthogonal of $\mathscr{M}$ with respect to $[\cdot,\cdot]_J$ is given by*

$$\mathscr{M}^{\perp} := \{x \in \mathscr{H}, \quad [x,y]_J = 0 \quad \forall y \in \mathscr{M}\},$$

and with respect to by $[\cdot,\cdot]$

$$\mathscr{M}^{[\perp]} := \{x \in \mathscr{H}, \quad [x,y] = 0 \quad \forall y \in \mathscr{M}\}.$$

Proposition 2. *Let $(\mathscr{H}, [\cdot,\cdot])$ be a Krein space and $\mathscr{M}$ be a subspace of $\mathscr{H}$. Then*

1. $$\mathscr{M}^{[\perp]} = J\mathscr{M}^{\perp},$$

2. $$\mathscr{M}^{\perp} = J\mathscr{M}^{[\perp]}, \tag{3}$$

3. $$(J\mathscr{M})^{[\perp]} = J\mathscr{M}^{[\perp]},$$

4. $$\mathscr{M}^{[\perp][\perp]} = \overline{\mathscr{M}}.$$

Example 1. *Let B be a compact subset of $\mathbb{R}^n$, and let B_1 and B_2 be a measurable partition of B. On B, we define the following signed measure:*

For $A \in \mathfrak{B}$, a fixed Borel subset of B, we define

$$\mu(A) = \int_A 1_{B_1} - 1_{B_2} \mathrm{d}\lambda,$$

where λ is the Lebesgue measure on $\mathbb{R}^n$, 1_{B_1} and 1_{B_2} are indicator functions of the sets B_1 and B_2, respectively. The variation of μ is given by

$$|\mu|(A) = \int_A 1_{B_1} + 1_{B_2} \mathrm{d}\lambda.$$

Let $\mathscr{H}$ be the set of square integrable functions with respect to the measure $|\mu|$:

$$\mathscr{H} = \left\{ f : B \to \mathbb{C} \text{ measurable}, \int_B |f|^2 \mathrm{d}|\mu| < \infty \right\},$$

equipped with the indefinite inner product

$$[f,g] := \int_B f\overline{g}d\mu = \int_{B_1} f\overline{g}d\lambda - \int_{B_2} f\overline{g}d\lambda \quad \forall f,g \in \mathscr{H}.$$

$(\mathscr{H}, [\cdot,\cdot])$ *is a Krein space, and its fundamental decomposition is given by*

$$\mathscr{K}_+ = \left\{f \in \mathscr{H},\, f|_{B_2} = 0\right\}, \text{ and } \mathscr{K}_- = \left\{f \in \mathscr{H},\, f|_{B_1} = 0\right\}.$$

2.2. Continuous K-frames for Hilbert spaces

For this section, we fix the measured space $(X, \mathfrak{B}, \mu)$ such that μ is a σ-finite positive measure.

Definition 4. *[15] Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \in \mathcal{B}(\mathcal{H})$ and $f : X \to \mathcal{H}$ be a function from X to $\mathcal{H}$. f is called a continuous K-frame (cK-frame) with regard to $\mathcal{H}$, if*

1. *f is weakly measurable (i.e. $x \in X \mapsto \langle f(x), y\rangle$ for every fixed $y \in \mathcal{H}$) and*
2. *there exist two constants $0 < A \leq B < \infty$ such that for each $y \in \mathcal{H}$,*

$$A\,\|K^*y\|^2 \leq \int_X |\langle y, f(x)\rangle|^2 d\mu \leq B\|y\|^2.$$

Remark 4. *If only the upper inequality is fulfilled, then we say that f is a c-Bessel mapping for $\mathcal{H}$.*

Definition 5. *[15] Let $\mathcal{H}$ be a Hilbert space, $f : X \to \mathcal{H}$ be a weakly measurable function, and $K \in \mathcal{B}(H)$. f is called a continuous K-atoms (cK-atoms) for $\mathcal{H}$ if we have:*

1. *For each $\eta \in L^2(X,\mu)$, $\int_X \eta f d\mu$ exists in $\mathcal{H}$ (i.e. $\int_X |\langle \eta(x)f(x), y\rangle| \mathrm{d}\mu < +\infty$, for all $y \in \mathcal{H}$);*
2. *There exist $g : X \to \mathcal{H}$ weakly measurable and $a > 0$ such that for each $h \in \mathcal{H}$,*

$$\int_X |\langle h, g(x)\rangle|^2 \leq a\|h\|,$$

and also

$$Kh = \int_X \langle h, g(x)\rangle f(x) d\mu(x),$$

i.e.

$$\langle Kh, y\rangle = \int_X \langle h, g(x)\rangle \langle f(x), y\rangle d\mu(x), \quad \forall y \in \mathcal{H}.$$

Theorem 1. *[15] Let $K \in \mathcal{B}(\mathcal{H})$. If $f : X \to \mathcal{H}$ is a weakly measurable function, then the following assertions are equivalent:*

1. *f is a local cK-atoms for $\mathcal{H}$.*
2. *f is a cK-frame for $\mathcal{H}$.*
3. *f is a c-Bessel mapping for $\mathcal{H}$, and there exists $g \in \mathcal{B}\big(\mathcal{H}, L^2(X,\mu)\big)$, such that*

$$Kh = \int_X g(h) f d\mu, \quad h \in \mathcal{H}.$$

Proposition 3. *[15] Let $K \in B(\mathcal{H})$ and $f : X \to \mathcal{H}$ be a weakly measurable function. Then f is a cK-frame for $\mathcal{H}$ if and only if the mapping*

$$T_f : L^2(X,\mu) \to \mathcal{H}, \quad T_f(g) = \int_X g f d\mu,$$

is a well-defined bounded linear operator with $\mathcal{R}(K) \subset \mathcal{R}(T_f)$.

2.3. The pseudo-inverse operator

Definition 6. *[6] Suppose we have two Hilbert spaces, $\mathcal{H}$ and $\mathcal{K}$, and a bounded operator $U : \mathcal{K} \to \mathcal{H}$ with a closed range $\mathcal{R}_U$. In that case, there exists a bounded operator $U^\dagger : \mathcal{H} \to \mathcal{K}$ such that*

$$UU^\dagger x = x, \forall x \in \mathcal{R}_U.$$

Proposition 4. *[6] Assume that $U : \mathcal{K} \to \mathcal{H}$ is bounded with a closed range. Under these conditions, the following statements are true:*

1. *The orthogonal projection of $\mathcal{H}$ onto the range of U is equal to $UU^\dagger$.*
2. *The orthogonal projection of $\mathcal{K}$ onto the range of $U^\dagger$ is equal to $U^\dagger U$.*

3. Continuous K-J-frames for Krein spaces

For the rest of the paper, $(X, \mathfrak{B}, \nu)$ will be a measured space such that ν is a signed measure.

Definition 7. *Suppose we have $\mathscr{H}$ a Krein space with a fundamental symmetry J, and K a bounded operator of $\mathscr{H}$. A function $F : X \to \mathscr{H}$ is called a continuous K-J-frame for $\mathscr{H}$ if it satisfies two conditions:*

1. *it is weakly measurable.*

2. there exist constants $0 < A \leq B < \infty$ *such that*

$$A\|K^{*_J}k\|_J^2 \leq \int_X |[F(x), k]|^2 \, \mathrm{d}|\nu|(x) \leq B\|k\|_J^2 \quad \text{for all } k \in \mathscr{H}. \tag{4}$$

Remark 5. • *A and B are referred to as frame bounds. The optimal lower frame bound is the largest constant A that satisfies equation (4), while the optimal upper frame bound is the smallest constant B that satisfies the same equation. A continuous K-J-frame is tight if $A\|K^{*_J}k\|_J^2 = \int_X |[F(x), k]|^2 \, \mathrm{d}|\nu|(x)$; we say that F is a continuous K-J-Parseval frame if F is tight and $A = 1$.*

- *If F verifies just the upper inequality, we say that F is a continuous Bessel family for $\mathscr{H}$.*

This theorem establishes, in a similar manner of theorem 3.4 [18], the relationship between the continuous K-J-frames in a given Krein space $\mathscr{H}$ and the continuous K-frames in its corresponding Hilbert space.

Theorem 2. *Let $(\mathscr{H}, [\cdot,\cdot])$ be a Krein space, J be a fundamental symmetry and $F : X \longrightarrow \mathscr{H}$ be a weakly measurable function. Then the following statements are equivalent:*

1. *F is a continuous K-J-frame in $(\mathscr{H}, [\cdot,\cdot])$ with frame bounds $A \leq B$.*
2. *JF is a continuous K-frame in the Hilbert space $(\mathscr{H}, [\cdot,\cdot]_J)$ with frame bounds $A \leq B$.*

Proof. The proof is straightforward and follows the same approach as in the discrete case. $JF : x \in X \longrightarrow JF(x) \in \mathscr{H}$ is obviously weakly measurable. If we assume that F is a continuous K-J-frame for the Krein space $\mathscr{H}$ with frame bounds $A \leq B$, we get

$$A\|K^{*_J}k\|_J^2 \leq \int_X |[F(x), k]|^2 \, \mathrm{d}|\nu|(x) \leq B\|k\|_J^2 \quad \text{for all } k \in \mathscr{H},$$

which is equivalent to

$$A\|K^{*_J}k\|_J^2 \leq \int_X |[JF(x), k]_J|^2 \, \mathrm{d}|\nu|(x) \leq B\|k\|_J^2 \quad \text{for all } k \in \mathscr{H}.$$

So we get the result. ◄

Example 2. *Consider the Krein space*

$$\mathscr{H} = \left\{ f : B \to \mathbb{C} \text{ measurable}, \int_B |f|^2 \mathrm{d}|\mu| < \infty \right\}$$

introduced in Example 1 with the same assumptions. But this time we assume that $B \subset \mathbb{R}$. Define K as a bounded operator on $\mathscr{H}$ that maps each $f \in \mathscr{H}$ to $itf \in \mathscr{H}$:

$$K : f \in \mathscr{H} \to itf \in \mathscr{H}.$$

We also define a function $F : \mathbb{R} \to \mathscr{H}$ such that $F(\omega)(t) = ite^{2\pi i\omega t}(1_{B_1}(t) - 1_{B_2}(t))$ for each $\omega \in \mathbb{R}$ and $t \in B$. And we find for each $f \in \mathscr{H}$ and $\omega \in \mathbb{R}$:

$$[f, F(\omega)] = \int_{B_1} -itf(t)e^{-2\pi i\omega t}\mathrm{d}\lambda(t) + \int_{B_2} -itf(t)e^{-2\pi i\omega t}\mathrm{d}\lambda(t) = \mathcal{F}\left(-itf1_B\right)(\omega),$$

where $\mathcal{F}\left(-itf1_B\right)(\omega)$ is the Fourier transform of $-itf1_B$. Consequently, we get

$$\int_{\mathbb{R}} |\,[f, F(\omega)]\,|^2 d\lambda(\omega) = \int_{\mathbb{R}} \left|\mathcal{F}\left(-itf1_B\right)\right|^2 d\lambda(\omega) =$$

$$= \|\mathcal{F}\left(-itf1_B\right)\|_2^2 = \|-itf1_B\|_2^2 = \|K^*f\|_2^2, \quad (f \in \mathscr{H}),$$

which shows that F is a continuous K-J-Parseval frame with respect to $(\mathbb{R}, \Sigma(\mathbb{R}), \lambda)$ for $\mathscr{H}$.

Example 3. *Let $L^2(\mathbb{R})$ be the Lebesgue space of complex-valued and square-integrable functions on $\mathbb{R}$. Consider on $L^2(\mathbb{R})$ the indefinite inner product*

$$\forall f, g \in L^2(\mathbb{R}) \quad [f, g] := \int_{\mathbb{R}} f(t)\overline{g(-t)}d\lambda(t)$$

and a fundamental symmetry

$$J : f \in L^2(\mathbb{R}) \longmapsto f(-t) \in L^2(\mathbb{R}).$$

$(L^2(\mathbb{R}), [\cdot, \cdot])$ is a Krein space with the fundamental symmetry J, and the inner product $[\cdot, \cdot]_J$ agrees with the classic inner product on $L^2(\mathbb{R})$. Let $g \neq 0$ be a function called the window function and consider the short-time Fourier transform of a function $f \in L^2(\mathbb{R})$ with respect to g defined as follows:

$$V_g f(x, \omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi it\omega}\mathrm{d}\lambda(t) = \int_{\mathbb{R}} f(t)\overline{M_\omega T_x g(t)}\mathrm{d}\lambda(t), \qquad \text{for } x, \omega \in \mathbb{R},$$

where M_ω is the modulation operator and T_x is the translation operator.
Let $K \in \mathcal{B}(L^2(\mathbb{R}))$ be a bounded operator on $L^2(\mathbb{R})$, so

$$\tilde{V}_{g,K} : (x, \omega) \in \mathbb{R}^2 \mapsto JKM_\omega T_x g \in L^2(\mathbb{R})$$

is a tight continuous K-J-frame for $(L^2(\mathbb{R}), [\cdot,\cdot])$ with respect to $(\mathbb{R}^2, \Sigma(\mathbb{R}^2), \lambda_2)$ for $(L^2(\mathbb{R}), [\cdot,\cdot])$. In fact:

$$\int_{\mathbb{R}^2} \left| \left[f, \tilde{V}_{g,K} \right] \right|^2 \mathrm{d}\lambda_2(x,\omega) = \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} K^* f(t) \overline{M_\omega T_x g(t)} \mathrm{d}\lambda(t) \right|^2 \mathrm{d}\lambda_2(x,\omega) =$$

$$= \|V_g K^* f\|^2 = \|K^* f\|^2 \|g\|^2.$$

Remark 6. *The definition given here depends on the fixed fundamental symmetry J, but using this definition, one can construct for every fundamental symmetry $\tilde{J}$ a continuous K-$\tilde{J}$-frame over $\mathscr{H}$.*

Proposition 5. *If F is a continuous K-J-frame on $\mathscr{H}$, and $\tilde{J}$ is another fundamental symmetry of $\mathscr{H}$, then $\tilde{J}JF$ will be a K-$\tilde{J}$-frame of $\mathscr{H}$ equipped this time with $\tilde{J}$ as a fundamental symmetry.*

Proof. Let $f \in \mathscr{H}$, and A and B be the frame bounds of F:

$$\|K^{*^{\tilde{J}}} f\|_{\tilde{J}} = \|\tilde{J} K^{[*]} \tilde{J} f\|_{\tilde{J}} = \|K^{[*]} \tilde{J} f\|_{\tilde{J}},$$

and by the fact that the norms $\|\cdot\|_J$ and $\|\cdot\|_{\tilde{J}}$ are equivalent, there exist $\alpha > 0$ and $\beta > 0$ such that $\|K^{[*]}\tilde{J} f\|_{\tilde{J}} \le \alpha \|K^{[*]}\tilde{J} f\|_J$ and $\|f\|_J \le \beta \|f\|_{\tilde{J}}$. So:

$$A/\alpha \|K^{*^{\tilde{J}}} f\|_{\tilde{J}} \le A\|K^{[*]}\tilde{J} f\|_J = A\|JK^{[*]}JJ\tilde{J} f\|_J \le A\|K^{*^J} J\tilde{J} f\|_J \le$$

$$\le \int_X \left| \left[\tilde{J}JF(x), f \right] \right|^2 \mathrm{d}|\nu|(x) \le B\|f\|_J \le B\beta \|f\|_{\tilde{J}},$$

which gives the desired result. ◄

To define the pre-frame and the frame operator associated to continuous K-J-frames in Krein spaces, we consider the Krein space $\mathscr{L}^2(X,\nu)$ (where ν is a signed measure) introduced in [19], as follows:

We consider X_+, X_-, the Hahn decomposition of X with respect to ν. The Krein space $(\mathscr{L}^2(X,\nu), [\cdot,\cdot])$ is given by

$$\mathscr{L}^2(X,\nu) := \mathscr{L}^2(X_+,\nu^+)[+]\mathscr{L}^2(X_+,\nu^-), \tag{5}$$

where the indefinite inner product is defined as

$$[f,g] := \int_{X_+} f\overline{g} d\nu^+ - \int_{X_-} f\overline{g} d\nu^-, \quad \forall f,g \in \mathscr{L}^2(X,\nu)$$

and the fundamental symmetry is given by

$$J_{\mathscr{L}^2(X,\nu)} : f \in \mathscr{L}^2(X,\nu) \mapsto f1_{X_+} - f1_{X_-} \in \mathscr{L}^2(X,\nu). \tag{6}$$

The associated Hilbert space to $\mathscr{L}^2(X,\nu)$ is $\mathscr{L}^2(X,|\nu|)$, the Hilbert space of complex-valued and square-integrable functions with respect to the positive measure $|\nu| = \nu^+ + \nu^-$, where ν^+ and ν^- are the positive and negative parts, respectively, of the signed measure ν.

Proposition 6. *If $(\mathscr{H}, [\cdot,\cdot])$ is a Krein space, J is a fundamental symmetry, $K \in \mathcal{B}(\mathscr{H})$ is a bounded operator and $F : x \in X \to F(x) \in \mathscr{H}$ is a continuous Bessel family in $\mathscr{H}$, then the linear map*

$$T_F : \mathscr{L}^2(X,\nu) \longrightarrow \mathscr{H}, \quad T_F(g) = \int_X g(x)F(x)d|\nu|$$

is well-defined and bounded. T_F is called the pre-frame operator of the continuous K-J-frame F.

Remark 7. *The converse is also true, meaning that if $F : X \to \mathscr{H}$ is weakly measurable map and T_F is well-defined and bounded, then F will be a Bessel family in $\mathscr{H}$.*

Demonstrating this result is simple.

Remark 8. *If F is a continuous Bessel family for $\mathscr{H}$, and T_F is the pre-frame operator of F, then the adjoint of T_F is given by*

$$T_F^{[*]}k = J_{\mathscr{L}^2(X,\nu)}[k,F], \quad \forall k \in \mathscr{H},$$

where

$$J_{\mathscr{L}^2(X,\nu)}[k,F] : x \in X \mapsto J_{\mathscr{L}^2(X,\nu)}(x)[k,F(x)] \in \mathbb{C}.$$

Theorem 3. *Let $(\mathscr{H}, [\cdot,\cdot])$ be a Krein space and J be a fundamental symmetry of $\mathscr{H}$. A weakly measurable function $F : X \to \mathscr{H}$ is a continuous K-J-frame for the Krein space $\mathscr{H}$ if and only if the pre-frame operator of F*

$$T_F : \mathscr{L}^2(X,\nu) \longrightarrow \mathscr{H}, \quad T_F(g) = \int_X g(x)F(x)d|\nu|$$

is bounded and $R(K) \subseteq R(JT_F)$.

Before starting the proof, an important lemma is stated.

Lemma 1. *[9] Let $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$ and $V \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ be bounded operators. The following statements are equivalent:*

1. *$R(U) \subset R(V)$;*

2. $UU^* \leq \lambda^2 VV^*$ for some $\lambda \geq 0$;

3. there exists a bounded operator $C \in (\mathcal{H}_1, \mathcal{H}_2)$ on $\mathcal{H}$ such that $U = VC$.

Proof. We suppose that F is a continuous K-J-frame for $\mathscr{H}$. So, there exist two constants $0 < A \leq B$ such that the inequality (4) is verified. By Proposition 6, the upper inequality is confirmed if and only if the pre-frame operator of F is bounded. On the other hand, the lower inequality can be represented as follows:

$$A[KK^{*_J}k, k]_J^2 \leq \int_X [[k, JF(x)]JF(x), k]_J d|\nu| = [JT_F T_F^{*_J} Jk, k]_J,$$

so

$$AKK^{*_J} \leq JT_F(JT_F)^{*_J},$$

and this is equivalent to the fact that $R(K) \subseteq R(JT_F)$. ◀

Proposition 7. *If $(\mathscr{H}, [\cdot, \cdot])$ is a Krein space, $K \in \mathcal{B}(\mathscr{H})$ is a bounded operator, and $F : X \to \mathscr{H}$ is a continuous K-J-frame in $\mathscr{H}$, then the operator*

$$S_F := T_F J_{\mathscr{L}^2(X,\nu)} T_F^{[*]}$$

is positive, self-adjoint and bounded.

S_F is called the frame operator, and it is given by the weak integral

$$S_F k = \int_X [k, F(x)]\, F(x) d|\nu|, \quad k \in \mathscr{H}.$$

Remark 9.

- *The proof of this result is straightforward.*
- *Moreover, S_F is not invertible in general.*

Proposition 8. *If $(\mathscr{H}, [\cdot, \cdot])$ is a Krein space, $K \in \mathcal{B}(\mathscr{H})$ is a bounded operator with a closed range, and $F : X \longrightarrow \mathscr{H}$ is a continuous K-J-frame for $\mathscr{H}$, then S_F is an invertible operator from $R(K)$ to $S_F(R(K))$.*

Proof. The proof is the as in the same discret case [18]. Indeed, since K has a closed range, there exists a pseudo-inverse $K^\dagger$ of K such that $KK^\dagger k = k, \forall k \in R(K)$, i.e. $KK^\dagger\big|_{R(K)} = I_{R(K)}$, and in particular $I_{R(K)}^{*_J} = \left(K^\dagger\big|_{R(K)}\right)^{*_J} K^{*_J}$. Hence, for any $k \in R(K)$ we obtain

$$\|k\|_J = \left\| \left(K^\dagger\Big|_{R(K)} \right)^{*_J} K^{*_J} k \right\|_J \leq \left\| K^\dagger \right\| \cdot \|K^{*_J} k\|_J. \tag{7}$$

Therefore,

$$B\|k\|_J^2 \geq [JS_F k, k]_J \geq A \left\|K^\dagger\right\|^{-2} \|k\|_J^2, \forall k \in R(K).$$

So, JS_F is invertible from $R(K)$ to $JS_F(R(K))$. Finally, the fact that J is invertible leads us to conclude the invertibility of the operator S_F from $R(K)$ to $S_F(R(K))$. Furthermore, S_F^{-1} is bounded and we have

$$B^{-1} \leq S_F^{-1} \leq A^{-1} \left\|K^\dagger\right\|^2.$$

◄

The theorems 1 and 2 leads to the fact that every continuous K-J-frame of the Krein space $(\mathscr{H}, [\cdot,\cdot])$ can be seen as a family of local cK-atoms for the associated $(\mathscr{H}, [\cdot,\cdot]_J)$,

Proposition 9. *If $F : X \longrightarrow \mathscr{H}$ is a continuous Bessel family for $(\mathscr{H}, [\cdot,\cdot])$, then the following assertions are equivalent:*

1. *F is a continuous K-J-frame for $(\mathscr{H}, [\cdot,\cdot])$;*
2. *JF is a family of local cK-atoms for the Hilbert space $(\mathscr{H}, [\cdot,\cdot]_J)$;*
3. *There exists $G \in \mathcal{B}\left(\mathscr{H}, L^2(X, |\nu|)\right)$ such that*

$$Kf = \int_X G(k)(x) JF(x) d|\nu|(x), \quad \forall k \in \mathscr{H}.$$

4. K-J-dual continuous Bessel family

This section introduces the notion of K-J-dual continuous Bessel family in the Krein spaces setting. This concept is a generalization of K-dual Bessel sequences in Krein spaces introduced in [18].

In this section, we fix $\mathscr{H}$ as a Krein space and J as a fundamental symmetry over $\mathscr{H}$.

Definition 8. *Assume that $F : X \to \mathscr{H}$ is a continuous Bessel family. A Bessel family $G : X \longrightarrow \mathscr{H}$ is called a K-J-dual continuous Bessel family of F if*

$$Kf = \int_X [f, G(x)] \pi_{R(K)} JF(x) d|\nu|(x), \quad \forall f \in \mathscr{H}.$$

Example 4. *We consider $\mathscr{H} = \ell^2(\mathbb{N}^*)$, with the indefinite form*

$$\forall x, y \in \mathscr{H} \quad [x, y] = \sum_{n\geq 1} (-1)^{n+1} x_n \overline{y_n}$$

and the fundamental symmetry

$$J : x = (x_n)_{n\in\mathbb{N}^*} \in \mathscr{H} \longmapsto ((-1)^{n+1} x_n)_{n\in\mathbb{N}^*} \in \mathscr{H}.$$

Note that $[x, y]_J = \langle x, y\rangle$, where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\ell^2(\mathbb{N}^*)$ given by

$$\langle x, y\rangle := \sum_{n\geq 1} x_n \overline{y_n}.$$

Denote by e_i the elementary vectors given by $e_i := (0, \dots, 1, 0 \dots)$ and consider a sequence of elements of $\mathscr{H}$, $\{k_j\}_{j\in\mathbb{N}^*}$ such that
$k_1 = e_1+e_2-e_3+e_4$, $k_2 = e_1-e_2+e_3+e_4$, and for every $j \geq 3$, $k_j = (-1)^{j+1} e_{j+2}$.

Let K be the operator defined on $\mathscr{H}$ by

$$K(e_1) = e_1 - e_3, \quad K(e_2) = e_2 - e_4, \quad K(e_j) = e_{j+2} \quad \forall j \geq 3.$$

We consider also Ω be a Borel subset of $\mathbb{R}$ and μ a signed measure on Ω, such that $|\mu|(\Omega) < +\infty$. Let $(\Omega_j)_{j\in\mathbb{N}^*}$ be a Borel partition of Ω such that every Ω_j has no null measure. So, we define

$$F : \omega \in \Omega \mapsto \sum_{j\geq 1} \frac{1}{\sqrt{|\mu|(\Omega_j)}} 1_{\Omega_j} k_j \in \mathscr{H},$$

and

$$G : \omega \in \Omega \mapsto \sum_{j\geq 1} \frac{1}{\sqrt{|\mu|(\Omega_j)}} 1_{\Omega_j} e_j \in \mathscr{H}.$$

It is easily seen that F is a continuous Bessel family with respect to $(\Omega, \mathfrak{B}(\Omega), \mu)$ and

$$\forall f, h \in \mathscr{H}, \quad \int_X [f, G(x)][\pi_{R(K)} JF(x), h] d|\mu|(x) = \sum_{j\geq 1} [f, e_j] \left[\pi_{R(K)} Jk_j, h\right] =$$

$$= [f, e_1] [e_1 - e_3, h] + [f, e_2] [e_2 - e_4, h] + \sum_{j\geq 3} [f, e_j] [e_{j+2}, h] = [Kf, h].$$

So, we conclude that for any $f \in \mathscr{H}$ we have

$$\int_X [f, G(x)] \pi_{R(K)} JF(x) d|\mu|(x) = Kf.$$

Proposition 10. *If $F : X \to \mathscr{H}$ is a continuous K-J-frame for $\mathscr{H}$, then F admits a K-J-dual continuous Bessel family.*

Before the proof, we cite this lemma:

Lemma 2. *[19] Let $(X, \mathfrak{B}, \nu)$ be a measured space, $(\mathscr{H}_i, [\cdot,\cdot]_i), i = 0, \ldots, 3$, be three Krein spaces, $\mathrm{F} : X \to \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ be a weakly measurable map, and $A \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ be a linear operator such that $\int_X \mathrm{F}(x)\mathrm{d}\nu(x) = A$. Then*

1. *$\int_X \mathrm{F}(x)k \ \mathrm{d}\nu(x) = Ak$ for all $k \in \mathscr{H}_1$.*
2. *For all linear operators $T \in \mathcal{B}(\mathscr{H}_0, \mathscr{H}_1)$ and $S \in \mathcal{B}(\mathscr{H}_2, \mathscr{H}_3)$,*

$$\int_X \mathrm{F}(x)T\mathrm{d}\nu(x) = AT \ \text{and} \ \int_X S\mathrm{F}(x)\mathrm{d}\nu(x) = SA.$$

Proof. F is a continuous K-J-frame for $\mathscr{H}$, so Proposition 9 asserts that there exist $G : X \to \mathscr{H}$ weakly measurable and $B > 0$ such that

$$\forall f \in \mathscr{H}, \quad \int_X |[f, G(x)]|^2 d|\nu|(x) \leq B\|f\|_J$$

and

$$\forall f \in \mathscr{H}, \quad Kf = \int_X [f, G(x)]JF(x)d|\nu|(x).$$

So, $g : X \to \mathscr{H}$ is a continuous Bessel family for $\mathscr{H}$. Furthermore, by Lemma 2, we have

$$\forall f \in \mathscr{H}, \quad Kf = \pi_{R(K)}Kf = \pi_{R(K)} \int_X [f, g(x)]JF(x)d|\nu|(x) =$$

$$\int_X [f, g(x)]\pi_{R(K)}JF(x)d|\nu|(x),$$

and we get the desired result. ◀

Remark 10. *The converse is not true: not every continuous Bessel family that admits a K-J-dual continuous Bessel family is a continuous K-J-frame. In fact if we go back to Example 4, we notice that for $f = e_1 + e_4$*

$$\|K^{*_J} f\|_J = 2, \ \text{and} \ \int_X |[f, F(X)]|^2 \, \mathrm{d}|\mu|(x) = |[f, k_1]|^2 + |[f, k_2]|^2 = 0.$$

So, F is not a continuous K-J-frame for $\mathscr{H}$.

Proposition 11. *Let $F : X \to \mathscr{H}$ be a continuous Bessel family, that admits a K-J-dual continuous Bessel family. Then $\pi_{R(K)}^{[*]}F : X \to \mathscr{H}$ is a continuous K-J-frame for $\mathscr{H}$.*

Proof. $\pi_{R(K)}^{[*]}F : X \to \mathscr{H}$ is a continuous Bessel family for $\mathscr{H}$, since $F : X \to \mathscr{H}$ is a continuous Bessel family and $\pi_{R(K)}$ is bounded. On the other hand, let G be a K-J-dual continuous Bessel family of F. Then for any $f \in \mathscr{H}$, we get

$$\|K^{*J}f\|_J^4 = \left|[K^{*J}f, K^{*J}f]_J^2\right| = \left|[KK^{*J}f, f]_J^2\right|$$

$$= \left|\int_X [K^{*J}f, G(x)] \left[\pi_{R(K)}JF(x), f\right]_J d|\nu|(x)\right|^2$$

$$\leq \int_X |[K^{*J}f, G(x)]|^2 \, d|\nu|(x) \int_X \left|\left[\pi_{R(K)}JF(x), f\right]_J\right|^2 d|\nu|(x)$$

$$\leq B_G \, \|K^{*J}f\|_J^2 \int_X \left|\left[\pi_{R(K)}^{[*]}F(x), f\right]\right|^2 d|\nu|(x).$$

B_G is an upper bound of the Bessel $G : X \to \mathscr{H}$. So $\pi_{R(K)}^{[*]}F : X \to \mathscr{H}$ is a continuous Bessel family that admits a lower continuous K-J-frame bound. So, $\pi_{R(K)}^{[*]}F : X \to \mathscr{H}$ is a continuous K-J-frame for $\mathscr{H}$. ◄

Remark 11. *Propositions 11 and 10 show that the operator $\pi_{R(K)}^{[*]}$ preserve continuous K-J-frames.*

In the case where K has a closed range, we can determine a K-J-dual continuous Bessel family of any continuous K-J-frame F more explicitly.

Proposition 12. *Let K be a bounded operator of $\mathscr{H}$ with a closed range and $F : X \to \mathscr{H}$ be a continuous K-J-frame for $\mathscr{H}$. Let A and B be the upper and lower bounds of F, respectively. The weakly measurable function given by $\tilde{F} = K^{[*]}J\left(S|_{R(K)}\right)^{-1}\pi_{S(R(K))}F$ forms a continuous K-J-dual Bessel family of F and it is called the canonical K-J-dual continuous Bessel family of F.*

Proof.

$K^{[*]}J\left(S|_{R(K)}\right)^{-1}\pi_{S(R(K))}F : X \longrightarrow \mathscr{H}$ is a continuous Bessel family. In fact,

$$\int_X \left|\left[f, K^{[*]}J\left(S|_{R(K)}\right)^{-1}\pi_{S(R(K))}F(x)\right]\right|^2 d|\nu| =$$

$$= \int_X \left| \left[J \left(\left(S_F|_{R(K)} \right)^{-1} \right)^{*_J} Kf, F(x) \right] \right|^2 d|\nu| \leq BA^{-1} \|K\|^2 \left\| K^\dagger \right\|^2 \|f\|_J^2.$$

Moreover, if we fix h and k, two elements of $\mathscr{H}$, then

$$\int_X \left[\left[f, K^{[*]} J \left(S|_{R(K)} \right)^{-1} \pi_{S(R(K))} F(x) \right] \pi_{R(K)} JF(x), g \right] d|\nu|(x)$$
$$= \int_X \left[Kf, J \left(S|_{R(K)} \right)^{-1} \pi_{S(R(K))} F(x) \right] \left[\pi_{R(K)} JF(x), g \right] d|\nu|(x)$$
$$= \left[Kf, J \left(S|_{R(K)} \right)^{-1} \pi_{S(R(K))} \int_X \left[\pi_{R(K)} Jg, F(x) \right] F(x) d|\nu| \right]$$
$$= \left[Kf, J \left(S|_{R(K)} \right)^{-1} S|_{R(K)} \left(\pi_{R(K)} Jg \right) \right]$$
$$= \left[Kf, J\pi_{R(K)} Jg \right] = \left[Kf, \pi_{R(K)}^{[*]} g \right] = [Kf, g].$$

So for all $f \in \mathscr{H}$,

$$Kf = \int_X \left[f, K^{[*]} J \left(S|_{R(K)} \right)^{-1} \pi_{S(R(K))} F(x) \right] \pi_{R(K)} JF(x) d|\nu|(x).$$

◀

Theorem 4. *Let $F : X \longrightarrow \mathscr{H}$ be a continuous K-J-frame for $\mathscr{H}$ and $\tilde{F} : X \longrightarrow \mathscr{H}$ be the canonical K-J-dual Bessel family of F. Then $G : X \longrightarrow \mathscr{H}$ is a continuous K-J-dual continuous Bessel family of F if and only if*

$$G = H + \tilde{F}$$

such that H is a continuous Bessel family, satisfying the condition

$$R(J_{L^2(X,\nu)} T_H^{[*]}) \subseteq T_F^{-1}(R(K)^{[\perp]}), \tag{8}$$

where T_F is the pre-frame operator of F and $T_H^{[*]}$ is the adjoint of the pre-frame operator T_H of H.

Proof. If G is a K-J-dual continuous Bessel family of F, then $F - G$ is a Bessel family and

$$\int_X \left[f, G(x) - \tilde{F}(x) \right] \pi_{R(K)} JF(x) d|\nu|(x) = 0$$

So, by the fact that

$$\pi_{R(K)} J \left(\int_X \left[f, G(x) - \tilde{F}(x) \right] F(x) d|\nu|(x) \right) = \int_X \left[f, G(x) - \tilde{F}(x) \right] \pi_{R(K)} JF(x) d|\nu|(x)$$

we get

$$T_F\left(J_{L^2(X,\nu)}T_{F-G}^{[*]}(f)\right) \in JR(K)^{\perp} = R(K)^{[\perp]} \quad \forall f \in \mathscr{H}.$$

Therefore $F - G$ is a Bessel family satisfying the condition

$$R\left(J_{L^2(X,\nu)}T_{F-G}^{[*]}\right) \subseteq T_F^{-1}\left(R(K)^{[\perp]}\right).$$

Reciprocally, let $G = \tilde{F} + H$, where H is a continuous Bessel family that verifies (8). Then G is a continuous Bessel family. In fact, for every $f \in \mathscr{H}$

$$\begin{aligned}\int_X |[f, G(x)]|^2 \, d|\nu|(x) &= \int_X \left|\left[f, \tilde{F}(X) + H(X)\right]\right|^2 d|\nu|(x) \\ &\leq \left[\left(\int_X \left|\left[f, \tilde{F}(x)\right]\right|^2 d|\nu|(x)\right)^{\frac{1}{2}} + \left(\int_X |[f, H(x)]|^2 \, d|\nu|(x)\right)^{\frac{1}{2}}\right]^2 \\ &\leq \left((BA^{-1})^{\frac{1}{2}} \left\|K^{\dagger}\right\| \|K\| + B_H\right)^2 \|f\|^2,\end{aligned}$$

where A and B are the frame bounds of F, and B_H is the upper bound of H. Moreover,

$$\int_X [f, g(x)] \, \pi_{R(K)} JF(x) d|\nu|(x) = Kf + \pi_{R(K)} JT_F J_{L^2(X,\nu)} T_H^{[*]} f = Kf,$$

which gives the desired result. ◄

5. Preservation of continuous K-J-frames by usual operations

5.1. Preservation by composition with bounded operator

In this first part of this section, we will study the image of continuous K-J-frames in a Krein space $\mathscr{H}$ under a bounded operator from $\mathcal{B}(\mathscr{H})$, and see the sufficient conditions that can be imposed on this bounded operator to ensure that this image remains a a continuous K-J-frame for $\mathscr{H}$.

Theorem 5. *Let $K \in \mathcal{B}(\mathscr{H})$ be with a dense range, $F : X \to \mathscr{H}$ be a continuous K-J-frame and $L \in \mathcal{B}(\mathscr{H})$ with a closed range. If $LF : X \to \mathscr{H}$ is a continuous K-J-frame for $\mathscr{H}$, then L is surjective.*

Proof. Since $LF : X \to \mathscr{H}$ is a continuous K-J-frame, by applying Theorem 3 we conclude that $R(K) \subseteq R(JT_{LF}) = R(JLT_F) \subseteq R(JL)$ and therefore $\mathscr{H} = \overline{R(K)} \subseteq \overline{R(JL)}$. The fact that J is a unitary operator and L has a closed range implies that

$$\overline{R(JL)} = \overline{JR(L)} = JR(L),$$

so $\mathscr{H} = JR(L)$ and we get $\mathscr{H} = R(L)$.

Theorem 6. *Let $K \in \mathcal{B}(\mathscr{H})$ be a bounded operator on $\mathscr{H}$ and let $F : X \to \mathscr{H}$ be a continuous K-J-frame for $\mathscr{H}$. If $L \in \mathcal{B}(\mathscr{H})$ is surjective with $LJK = JKL$, then $LF : X \to \mathscr{H}$ is a continuous K-J-frame for $\mathscr{H}$.*

Proof. L is bounded, so it obviously preserves every continuous Bessel family for $\mathscr{H}$, therefore LF is a continuous Bessel family. On the other hand, the fact that F is a continuous K-J-frame for $\mathscr{H}$ ensures that T_F the pre-frame operator is bounded and

$$R(JK) \subseteq R(T_F), \tag{9}$$

and, as L is surjective and $LJK = JKL$, it follows that L stabilises the set $R(JK)$. Indeed, we have

$$LR(JK) = R(LJK) = R(JKL) = R(JK),$$

so by composing by L in (9) we get $LR(JK) = R(JK) \subseteq LR(T_F) = R(T_{LF})$. Then we get the result by Theorem 3. ◄

So from Theorems 5 and 6 we have

Corollary 1. *If $K \in \mathcal{B}(\mathscr{H})$ is a bounded operator with a dense range, $F : X \to \mathscr{H}$ is a continuous K-J-frame for $\mathscr{H}$ and $L \in \mathcal{B}(\mathscr{H})$ is such that $LJK = JKL$, then L is surjective if and only if LF is a continuous K-J-frame for $\mathscr{H}$.*

5.2. Preservation of continuous K-J-frames by sum

In this part, we will search for conditions that guarantee that the sum of two continuous K-J-frames in $\mathscr{H}$ remains a continuous K-J-frame in $\mathscr{H}$, and also the addition of a continuous Bessel operator in $\mathscr{H}$ to a continuous K-J-frame results to a continuous K-J-frame in $\mathscr{H}$.

Theorem 7. *Let F and G be two continuous K-J-frames in a Krein space $\mathscr{H}$ with lower frame bounds A_1 and A_2, respectively, and upper frame bounds B_1 and B_2, respectively, and the corresponding pre-frame operators be T_F and T_G, respectively. If $T_F T_G^{*J}$ is a J-positive operator, then $F + G$ is a continuous K-J-frame in $\mathscr{H}$.*

Proof. Let $f \in \mathscr{H}$. We have

$$\int_X |[F(x) + G(x), f]|^2 d|\nu|(x) = [J(T_F + T_G)(T_F + T_G)^{*J} Jf, f]_J$$

$$= \left[JT_F T_F^{*J} Jf, f\right]_J + \left[JT_F T_G^{*J} Jf, f\right]_J + \left[JT_G T_F^{*J} Jf, f\right]_J + \left[JT_G T_G^{*J} Jf, f\right]_J$$

$$\geq \left[JT_F T_F^{*J} Jf, f\right]_J + \left[JT_G T_G^{*J} Jf, f\right]_J$$

$$= \int_X |[F(x), f]|^2 d|\nu|(x) + \int_X |[G(x), f]|^2 d|\nu|(x) \geq (A_1 + A_2)\|K^{*_J} f\|_J.$$

The existence of the upper bound is obvious. ◀

Theorem 8. *Let $F : X \to \mathscr{H}$ be a continuous K-J-frame in a Krein space $\mathscr{H}$ and $G : X \to \mathscr{H}$ be a continuous Bessel family in $\mathscr{H}$. Let their corresponding pre-frame operators be T_F and T_G, respectively, such that $R(T_F)$ is closed and $R(T_G)$ is a subset of $R(T_F)$. Then there exists $r > 0$ such that for every $\alpha \in \mathbb{C}$ with $|\alpha| < r$, $\alpha G + F$ is a continuous K-J-frame in $\mathscr{H}$.*

Proof. Let $T_F^\dagger$ be the pseudo-inverse of T_F and let $\alpha \in \mathbb{C}$ be a constant such that

$$\|\alpha T_G\| < \|T_F^\dagger\|^{-1}.$$

So, the operator $(\alpha T_F^\dagger T_G + I)$ is invertible, where I is the identity operator. In particular, we have $R(\alpha T_F T_F^\dagger T_G + T_F) = R(T_F)$. Since $T_F T_F^\dagger$ is the projection onto $R(T_F)$, we get

$$R(\alpha T_G + T_F) = R(T_F).$$

Therefore

$$R(K) \subseteq R(J(\alpha T_G + T_F)),$$

since F is a continuous K-J-frame. By Theorem 3 we conclude that $\alpha G + F$ is also a continuous K-J-frame in $\mathscr{H}$. ◀

Corollary 2. *Let $F : X \to \mathscr{H}$ be a continuous frame in a Krein space $\mathscr{H}$ and $G : X \to \mathscr{H}$ be a continuous Bessel family in $\mathscr{H}$. There exists an $r > 0$ such that for every $\alpha \in \mathcal{D}(0, r) := \{\alpha \in \mathbb{C} \mid |\alpha| < r\}$, $\alpha G + F$ is a continuous frame in $\mathscr{H}$.*

6. Continuous K-J-frames in Krein spaces arising from a non-regular $\mathfrak{G}$-metric

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathfrak{G}$ be an injective self-adjoint operator with a domain $\mathcal{D}(\mathfrak{G}) \subset \mathcal{H}$.

We can use $\mathfrak{G}$ to build a Krein space on a subspace of $\mathcal{H}$ or a larger space containing H. We will briefly state the method to do this in a few lines. First, let $\mathfrak{G} = U|\mathfrak{G}|$ be the polar decomposition of $\mathfrak{G}$. The fact $N(\mathfrak{G}) = \{0\}$ implies that U is a unitary self-adjoint operator. We consider also $E(\lambda)$, $-\infty < \lambda < +\infty$, the resolution of the identity associated to $\mathfrak{G}$, $\mathfrak{G} = \int_{\mathbb{R}} \lambda \mathrm{d}E(\lambda)$, the integral representation of $\mathfrak{G}$, and

$$[f, g] := \langle f, \mathfrak{G} g \rangle, \quad f, g \in \mathcal{D}(\mathfrak{G}),$$

a non-degenerate inner product on $\mathcal{D}(\mathfrak{G})$. We define the fundamental decomposition for $\mathcal{D}(\mathfrak{G})$ by $\mathcal{D}(\mathfrak{G}) = \mathcal{D}_+ \oplus \mathcal{D}_-$, where

$$\mathcal{D}_+ := E(0,\infty)\mathcal{D}(\mathfrak{G}), \quad \mathcal{D}_- := E(-\infty,0)\mathcal{D}(\mathfrak{G}), \tag{10}$$

and the fundamental symmetry

$$J = E(0,\infty) - E(-\infty,0). \tag{11}$$

Note that $J^2 = Id$, where Id is the identity operator on $\mathcal{H}$. So, by (10) and the uniqueness of the polar decomposition ([16][Chapter VI, Subsect. 2.7]), we deduce that $U = J$ and

$$[f,g]_J := [f, Jg] = \langle f, |\mathfrak{G}|g\rangle, \quad f,g \in \mathcal{D}(\mathfrak{G}).$$

By considering the closure of the norm defined by $[\cdot,\cdot]_J$ and extending the operator J to this closure, we obtain a Krein space denoted as $(\mathcal{H}_\mathfrak{G}, [\cdot,\cdot])$.

This Krein space has a fundamental symmetry represented by the operator J and a fundamental decomposition $\mathcal{H}_\mathfrak{G} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are the closures of $\mathcal{D}_+$ and $\mathcal{D}_-$, respectively. We refer to $\mathcal{H}_\mathfrak{G}$ as the Krein space derived from $\mathfrak{G}$, and $\mathfrak{G}$ is called a Gram operator.

To introduce continuous K-J-frames for $\mathcal{H}_\mathfrak{G}$ by using the continuous K-J-frames of $\mathcal{H}$, we distinguish two cases:

The first case, where the Gram operator $\mathfrak{G}$ is bounded and invertible. The inner product $\langle\cdot,\cdot\rangle$ is, of course, equivalent to the one defined by $[\cdot,\cdot]_J$, which implies that $\mathcal{H}_\mathfrak{G}$ coincides with $\mathcal{H}$. The operator K remains bounded on the Krein space $\mathcal{H}_\mathfrak{G}$, and every continuous K-J-frame F for $\mathcal{H}$ is also a continuous K-J-frame for $(\mathcal{H}_{W\mathfrak{G}}, [\cdot,\cdot]_J)$, the associated Hilbert space to our Krein space $\mathcal{H}_\mathfrak{G}$. By Theorem 2, JF will be a continuous K-J-frame for $(\mathcal{H}_\mathfrak{G}, [\cdot,\cdot])$.

The second case, where the Gram operator $\mathfrak{G}$ is unbounded or not invertible, or both. In this case, there are two constraints. The first is that the operator K may not be bounded on $\mathcal{H}_\mathfrak{G}$. The second is that if $K = Id$, no continuous K-J-frame for the Hilbert space $\mathcal{H}$ can be a continuous K-J-frame for the Krein space $\mathcal{H}_\mathfrak{G}$ (see [19] Section 4). Therefore, the transfer of continuous K-J-frames of $\mathcal{H}$ to continuous K-J-frames of $\mathcal{H}_\mathfrak{G}$ is not obvious as in the first case.

First let us recall some properties of the Krein space $\mathcal{H}_\mathfrak{G}$:

Theorem 9. *[17] Let $\mathfrak{G} : \mathcal{D}(\mathfrak{G}) \longrightarrow \mathcal{H}$ be an injective self-adjoint operator on the Hilbert space $\mathcal{H}$. Then*

1. *The set* $\mathcal{D}(\sqrt{|\mathfrak{G}|})$ *is complete with respect to the norm* $\|\cdot\|_J$ *if and only if* $0 \notin \sigma(\mathfrak{G})$. *In this situation, the set* $\mathcal{H}_{\mathfrak{G}}$ *can be identified with* $\mathcal{D}(\sqrt{|\mathfrak{G}|})$.

2. *The operator* $\sqrt{|\mathfrak{G}|} : (\mathcal{D}(\sqrt{|\mathfrak{G}|}), [\cdot,\cdot]_J) \longrightarrow (\mathcal{H}, \langle\cdot,\cdot\rangle)$ *is isometric and it admits an extension to a* J*-unitary operator* $U := \overline{\sqrt{|\mathfrak{G}|}} : \mathcal{H}_{\mathfrak{G}} \longrightarrow \mathcal{H}$.

Like in [18], we will assume that K commutes with $\sqrt{|\mathfrak{G}|}$ to ensure that it remains bounded in $\mathcal{H}_{\mathfrak{G}}$

Proposition 13. *[18] We suppose that* K *is a bounded operator for* $\mathcal{H}$ *that satisfies*

$$K|_{\mathcal{D}(\mathfrak{G})} : \mathcal{D}(\mathfrak{G}) \longrightarrow \mathcal{D}(\mathfrak{G}); \quad \forall f \in \mathcal{D}(\mathfrak{G}) \quad K\sqrt{|\mathfrak{G}|}f = \sqrt{|\mathfrak{G}|}\, K|_{\mathcal{D}(\mathfrak{G})}\, f. \tag{12}$$

So, $K|_{\mathcal{D}(\mathfrak{G})}$ *is a bounded operator on* $\mathcal{D}(\mathfrak{G})$ *equipped with the norm defined by the inner product* $[\cdot,\cdot]_J$

The main result which ensures the desired transfer is similar to the result given by Theorem 5.4 in [18]:

Theorem 10. *Let* $K \in \mathcal{B}(\mathcal{H})$ *be a bounded operator that satifies (12). A weakly measurable function* $F : X \to \mathcal{H}$ *will be a continuous* K*-frame for the Hilbert space* $\mathcal{H}$ *with frame bounds* $A \leq B$ *if and only if* $\overline{JU^{-1}F} : X \to \mathcal{H}_{\mathfrak{G}}$ *is a continuous* $\widetilde{K}$*-J-frame for the Krein space* $\mathcal{H}_{\mathfrak{G}}$ *with the same frame bounds.* $\widetilde{K}$ *refers to the bounded operator extending* $K|_{\mathcal{D}(\mathfrak{G})}$ *to* $\mathcal{H}_{\mathfrak{G}}$ *and* U *is the* J*-unitary operator* $U := \overline{\sqrt{|\mathfrak{G}|}} : \mathcal{H}_{\mathfrak{G}} \longrightarrow \mathcal{H}$ *given in Theorem 9.*

The proof remains the same as the one in [18] for Theorem 5.4.

Example 5. *Let* $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ *be a sequence of complex numbers. We can define*

$$\mathcal{D}(\mathfrak{G}_\alpha) := \left\{ x = (x_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N}) : \sum_{n\geq 0} |\alpha_n|^2 |x_n|^2 < \infty \right\}$$
$$\mathfrak{G}_\alpha : \mathcal{D}(\mathfrak{G}_\alpha) \longrightarrow l^2(\mathbb{N}), \quad (\mathfrak{G}_\alpha x)_n := \alpha_n x_n.$$

Moreover, we assume that for every $n \in \mathbb{N}$, α_n *is a non-zero real number. Thus,* $\mathfrak{G}_\alpha^* = \mathfrak{G}_\alpha$ *and* $\operatorname{Ker}(\mathfrak{G}_\alpha) = \{0\}$. *Therefore,* W_α *defines a Gram operator for the Krein space*

$$\mathcal{H}_{\mathfrak{G}_\alpha} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{n\geq 0} |\alpha_n||x_n|^2 < \infty \right\},$$

equipped with the indefinite product $[x,y] = \sum_{n\geq 0} \alpha_n x_n \overline{y_n}$ *and with the fundamental symmetry* J *given by* $(Jx)_n := \operatorname{sign}(\alpha_n)x_n$*. The operator* U *introduced in Theorem 9 is explicitly given as follows:*

$$U = \mathfrak{G}_{|\alpha|^{1/2}} : \mathcal{H}_{\mathfrak{G}_\alpha} \longrightarrow l^2(\mathbb{N}), \quad \left(\mathfrak{G}_{|\alpha|^{1/2}} f\right)_n := \sqrt{|\alpha_n|}x_n,$$
$$U^{-1} = \mathfrak{G}_{|\alpha|^{-1/2}} : l^2(\mathbb{N}) \longrightarrow \mathcal{H}_{\mathfrak{G}_\alpha}, \quad \left(\mathfrak{G}_{|\alpha|^{-1/2}} f\right)_n := \frac{1}{\sqrt{|\alpha_n|}}x_n.$$

Now let $\beta \in \mathbb{C}^{\mathbb{N}}$ *be a bounded sequence and* $\mathfrak{G}_\beta$ *be the operator defined as* $\mathfrak{G}_\alpha$ *by*

$$\mathfrak{G}_\beta : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N}), \quad (\mathfrak{G}_\beta x)_n := \beta_n x_n.$$

$\mathfrak{G}_\beta$ *is bounded and commutes* $\mathfrak{G}_{|\alpha|^{1/2}}$*:*

$$\mathfrak{G}_\beta|_{\mathcal{D}(\mathfrak{G}_\alpha)} : \mathcal{D}(\mathfrak{G}_\alpha) \longrightarrow \mathcal{D}(\mathfrak{G}_\alpha), \quad \forall x \in \mathcal{D}(\mathfrak{G}_\alpha) \quad \mathfrak{G}_\beta \mathfrak{G}_{|\alpha|^{1/2}} x = \mathfrak{G}_{|\alpha|^{1/2}} \mathfrak{G}_\beta x.$$

Therefore, $\mathfrak{G}_\beta$ *admits a unique extension to a bounded operator of* $\mathcal{H}_{\mathfrak{G}_\alpha}$*, denoted by* $\widetilde{\mathfrak{G}_\beta}$*. Taking into consideration Theorem 10, we conclude that if* $F : X \to l^2(\mathbb{N})$ *is a continuous* $\mathfrak{G}_\beta$*-frame for* $l^2(\mathbb{N})$*, then* $JU^{-1}F : X \to l^2(\mathbb{N})$ *will be a continuous* $\widetilde{\mathfrak{G}_\beta}$*-frame for the Krein space* $\mathcal{H}_{\mathfrak{G}_\alpha}$*.*

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