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Stefan Problems With Unbounded Heat Conduction and Integrable Data

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Abstract. In this paper, we prove the existence of renormalized solution for a class of Stefan-type problems

$$\frac{\partial b(u)}{\partial t} - div(A(t,x,u)Du) = f,$$

where the matrix $A(t,x,s) = (a_{ij}(t,x,s))_{\substack{1\le i\le N\\ 1\le j\le N}}$ is not controlled with respect to u, $f \in L^1(Q)$, and b is a maximal monotone graph.

Key Words and Phrases: renormalized solutions, below-up, Stefan-type, $L^1 - data$.

2010 Mathematics Subject Classifications: 47A15, 46A32

1. Introduction

This paper investigates the existence of a solution for a class of Stefan type problems of the form

$$\begin{cases} \frac{\partial b(u)}{\partial t} - div(A(t,x,u)Du) = f & \text{in } \Omega \times (0,T), \\ b(u)(t=0) = b_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times]0,T[\,, \end{cases} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \ge 1)$, and T, m are two positive numbers, b is a maximal monotone graph on $\mathbb{R}$, and b^{-1} is continuous on $\mathbb{R}$. Moreover, the matrix

$$A : (t,x,s) \longrightarrow A(t,x,s) = (a_{ij}(t,x,s))_{\substack{1\le i\le N\\ 1\le j\le N}} \tag{2}$$

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is a Carathéodory function from $Q \times (-\infty, m)$ into $\mathbb{R}_s^{N\times N}$, the set of $N \times N$ symmetric matrices, which blows up when $s \to m^-$ and satisfies the coercivity condition.

Under the assumptions on the operator b, the problem (1) is of a Stefan problem class. The applications of the Stefan problem can be found in many areas, among which we mention the Stefan liquid diffusion problem (see [12]).

When we study this problem, we find two difficulties. Due to these difficulties, the problem (1) does not admit, in general, a weak solution. In order to overcome these difficulties, we are using the framework of renormalised solutions. This concept was presented by DiPerna and Lions [8] (see also Lions [9] for a few applications to fluid mechanics models). We refer the reader to [6, 10, 11] for elliptic problems and to [4, 3] for parabolic equations.

By contrast, to prove the existence of a normalized solution, we use the technique developed by D. Blanchard and A. Porretta [2], K. Ammar and P. Wittbold [7]. This technique is based on introducing the approximate problem. Then, thanks to the theory of semi-groups, we show the weak solution of the approximate problem. Finally, we pass to the limit in the approximate problem to establish the existence of renormalized solution of the problem (1).

In the case where the field is a Leray-Lions operator and $b(u) = b(x, u)$, the existence of renormalized solutions has been proved in [4] and [1] in the weighted Sobolev space. In the case where the field is a Leray-Lions operator and b is a maximal monotone graph on $\mathbb{R}$, the existence of renormalized solutions has been established in [2, 7].

The main purpose of this paper is to study the existence of renormalized solution for the problem (1), and gives a sense to the flux

$$A\,(t, x, u)\, DuDu$$

in the set $\{(t, x) \in Q\ ;\ u\,(t, x) = m\}$.

The paper is structured as follows. In Section 2, we make assumptions and provide our main result. In Section 4, we give the proof of main result.

2. Basic assumption and main result

In this paper, we study a class of Stefan problems

$$\begin{cases} \frac{\partial b(u)}{\partial t} - div(A(t,x,u)Du) = f & \text{in } Q, \\ b(u)(t=0) = b_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times]0,T[\,, \end{cases} \tag{3}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 1)$, $Q = \Omega \times]0, T[$, where $T > 0$ and m is a positive real number. We assume that

$$b : \mathbb{R} \to \mathcal{P}(\mathbb{R}) \text{ is a maximal monotone graph such that } 0 \in b(0). \tag{4}$$

We denote by b^{-1} the inverse function of b, and

$$b_0 \in L^1(Q), \tag{5}$$

$$f \in L^1(Q). \tag{6}$$

$$A : (t, x, s) \longrightarrow A(t, x, s) = (a_{ij}(t, x, s))_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}} \tag{7}$$

is a Caracthéodory function from $Q \times (-\infty, m)$ into $\mathbb{R}_s^{N \times N}$, the set of $N \times N$ symetric matrices, such that there exist two positive functions β and γ in $\mathcal{C}^0((-\infty, m))$ which satisfy

$$\lim_{s \longrightarrow m^-} \beta(s) = +\infty \ ; \ \ \beta(s) \geq \alpha > 0 \quad \forall s \in (-\infty, m), \tag{8}$$

$$\int_0^m \gamma(s) ds < +\infty, \tag{9}$$

$$\beta(s) |\xi|^2 \leq A(t, x, s)\xi\xi \leq \gamma(s) |\xi|^2. \tag{10}$$

For any positive real number ε, we define the function b_ε by

$$\sigma_\varepsilon(r) = \begin{cases} 1 & \text{if} \ \ r \leq m - 2\varepsilon \\ 1 - (r - m + \varepsilon) & \text{if} \ \ m - 2\varepsilon \leq r \leq m - \varepsilon \\ 0 & \text{if} \ \ r \geq m - \varepsilon. \end{cases} \tag{11}$$

We define for a fixed $n \geq 1$

$$\theta_n(s) = \frac{1}{n}(T_n(s - T_n(s)), \tag{12}$$

and $h(s) = 1 - |\theta_n(s)|$ for all $s \in \mathbb{R}$.

Definition 1. *A measurable function u defined on Ω is a renormalized solution of problem (3) if:*

$$T_k(u) \in L^2(0, T; H_0^1(\Omega)) \ \forall k \geq 0, \tag{13}$$

$$u \leq m \qquad a.e. \ \ in \ Q, \tag{14}$$

$$\text{for every } k \geq 0 \quad \chi_{\{-k<u<m\}} A(t; x, u) Du.Du \in L^2(\Omega), \tag{15}$$

for every $i = 1, ..., N$,

$$\lim_{p\to+\infty}\frac{1}{p}\int_{\{-2p<u<-p\}}A(t,x,u)Du.Dudxdt = 0, \tag{16}$$

for any $\varphi \in C_c^\infty([0,T])$

$$\lim_{p\to+\infty} p\sum_{i=1}^{N}\int_Q \varphi\chi_{\{m-2/p<u<m-1/p\}}A(t,x,u)Du.Dudxdt \tag{17}$$
$$= \int_Q f\varphi\chi_{\{u=m\}}dxdt,$$

there exists $\varrho_u \in L^1(Q)$ *such that* $\varrho_u \in b(u)$ *a.e. in* Q, *and* u *satisfies*

$$-\int_Q \varphi_t \int_0^{\varrho_u} S'(b^{-1}(r))drdtdx - \int_\Omega \varphi(0)\int_0^{b_0} S'(b^{-1}(r))dr)dx$$
$$+\int_Q A(t,x,u)Du.D\left(S'(u)\varphi\right)dxdt$$
$$= \int_Q fS'(u)\varphi dxdt \tag{18}$$

for every function S *in* $W^{2,\infty}(\mathbb{R})$ *such that supp*(S') *is compact and* $S'(m) = 0$, *and for any* $\varphi \in C_c^\infty\left([0,T]\times\bar{\Omega}\right)$ *such that* $S'(u)\varphi \in L^2(0,T;H_0^1(\Omega))$.

Remark 1. *Conditions (13) and (15) provide that all terms in (18) are well defined.*

The assumption (17) has been established in [5] when $b(u) = u$.

Theorem 1. *Under the assumptions (4) -(10) there exists at least a renormalized solution* u *of problem (3).*

3. Proof of main result

3.1. Step 1. Approximation problem

For $\varepsilon > 0$, we consider the field of matrices

$$A^\varepsilon(t,x,s) = \sigma_\varepsilon(s)A(t,x,s) + (1-\sigma_\varepsilon(s))\beta(m-\varepsilon)I, \tag{19}$$

where σ_ε is the function defined in (11) and I is a diagonal matrix. Indeed, in (19) we use the convention

$$\sigma_\varepsilon(s)A(t,x,s) = 0 \text{ for } s \geq m-\varepsilon.$$

Due to the assumptions (8) and (10), we have

$$\beta(s)\,|\xi|^2 \le A^\varepsilon(t,x,s)\xi\xi \le (\gamma\,(s)\,b_\varepsilon(s) + \sup_{r\in(0,m-\varepsilon)} \beta(r))\,|\xi|^2\,. \tag{20}$$

Finally, $\exists\,(f_\varepsilon)_{\varepsilon>0} \in L^\infty\,(Q)$ such that

$$f_\varepsilon \to f \ \text{ in } L^1(Q), \tag{21}$$

and $\exists\,(b_0^\varepsilon)_{\varepsilon>0} \in L^\infty\,(\Omega)$ such that

$$b_0^\varepsilon \to b_0 \ \text{ in } L^1(\Omega). \tag{22}$$

Proposition 1. *The regularized problem*

$$\begin{array}{ll} \frac{\partial b(u^\varepsilon)}{\partial t} - div(A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon) = f_\varepsilon & in\ Q, \\ b(u^\epsilon)(t=0) = b_0^\varepsilon & in\ \Omega, \\ u^\epsilon = 0 & in\ \partial\Omega\times\,]0,T[\,. \end{array} \tag{23}$$

admits a weak solution u^ε in the sense that u satisfies

$$T_k(u^\varepsilon) \in L^2(0,T;H_0^1(\Omega)\cap L^\infty(Q)\forall k\ge 0,$$

there exists $\varrho_{u^\varepsilon} \in L^1(Q)$ such that $\varrho_{u^\varepsilon} \in b(u^\varepsilon)$ a.e. in Q, $\varrho_{u^\varepsilon}(t) \in L^1(Q)$, and u^ε satisfies

$$\begin{aligned} -\int_Q \varphi_t \varrho_{u^\varepsilon} dxdt - \int_\Omega \varphi(0) b_0^\varepsilon dx + \int_Q A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon D\varphi dxdt \\ = \int_Q f_\varepsilon \varphi dxdt \end{aligned} \tag{24}$$

for every function S in $W^{2,\infty}(\mathbb{R})$ such that $supp(S')$ is compact, and for any $\varphi \in C_c^\infty\left([0,T]\times\bar{\Omega}\right)$ such that $S'(u)\varphi \in L^2(0,T;H_0^1(\Omega))$. Moreover, we have

$$\|\varrho_{u^\varepsilon}(t) - \varrho_{u^\eta}(t)\|_{L^1(\Omega)} \le \|f^\varepsilon - f^\eta\|_{L^1(\Omega)} + \|b_0^\varepsilon - b_0^\eta\|_{L^1(\Omega)} \ \ \forall t\ge 0. \tag{25}$$

Proof. The condition (4) made a difficulty to prove this proposition. The idea of proof is to approach the graph b by b_k, where

$$b_k = \mathfrak{R} + ks,$$

and $\mathfrak{R}$ is the Yoshida approximate of b. The proof is similar to that in [2]. ◀

Remark 2. *Any weak solution is a renormalized solution. Indeed, for any* $S \in W^{2,\infty}(\mathbb{R})$ *and any* $\varphi \in C_c^\infty((0,T)\times\Omega)$ *such that* $S'(u^\varepsilon)\varphi \in L^2(0,T;H_0^1(\Omega))$*, we can choose* $S'(u^\varepsilon)\varphi$ *as a test function in (25). Using the integration-by-parts formula (see [3]), we have*

$$-\int_Q \varphi_t \int_0^{\varrho_{u^\varepsilon}} S'(b^{-1}(r))drdtdx - \int_\Omega \varphi(0)\int_0^{b_0^\varepsilon} S'(b^{-1}(r))dr)dx + \int_Q A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon D\left(S'(u^\varepsilon)\varphi\right)dxdt = \int_Q fS'(u^\varepsilon)\varphi dxdt \tag{26}$$

for any $S \in W^{2,\infty}(\mathbb{R})$ *and any* $\varphi \in C_c^\infty((0,T)\times\Omega)$.

3.2. Step 2. A priori estimate

The test function φ is always equal to $\varphi = \min(\frac{(T-\delta-t)^+}{\delta},1)$.
Choosing $S'(r) = T_k(u^\varepsilon)$ in (26), we have

$$\frac{1}{\delta}\int_{T-2\delta}^{T-\delta}\int_\Omega\int_0^{\varrho_{u^\varepsilon}} S'(b^{-1}(r))drdxdt) + \int_0^{T-2\delta}\int_\Omega A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)DT_k(u^\varepsilon)dxdt \leq k\left[\|f_\varepsilon\|_{L^1(Q)} + \|b_\varepsilon(u_0^\epsilon)\|_{L^1(\Omega)}\right]. \tag{27}$$

Letting δ tend to 0, we have

$$\int_0^T\int_\Omega A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)DT_k(u^\varepsilon)dxdt \leq k\left[\|f_\varepsilon\|_{L^1(Q)} + \|b_\varepsilon(u_0^\epsilon)\|_{L^1(\Omega)}\right].$$

Thanks to (10) and $f_\varepsilon \in L^1(Q)$, we have

$$\alpha\int_Q |DT_k(u^\varepsilon)|^2 dxdt \leq Ck, \tag{28}$$

$$\int_Q |A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)|^2 dxdt \leq C,$$

and

$$X^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon) \in (L^2(Q))^N, \tag{29}$$

where $X^\varepsilon(x,s) = \left(x_{ij}^\varepsilon(x,s)\right)_{\substack{1\leq i\leq N\\ 1\leq j\leq N}}$ is the square root of the matrix $A^\varepsilon(x,s)$.

Let $k \geq 0$. Note that

$$
\begin{aligned}
|\{|u^\varepsilon| > k\}| &= k^{-2}(\int_Q |T_k(u^\varepsilon)|^2\,dxdt \\
&\le Ck^{-2}.
\end{aligned}
$$

Therefore, we have

$$\lim_{k\to+\infty} meas\,\{|u^\varepsilon| > k\} = 0.$$

Then, from (25) we deduce

$$\varrho_{u^\varepsilon} \text{ strongly converges to } \varrho_u \text{ in } L^\infty(0,T;L^1(\Omega)). \tag{30}$$

Since b^{-1} is continuous, this implies, up to a subsequence,

$$u^\varepsilon \to u \;\; a.e. \;\text{ in } Q, \tag{31}$$

where $u := b^{-1}(\varrho_u)$. Next, from (28) we can extract a subsequence, still denoted by itself, such that

$$T_k\,(u^\varepsilon) \rightharpoonup v_k \text{ weakly in } \; L^2(0,T;H_0^1(\Omega)), \tag{32}$$

strongly in $L^2(Q)$ and $a.e.$ in Q. It can be seen that $T_k\,(u^\varepsilon)$ is a Cauchy sequence which converges in measure in Q. We have, for any $k > 0$,

$$T_k\,(u^\varepsilon) \rightharpoonup T_k\,(u) \text{ weakly in } L^2(0,T;H_0^1(\Omega)).$$

Now using $S'(u^\varepsilon) = T_{2m}^+(s) - T_m^+(s)$ in (26), leads to

$$
\begin{aligned}
&\frac{1}{\delta}\int_{T-2\delta}^{T-\delta}\int_\Omega\int_0^{\varrho_{u^\varepsilon}} S'(b^{-1}(r))drdxdt)+ \\
&\int_0^{T-2\delta}\int_\Omega A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)D\left(T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)\right)dxdt \\
&\le k\left[\|f_\varepsilon\|_{L^1(Q)} + \|b_\varepsilon(u_0^\epsilon)\|_{L^1(\Omega)}\right],
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
T_{2m}^+(u) - T_m^+(u) &= 0 \qquad a.e. \;\text{ in } Q, \\
u &\le m \qquad a.e. \;\text{ in } Q.
\end{aligned} \tag{34}
$$

We define two sequences of auxiliary functions:

$$v^\varepsilon = \int_0^{(u^\varepsilon)^+} (\gamma\,(s)\,\sigma_\varepsilon(s) + (1-\sigma_\varepsilon(s))\beta(m-\varepsilon)ds \tag{35}$$

and

$$d^{\varepsilon} = \int_{0}^{(u^{\varepsilon})^{+}} (\beta(s)\,\sigma_{\varepsilon}(s) + (1-\sigma_{\varepsilon}(s))\beta(m-\varepsilon)ds. \tag{36}$$

For every $k \geq 0$ we have $T_k(v^{\varepsilon}) \in L^2(0,T;H_0^1(\Omega))$ and $T_k(d^{\varepsilon}) \in L^2(0,T;H_0^1(\Omega))$ with

$$\nabla T_k(v^{\varepsilon}) = \chi_{\{v^{\varepsilon}<k\}}\left[(\gamma(u^{\varepsilon})\,\sigma_{\varepsilon}(u^{\varepsilon}) + (1-\sigma_{\varepsilon}(u^{\varepsilon}))\beta(m-\varepsilon)\right]\nabla T_{k/\alpha}(u^{\varepsilon})^{+} \tag{37}$$

and

$$\nabla T_k(d^{\varepsilon}) = \chi_{\{d^{\varepsilon}<k\}}\left[(\beta(u^{\varepsilon})\,\sigma_{\varepsilon}(u^{\varepsilon}) + (1-\sigma_{\varepsilon}(u^{\varepsilon}))\beta(m-\varepsilon)\right]\nabla T_{k/\alpha}(u^{\varepsilon})^{+}. \tag{38}$$

Let us now take $S'(u^{\varepsilon}) = T_n(d^{\varepsilon} - (u^{\varepsilon})^{-})$ in (26). Then we have

$$\int_Q A^{\varepsilon}(t,x,u^{\varepsilon})Du^{\varepsilon}.T_n(d^{\varepsilon} - (u^{\varepsilon})^{-})dxdt \leq C. \tag{39}$$

Since the supports of d^{ε} and $(u^{\varepsilon})^{-}$ are disjoint, we deduce, using (38),

$$\sum_{i=1}^{N}\int_Q \chi_{\{d^{\varepsilon}<k\}}\left[\beta(u^{\varepsilon})\sigma_{\varepsilon}(u^{\varepsilon}) + (1-\sigma_{\varepsilon}(u^{\varepsilon}))\right]\left(A^{\varepsilon}(t,x,u^{\varepsilon})D\,(u^{\varepsilon})^{+}\right).DT_{\frac{n}{\alpha}}(u^{\varepsilon})^{+}dxdt$$
$$+\sum_{i=1}^{N}\int_Q \chi_{\{(u^{\varepsilon})^{-}<k\}}\sum_{j=1}^{N} A^{\varepsilon}(t,x,u^{\varepsilon})D\,(u^{\varepsilon})^{-}.DT_n(u^{\varepsilon})^{-}dxdt \leq C. \tag{40}$$

Now the definition (19) of A^{ε} together with assumptions (10) show that

$$(1-\sigma_{\varepsilon}(s))\beta(m-\varepsilon)\,|\xi|^2 + \beta(s)\sigma_{\varepsilon}(s)\,|\xi|^2 \leq A^{\varepsilon}(t,x,u^{\varepsilon})\xi_i.\xi_j$$

for any $s \in \mathbb{R}$, any $\xi \in \mathbb{R}^N$ and $a.e.$ in Q.

Then (19) and (40) yield

$$\int_Q |DT_n(d^{\varepsilon})|^2\,dxdt + \alpha\int_Q \left|DT_n((u^{\varepsilon})^{-})\right|^2 dxdt \leq C. \tag{41}$$

Since the supports of d^{ε} and $(u^{\varepsilon})^{-}$ are disjoint, we deduce

$$\min(1,\alpha)\int_Q \left|T_n(d^{\varepsilon} - (u^{\varepsilon})^{-})\right|^2 dxdt \tag{42}$$

$$\leq C.$$

Poincaré's inequality and (42) lead to

$$n^2 meas\left\{(t,x)\in Q;\ \left|d^\varepsilon-(u^\varepsilon)^-\right|>n\right\}=0,$$

where C does not depend on n and ε, and we obtain

$$\lim_{n\to+\infty}\sup_\varepsilon\left\{(t,x)\in Q;\ \left|d^\varepsilon-(u^\varepsilon)^-\right|>n\right\}=0. \tag{43}$$

To obtain the analog of (43) with

$$v^\varepsilon=d^\varepsilon+\int_0^{(u^\varepsilon)^+}(\gamma(s)-\beta(s)\sigma_\varepsilon(s)ds\leq d^\varepsilon+\int_0^m(\gamma(s)-\beta(s)\sigma_\varepsilon(s)ds, \tag{44}$$

where $\int_0^m(\gamma(s)-\beta(s))\sigma_\varepsilon(s)ds<+\infty$, from (9) and (43) we get

$$\lim_{n\to+\infty}\sup_\varepsilon\left\{(t,x)\in Q\ ;\ \left|v^\varepsilon-(u^\varepsilon)^-\right|>n\right\}=0. \tag{45}$$

Next, by (42) and the same procedures as above, we deduce

$$d^\varepsilon\to d\ a.e.\ \text{in}\ Q, \tag{46}$$

where d is a measurable function. Then, by (31), (44) and (46), we have

$$v^\varepsilon\to v\ a.e.\ \text{in}\ Q, \tag{47}$$

where $v=d+\int_0^{(u)^+}(\gamma(s)-\beta(s)\sigma_\varepsilon(s)ds$ and v is a measurable positive function. Consequently, by the definitions (11) of σ_ε and (35) of v^ε, the convergences (32) and (47), we have

$$v=\int_0^{(u)^+}\gamma(s)ds\ a.e.\ \text{in}\quad\{(x,t)\in Q\ ;\ u\,(x,t)<m\}. \tag{48}$$

However, as far as we know, we cannot expect to have a similar identification on the subset $\{(t,x)\in Q;\, u\,(t,x)=m\}$.
Now, we choose $S'(u^\varepsilon)=\theta_n\left(v^\varepsilon-(u^\varepsilon)^-\right)$ in (26). Then

$$\frac{1}{n}\int_{\left\{n\leq\left|v^\varepsilon-(u^\varepsilon)^-\right|\leq 2n\right\}}A^\varepsilon(t,x,u^\varepsilon)D\,(u^\varepsilon)\,.DT_n(v^\varepsilon-(u^\varepsilon)^-)dxdt\leq$$

$$\int_{\left\{\left|v^\varepsilon-(u^\varepsilon)^-\right|>n\right\}}|f_\varepsilon|\,dxdt++\int_\Omega\int_0^{|u_0|}\left|b'_\varepsilon(r)\theta_n\left(G^\varepsilon(r)-(r)^-\right)\right|drdx. \tag{49}$$

For the second term, we recall that the supports of $G^{\varepsilon}(r)$ and r^{-} are disjoint and, using $f_{\varepsilon} \in L^1(Q)$ and (45), we obtain

$$\lim_{n\to+\infty} \sup_{\varepsilon} \frac{1}{n} \int_{\left\{n\leq\left|v^{\varepsilon}-(u^{\varepsilon})^{-}\right|\leq 2n\right\}} A^{\varepsilon}(t,x,u^{\varepsilon})D\left(u^{\varepsilon}\right).DT_n(v^{\varepsilon}-(u^{\varepsilon})^{-})dxdt = 0. \quad (50)$$

Repeating the above argument with $S'(r) = \theta_n(r)$, we have

$$\lim_{n\to+\infty} \sup_{\varepsilon} \frac{1}{n} \int_{\{n\leq|u^{\varepsilon}|\leq 2n\}} A^{\varepsilon}(t,x,u^{\varepsilon})D\left(u^{\varepsilon}\right).DT_n(u^{\varepsilon})dxdt = 0. \quad (51)$$

To prove the existence of the weak limit of the field, we need to show that $A^{\varepsilon}(t,x,u^{\varepsilon})Du^{\varepsilon}$ is bounded in $L^2(Q)$ for every $i = 1, ..., N$ in the subset, where $v^{\varepsilon} - (u^{\varepsilon})^{-}$ is truncated. Indeed, we plug the test function $T_k(v^{\varepsilon})$ in (26), and using (37) we obtain

$$\int_{\{|v^{\varepsilon}|\leq k\}} A^{\varepsilon}(t,x,u^{\varepsilon})D\left(u^{\varepsilon}\right).D\left(u^{\varepsilon}\right)^{+}\left[\left(\gamma\left(u^{\varepsilon}\right)\sigma_{\varepsilon}(u^{\varepsilon}) + (1-\sigma_{\varepsilon}(s))\beta(m-\varepsilon)\right]dxdt \leq C.$$

By the definition (19) of $A^{\varepsilon}(x,s)$ and (10), we get

$$A^{\varepsilon}(t,x,s)\xi.\xi \leq \left(\gamma\left(s\right)\sigma_{\varepsilon}(s) + (1-\sigma_{\varepsilon}(s))\beta(m-\varepsilon)\left|\xi\right|^2 \quad (52)$$

for any $s \in \mathbb{R}$, any $\xi \in \mathbb{R}^N$ and *a.e.* in Ω. Using (52) with $\xi = X^{\varepsilon}(x,u^{\varepsilon})D\left(u^{\varepsilon}\right)^{+}$, we obtain

$$\int_{\{|v^{\varepsilon}|\leq k\}} \left|A^{\varepsilon}(t,x,s)D\left(u^{\varepsilon}\right)^{+}\right|^2 dxdt \leq C,$$

and then for any $k \geq 0$

$$\chi_{\{v^{\varepsilon}<k\}} A^{\varepsilon}(t,x,s)D\left(u^{\varepsilon}\right)^{+} \text{ is bounded in } L^2(Q),$$

uniformly in ε.

Now, since $\chi_{\left\{\left|v^{\varepsilon}-(u^{\varepsilon})^{-}\right|<k\right\}} = \chi_{\{0\leq v^{\varepsilon}<k\}} + \chi_{\{0\leq u^{\varepsilon}<k\}}$ *a.e.* in Ω using the continuous character of $A^{\varepsilon}(t,x,s)$ for $s \in (-\infty, 0]$ and the estimate (28), we have

$$\chi_{\{|v^{\varepsilon}-(u^{\varepsilon}))^{-}|<k\}} A^{\varepsilon}(t,x,u^{\varepsilon})D\left(u^{\varepsilon}\right)^{+} \text{ is bounded in } \left(L^2(Q)\right)^N \quad (53)$$

uniformly in ε.

Using the estimates (53) and (29), we extract another subsequence, still denoted by ε, such that

$$h_n(v^{\varepsilon} - (u^{\varepsilon})^{-})A^{\varepsilon}(t,x,u^{\varepsilon})D\left(u^{\varepsilon}\right) \to \psi_n \quad (54)$$

$$\text{weakly in } (L^2(Q))^N,$$

$$X^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon) \to Y_k$$
$$\text{weakly in } L^2(Q)$$

as ε tends to 0, where for any $k \geq 0$, $n \geq 1$, $\psi_n \in L^2(Q)$ and $Y_k \in L^2(Q)$.

Next we identify ψ_n on the subset where $u < m$. Let h be a $C^\infty(\mathbb{R})-$function such that $supp(h)$ is compact in $(-M, l)$ with $l < m$ and $M > 0$. Then, using the fact that $h(s)A^\varepsilon(t,x,u^\varepsilon) = h(s)A(t,x,T_l(s^+) - T_M(s^-))$ for ε small enough and the convergences (32) and (47), we have

$$h(u^\varepsilon)h_n(v^\varepsilon - (u^\varepsilon)^-)A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon \to h(u)h_n(v - u^-)A(t,x,u)Du$$
$$\text{weakly in } \left(L^2(Q)\right)^N \tag{55}$$

as ε tends to 0 and where Du stands for $DT_l(u^+) - DT_M(u^+)$. It follows from (55) and (54) that

$$\psi_n = h_n(v - u)\, A(t,x,u)Du \quad a.e. \text{ in } \quad \{(t,x) \in Q \; ; \; u\,(t,x) < m\}\,, \tag{56}$$

since $l < m$ and M are arbitrary.

Now remark that on the subset $\{(t,x) \in Q \; ; \; u\,(t,x) < m\}$, we have

$$0 \leq v = \int_0^{(u)^+} \gamma(s)ds < \int_0^m \gamma(s)ds,$$

and then for $n > \int_0^m \gamma(s)ds$, we have $h_n\,(v - u) = h_n\,(-u)$ on $\{(t,x) \in Q \; ; \; u\,(t,x) < m\}$. It follows from (56) that

$$\psi_n = h_n\,(-u)\, A(t,x,u)Du \text{ a.e.} in \;\; \{(t,x) \in Q \; ; \; u\,(t,x) < m\}\,,$$

which in turn implies that

$$\chi_{\{-k<u<m\}}A(t,x,u)Du \;\in \left(L^2(Q)\right)^N. \tag{57}$$

To identify Y_n, we use ψ_n defined above. We have for every $k \geq 0$

$$h_n(v^\varepsilon - (u^\varepsilon)^-)A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon) \to \psi_n^k \text{ weakly in } \left(L^2(Q)\right)^N.$$

We can write

$$h_n(v^\varepsilon - (u^\varepsilon)^-)X^\varepsilon(t,x,u^\varepsilon)T_k(u^\varepsilon) = h_n(v^\varepsilon - (u^\varepsilon)^-)\left(X^\varepsilon(t,x,u^\varepsilon)\right)^{-1} A^\varepsilon(t,x,u^\varepsilon)T_k(u^\varepsilon)$$

for some technique developed in ([4]). Then we deduce

$$Y_k = \chi_{\{u<m\}}X(t,x,u)T_k(u) \;\; a.e.\text{in } Q.$$

3.3. Step 3. Strong convergence of the field

Let $\xi \in C_0^\infty(0,T]$ be such that $0 \leq \xi \leq 1$. We choose $S'(r) = h_n(r)T_k(r)$ and $\xi = \varphi$ in (26):

$$\int_Q A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon).DT_k(u^\varepsilon)dxdt \leq \int_Q \xi_t \int_0^{\varrho_{u^\varepsilon}} S'(b^{-1}(r))drdxdt+$$
$$\int_\Omega \xi(0)\int_0^{b_0^\varepsilon} S'(b^{-1}(r))drdxdt$$
$$+\int_Q \xi f_\varepsilon h_n(u^\varepsilon)T_k(u^\varepsilon)dsdxdt + k\frac{1}{n}\int_{\{n<|u^\varepsilon|<2n\}} A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon).DT_k(u^\varepsilon)dtdx. \tag{58}$$

We pass to the limit as ε tends to 0 in (58) and using (30) and (31), we have

$$\limsup_{\varepsilon\to 0}\int_Q A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon).DT_k(u^\varepsilon)dxdt$$
$$\leq \int_Q \xi_t\int_0^{\varrho_u} h_n(b^{-1}(r))T_k(b^{-1}(r))drdxdt + \int_\Omega \xi(0)\int_0^{b_0} h_n(b^{-1}(r))T_k(b^{-1}(r))drdxdt$$
$$+\int_Q \xi f h_n(u)T_k(u)dsdxdt$$
$$+\limsup_{\varepsilon\to 0} k\frac{1}{n}\int_{\{n<|u^\varepsilon|<2n\}}\int_{\{n<|u^\varepsilon|<2n\}} A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon).DT_k(u^\varepsilon)dxdt.$$

Using (51), we have

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0} k\frac{1}{n}\int_{\{n<|u^\varepsilon|<2n\}} A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon).DT_k(u^\varepsilon)dxdt = 0. \tag{59}$$

Now using (59), we pass to the limit as n tends to $+\infty$ and we obtain

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_Q A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon).DT_k(u^\varepsilon)dxdt \leq$$
$$\int_Q \xi_t\int_0^{\varrho_u} T_k(b^{-1}(r))drdxdt + \int_Q \xi(0)\int_0^{b_0} T_k(b^{-1}(r))drdxdt+$$
$$\int_Q \xi f T_k(u)dxdt. \tag{60}$$

Now we use $S_n'(r) = h_n(G^\varepsilon(r^+) - r^-)$ in (26). Then we have

$$\begin{aligned}
&- \|\varphi\|_{L^\infty(\Omega)} \frac{1}{n} \int_{\{n<|v^\varepsilon-(u^\varepsilon)^-|<2n\}} A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon.D(v^\varepsilon - (u^\varepsilon)^-)dxdt \\
&\le - \int_Q \varphi_t \int_0^{\varrho_{u^\varepsilon}} S_n'(b^{-1}(s))dsdxdt - \int_Q \varphi(0) \int_0^{b_0^\varepsilon} S_n'(b^{-1}(s))dsdxdt \\
&- \int_Q \varphi f h_n(v^\varepsilon - (u^\varepsilon)^-)dxdt + \int_Q A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon.D\varphi h_n(v^\varepsilon - (u^\varepsilon)^-)dxdt \\
&\le \|\varphi\|_{L^\infty(\Omega)} \frac{1}{n} \int_{\{n<|v^\varepsilon-(u^\varepsilon)^-|<2n\}} A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon.D(v^\varepsilon - (u^\varepsilon)^-)dxdt. \qquad (61)
\end{aligned}$$

Now we need to pass to the limit in ε. First, we remark that for $n > \int_0^m \gamma(s)ds$

$$h_n(G^\varepsilon(s^+) - s^-) \to \chi_{\{s<0\}} h_n(-s^-) + \chi_{\{0<s<m\}} h_n(s^+) \qquad (62)$$

as ε tends to 0. As a consequence of (62), it follows that

$$\begin{aligned}
&\int_Q \varphi_t \int_0^{\varrho_{u^\varepsilon}} h_n(G^\varepsilon(\left(b^{-1}(s)\right)^+) - \left(b^{-1}(s)\right)^-)dsdxdt \\
&\quad \to \int_Q \varphi_t \left[\int_0^{-\varrho_u^-} h_n(b^{-1}(s))ds + T_m^+(\varrho_u) \right] dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\int_Q \varphi_t \int_0^{b_0^\varepsilon} h_n(G^\varepsilon(\left(b^{-1}(s)\right)^+) - \left(b^{-1}(s)\right)^-)dsdxdt \\
&\quad \to \int_\Omega \varphi(0) \left[\int_0^{-b_0^-} h_n(b^{-1}(s))ds + T_m^+(b_0) \right] dxdt,
\end{aligned}$$

as ε tends to 0. Secondly, from (56) we get

$$\int_Q A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon.D\varphi h_n(v^\varepsilon - (u^\varepsilon)^-)dxdt \to \int_Q \psi_n.D\varphi dxdt.$$

On the other hand, we use (56) and the inequalities

$$\begin{aligned}
\int_Q f_\varepsilon \varphi dxdt - \|\varphi\|_{L^\infty(\Omega)} \int_{\{|v^\varepsilon-(u^\varepsilon)^-|>n\}} |f_\varepsilon|\, dxdt &\le \int_Q \varphi f_\varepsilon h_n(v^\varepsilon - (u^\varepsilon)^-)dxdt \\
&\le \int_Q f_\varepsilon \varphi dxdt + \|\varphi\|_{L^\infty(\Omega)} \int_{\{|v^\varepsilon-(u^\varepsilon)^-|>n\}} |f_\varepsilon|\, dxdt.
\end{aligned}$$

Setting

$$\kappa_1(n)=\frac{1}{n}\sup_{\varepsilon}\frac{1}{n}\int_{\{n<|v^\varepsilon-(u^\varepsilon)^-|<2n\}}A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon.D(v^\varepsilon-(u^\varepsilon)^-)dxdt$$

and

$$\kappa_2(n)=\sup_{\varepsilon}\int_{\{|v^\varepsilon-(u^\varepsilon)^-|>n\}}|f_\varepsilon|\,dxdt,$$

we pass to the limit:

$$\begin{aligned}
-\|\varphi\|_{L^\infty(\Omega)}(\kappa_1(n)+\kappa_2(n))&\leq-\int_{\{u(t,x)=m\}}\psi_n.D\varphi dxdt\\
&-\int_\Omega\varphi(0)\left[\int_0^{-b_0^-}h_n(b^{-1}(s))ds+T_m^+(b_0)\right]dx\\
&+\int_Q\varphi_t\left[\int_0^{-\varrho_u^-}h_n(b^{-1}(s))ds+T_m^+(\varrho_u)\right]dxdt\\
&+\int_{u(t,x)<m}A(t,x,u)Du.D\varphi h_n(v-(u)^-)dxdt-\\
&-\int_Q f_\varepsilon\varphi dxdt\leq\|\varphi\|_{L^\infty(\Omega)}(\kappa_1(n)+\kappa_2(n)).
\end{aligned}$$

Now, let u_{0j} be a sequence of $C_0^\infty(\Omega)$ such that

$$u_{0j}\to u_0 \text{ strongly in } L^1(\Omega)$$

and

$$u(t)=u_{0j}\text{ for } t<0.$$

We choose $\varphi=\xi\frac{1}{h}\int_{t-h}^t T_k(u(\tau))d\tau$ as a test function in (26). Then

$$\begin{aligned}
-k(\kappa_1(n)+\kappa_2(n))\leq&\int_{\{u(t,x)=m\}}\psi_n.D(\frac{1}{h}\int_{t-h}^t T_k(u(\tau))d\tau)dxdt-\\
&\int_\Omega\varphi(0)\left[\int_0^{-b_0^-}h_n(b^{-1}(s))ds+T_m^+(b_0)\right]dxdt\\
-\int_Q\frac{\partial}{\partial t}(\xi\frac{1}{h}\int_{t-h}^t T_k(u(\tau))d\tau)&\left[\int_0^{-\varrho_u^-}h_n(b^{-1}(s))ds+T_m^+(\varrho_u)\right]dxdt \qquad (63)\\
\int_{u(t,x)<m}A(t,x,u)Du.D&\left(\frac{1}{h}\int_{t-h}^t T_k(u(\tau))d\tau\right)h_n(v-(u)^-)dxdt-
\end{aligned}$$

$$-\int_Q f\varphi dxdt \leq k(\kappa_1(n)+\kappa_2(n)).$$

To control the parabolic term in the previous inequality, we now apply Lemma 2.3 of [2] with $w = u$, $F(u) = u$, $B(r) = \int_0^{-r} h_n(b^{-1}(s))ds + T_m^+(r)$. Letting h tend to 0 in (61), we have

$$\limsup_{h\to 0} -\int_Q \frac{\partial}{\partial t}\left(\xi\frac{1}{h}\int_{t-h}^{t} T_k(u(\tau))\mathrm{d}\tau\right)$$

$$-\left[\int_0^{-\varrho_u^-} h_n(b^{-1}(s))ds + T_m^+(\varrho_u)\right]\mathrm{d}x\ \mathrm{d}t-$$

$$\int_\Omega \varphi(0)\left[\int_0^{-b_0^-} h_n(b^{-1}(s))ds + T_m^+(b_0)\right]\mathrm{d}x$$

$$\leq -\int_Q \xi_t\left[\left(\int_0^{-\varrho_u^-} h_n(b^{-1}(s))ds + T_m^+(\varrho_u)\right)T_k(u)\mathrm{d}r-\right.$$

$$\left.\int_0^u T_k'(r)\left(\int_0^{-r^-} h_n(b^{-1}(s))ds + T_m^+(r)\right)\mathrm{d}r\right]\mathrm{d}x\ \mathrm{d}t-$$

$$\int_\Omega \xi(0)\left[\left(\int_0^{-b_0^-} h_n(b^{-1}(s))ds + b(T_m^+(b_0)\right)T_k\left(u_{0j}\right)\mathrm{d}r\right.$$

$$\left.-\int_0^{u_{0j}} T_k'(r)\left(\int_0^{-r^-} h_n(b^{-1}(s))ds + T_m^+(r)\right)\mathrm{d}r\right]\mathrm{d}x.$$

We can easily prove that when h tends to 0, then $\frac{1}{h}\int_{t-h}^{t} T_k(u(\tau)d\tau \to T_k(u)$ strongly in $L^2(0,T;H_0^1(\Omega))$. From (63) and letting j go to infinity, we get

$$-k(\kappa_1(n)+\kappa_2(n))$$

$$\leq -\int_Q \xi_t\left[\left(\int_0^{-\varrho_u^-} h_n(b^{-1}(s))ds + T_m^+(\varrho_u)\right)T_k(u)\mathrm{d}r-\right.$$

$$\int_0^u T_k'(r)\left(\int_0^{-r^-} h_n(b^{-1}(s))ds + T_m^+(r)\right)\mathrm{d}r\mathrm{d}x\ \mathrm{d}t$$

$$-\int_\Omega \xi(0)\left[\left(\int_0^{-b_0^-} h_n(b^{-1}(s))ds + b(T_m^+(b_0)\right)T_k\left(u_{0j}\right)\mathrm{d}r-\right.$$

$$\int_0^{u_{0j}} T_k'(r)\left(\int_0^{-r^-} h_n(b^{-1}(s))ds + T_m^+(r)\right)\mathrm{d}r\Bigg]\,\mathrm{d}x$$

$$+\int_{\{u(t,x)=m\}} \psi_n.DT_k(u)dxdt+$$

$$\int_{u(t,x)<m} A(t,x,u)Du.DT_k(u)h_n(v-(u)^-)dxdt$$

$$-\int_Q f\xi T_k\,(u)\,dxdt.$$

Finally, let n go to infinity. Observe first that, by definition of $T_k\,(s)$, we have

$$\chi_{\{u=m\}}\psi_n.DT_k(u)dxdt = 0\ .$$

Thanks to (51) and (45), we have

$$\kappa_1(n)\to 0 \text{ and } \kappa_2(n)\to 0.$$

Since $h_n(s)\to 1$ for every $n>\int_0^m\gamma(s)ds$, the inequality (3.3) yields

$$\begin{aligned} &0\\ \le &-\int_Q \xi_t\left[\left(-\varrho_u^- + T_m^+(\varrho_u)\right)T_k(u)\mathrm{d}r - \int_0^u T_k'(r)\left(b(-r^-)+b(T_m^+(r))\right)\mathrm{d}r\right]\mathrm{d}x\ \mathrm{d}t\\ &-\int_\Omega \xi(0)\left[\left(-b_0^- + T_m^+(b_0)\right]T_k\,(u_0)\,\mathrm{d}r - \int_0^{u_{0j}} T_k'(r)\left(b(-r^-)+b(T_m^+(r))\right)\mathrm{d}r\right]\mathrm{d}x\\ &\int_{u(t,x)<m} A(t,x,u)Du.DT_k\,(u)\,dxdt - \int_Q f\xi T_k\,(u)\,dxdt. \end{aligned} \tag{64}$$

Now, remark that for every $s\le m$ we have

$$\begin{aligned} &\left(b(-r^-)+b(T_m^+(s))\right)T_k(s) - \int_0^{T_k(s)}\left(b(-r^-)+b(T_m^+(r))\right)\mathrm{d}r\\ &\qquad = \int_0^s T_k\left(b^{-1}(r)\right)dr. \end{aligned} \tag{65}$$

From (65) and putting together (64) and (60), we obtain

$$\limsup_{\varepsilon\to 0}\int_Q \xi A(t,x,u)Du.DT_k\,(u)\,dxdt \le \tag{66}$$

$$\int_{\{u(t,x)<m\}} \xi A(t,x,u)Du.DT_k\,(u)\,dxdt. \tag{67}$$

By Minty trick lemma, we conclude that for any $k \geq 0$ and any $0 < \tau < T$

$$\chi_{\{u^\varepsilon < m\}} X^\varepsilon(t,x,u^\varepsilon) Du^\varepsilon . DT_k(u^\varepsilon) \to \chi_{\{u<m\}} X(t,x,u) Du . DT_k(u) \quad \text{strongly in } L^2(0,\tau; H_0^1(\Omega)) \tag{68}$$

for every $i = 1, ..., N$. Remark that (68) implies

$$T_k(u^\varepsilon) \to T_k(u) \text{ strongly in } L^2(0,\tau; H_0^1(\Omega)). \tag{69}$$

3.4. Step 4. End of the proof

In this step, we prove that u is a renormalized solution in the sense of definition. It is easy to prove that u satisfies (13)-(15).

Firstly, we prove that u satisfies (18). Let $S \in W^{2,\infty}(\mathbb{R})$, and $supp(S') \subset (-L, L)$ be compact. Then we have

$$\begin{aligned} -\int_Q \varphi_t \int_0^{\varrho_{u^\varepsilon}} S'(b^{-1}(r)) dr dt dx - \int_\Omega \varphi(0) \int_0^{b_0^\varepsilon} S'(b^{-1}(r)) dr) dx \\ + \int_Q A^\varepsilon(t,x,u^\varepsilon) Du^\varepsilon . D\varphi S'(u^\varepsilon) dx dt \\ + \int_Q A^\varepsilon(t,x,u^\varepsilon) Du^\varepsilon . Du^\varepsilon S''(u^\varepsilon) dx dt \\ = \int_Q f S'(u^\varepsilon) \varphi dx dt. \end{aligned} \tag{70}$$

Now we take the limit as ε tends to 0 in (70).

***Limits of first and second terms in* (70)**

By *(30) and (22), we have*

$$\int_Q \varphi_t \int_0^{\varrho_{u^\varepsilon}} S'(b^{-1}(r)) dr dt dx \to \int_Q \varphi_t \int_0^{\varrho_u} S'(b^{-1}(r)) dr dt dx,$$

$$\int_\Omega \varphi(0) \int_0^{b_0^\varepsilon} S'(b^{-1}(r)) dr) dx \to \int_\Omega \varphi(0) \int_0^{b_0} S'(b^{-1}(r)) dr) dx.$$

Limits of second and third terms in (70)

Since $supp(S') \subset (-L, L)$, we can replace u^ε by $T_L(u^\varepsilon)$ in the second and third terms of (70). Then, due to (31) and (68), we have

$$S''(T_L(u^\varepsilon)) A^\varepsilon(t,x,u^\varepsilon) DT_L(u^\varepsilon) . DT_L(u^\varepsilon) \rightharpoonup S''(T_L(u)) A(t,x,u) DT_L(u) . DT_L(u) \text{ weakly in } L^1(Q).$$

In view of (31) and (69), we have

$$S^{'}(T_L(u^\varepsilon))A^\varepsilon(t,x,u^\varepsilon)DT_L(u^\varepsilon) \rightharpoonup$$
$$S'(T_L(u))A(t,x,u)DT_L(u) \quad \text{weakly in } L^2(Q,\omega_i^*)$$

for every $i = 1, ..., N$.

Limit of the right-hand side of (70)

Due to (31) and (21), we have

$$f_\varepsilon S(u^\varepsilon) \to fS(u) \text{ strongly in } L^1(Q).$$

Secondly, we prove that u satisfies (16). We choose $S'(r) = \theta_p(-r^-)$ in (26) for a fixed integer $p \geq 1$ and we do the same procedure as in Step 2. Then we obtain

$$\lim_{p\to+\infty} \frac{1}{p} \int_{\{-2p<u<-p\}} A(t,x,u)Du.Dudxdt = 0.$$

Finally, to etablish (17), we take $S'(r) = (1 - \sigma_{1/p}(r^+))$, where p is a fixed integer ≥ 1, and for any $\varphi \in C_c^\infty([0,T])$ in (26), we have

$$- \int_Q \varphi_t \int_0^{\varrho_u} S'(b^{-1}(r))drdtdx - \int_\Omega \varphi(0) \int_0^{b_0} S'(b^{-1}(r))dr)dx$$
$$+p \int_Q \chi_{\{m-2/p<u<m-1/p\}} A(t,x,u)Du.Du\varphi dxdt$$
$$= \int_Q f_\varepsilon(1 - \sigma_{1/p}(u^+))\varphi dxdt.$$

Now, as p tends to $+\infty$, $(1 - \sigma_{1/p}(u^+)) \to \chi_{\{u=m\}}$ *a.e.* in Q, we have

$$\lim_{p\to+\infty} p \int_Q \chi_{\{m-2/p<u<m-1/p\}} A(t,x,u)Du.Du\varphi dxdt$$
$$= \int_Q f\chi_{\{u=m\}}\varphi dxdt,$$

which is (17).

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