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Spectral Analysis for the Almost Periodic Sturm–Liouville Operator with Impulse

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Abstract. This paper focuses on the impulsive Sturm–Liouville operator on the whole axis with complex almost periodic potentials and the discontinuous coefficient on the right–hand side. We study the most important characteristics of the fundamental solutions of the Sturm–Liouville equation. Using the impulsive condition, we determine the transfer matrix. Further, using the impulsive condition and transfer matrix, we construct Green's function and obtain the resolvent of the impulsive Sturm–Liouville operator. Also, we investigate the eigenvalues of the impulsive Sturm–Liouville operator.

Key Words and Phrases: impulsive conditions, Sturm–Liouville operators, spectral singularities, almost periodic functions.

2010 Mathematics Subject Classifications: 34B10, 34B15

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1. Introduction

In this work, we consider the Sturm–Liouville equation on the whole axis:

$$-y'' + q(x)\,y = \lambda^2 \rho(x)\,y, \quad x \in (-\infty, 0) \cup (0, \infty) \tag{1}$$

with the impulsive condition

$$\begin{bmatrix} y(0^+) \\ y'(0^-) \end{bmatrix} = B \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}, \quad B = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}, \tag{2}$$

where α_1, α_2, α_3, α_4 are complex numbers such that $\det B \neq 0$,

$$q(x) = \sum_{n=1}^{\infty} q_n e^{i\Lambda_n x} \tag{3}$$

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and the condition

$$\sum_{n=1}^{\infty} |q_n| < \infty \tag{4}$$

is satisfied.

The set of exponents is a countable set of positive real numbers closed with respect to the addition

$$M = \{\Lambda_1,\, \Lambda_2,\, \Lambda_3,\, ...\Lambda_n, ...\}\,,\ \ \Lambda_n > 0,\ n \in N. \tag{5}$$

The density function $\rho\,(x)$ has the form

$$\rho\,(x) = \begin{cases} 1, & x < 0 \\ \beta^2, & x > 0 \end{cases} \qquad \beta > 0,\, \beta \neq 1. \tag{6}$$

In mathematical physics and quantum mechanics, boundary–value problems with discontinuities inside an interval are of great interest. Additional conditions, such as impulsive conditions, are imposed at the points of discontinuity to address these inner discontinuities. The theory of impulsive differential equations was studied in detail in applied mathematics by Bainov and Simeonov in 1995 [1]. Many authors have also studied the spectral theory of impulsive differential equations. Recently, the physical meaning and potential applications of spectral singularities of impulsive differential equations have been introduced and studied by Mostafazadeh [2], who provided the physical meanings of eigenvalues and spectral singularities of the Schrödinger equation at a single point.

In this work, we are concerned with the impulsive Sturm-Liouville operator on the whole axis. We investigate the fundamental solutions to this problem, analyze the spectrum, and solve the inverse problem.

A similar problem where discontinuity conditions (2) are absent, i.e., $\alpha_1 = \alpha_4 = 1,\ \alpha_2 = \alpha_3 = 0$, was completely solved for $\Lambda_n = n$ in [3], and considered for almost periodic coefficients in [4]. Also, some aspects of direct and inverse problems for differential operators with real-valued functions $q\,(x)$ satisfying the condition $\int\limits_{-\infty}^{\infty} (1 + |x|)\, |q\,(x)|\, dx < \infty$, with $\rho\,(x) = 1,$ were considered in [5].

A similar impulse problem for differential operators with real-valued potential function $q\,(x)$ with $\int\limits_{0}^{\infty} x\, |q\,(x)|\, dx < \infty$ on the half–axis was considered in [6].

Finally, we note that an operator generated by a finite sum in (3) $\rho\,(x) \equiv 1$ was studied by P. Sarnak [7].

Inverse problems of differential equations with various effects are the subject of numerous works [8]–[29].

Further, we will write $n \gg m$ or $n \ll m$ if $\alpha_n > \alpha_m$ or $\alpha_n < \alpha_m$, respectively. The symbol $\sum\limits_{n:n>m}$ will be used for summing over all n such that $\alpha_n > \alpha_m$. We also will use $n \oplus m = \gamma$ if $\alpha_n + \alpha_m = \alpha_\gamma$. For any λ_0 the limit

$$\angle \lim_{\lambda\to\lambda_0} f(x,\lambda)(\lambda-\lambda_0) = \begin{cases} 0,\ \lambda \notin M \\ f_n(x),\ \lambda_0 = \alpha_n, n \in N \end{cases}$$

exists and is uniform with respect to x.

In the sequel, the symbol $\angle \lim$ means the limit in a non-tangent direction as λ approaches λ_0 in a manner such that $\delta < \arg(\lambda-\lambda_0) < \pi - \delta$for any fixed $\delta > 0$.

2. Particular solutions to the equation $Ly = \lambda^2 y$

In this section, we study the solutions of the main equation (1) that will be needed later. We can prove the existence of these solutions if condition (4) is fulfilled for the potential. This will be a unique restriction on the potential, and later on, we will consider it to be fulfilled.

Theorem 1. *The problem* (1)–(2) *with the potential* $q(x)$ *of the form* (3) *and* $\rho(x)$ *defined by* (6) *has fundamental solutions of the form*

$$f_1^{\pm}(x,\lambda) = e^{\pm i\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{\Lambda_n \pm 2\lambda} e^{i\Lambda_\alpha x}\right) \qquad for \quad x < 0,$$

satisfying the asymptotic condition

$$\lim_{\operatorname{Im} x\to-\infty} f_1^{\pm}(x,\lambda)\, e^{\mp i\lambda x} = 1 \qquad for \quad \pm \operatorname{Im}\lambda > 0,$$

and

$$f_2^{\pm}(x,\lambda) = e^{\pm i\beta\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{\Lambda_n \pm 2\beta\lambda} e^{i\Lambda_\alpha x}\right) \qquad for \quad x > 0,$$

satisfying the asymptotic condition

$$\lim_{\operatorname{Im} x\to+\infty} f_2^{\pm}(x,\lambda)\, e^{\mp i\beta\lambda x} = 1 \qquad for \quad \pm \operatorname{Im}\lambda > 0.$$

Here the numbers $V_{n\alpha}$ are defined by the relations

$$\Lambda_\alpha(\Lambda_\alpha - \Lambda_n) V_{n\alpha} + \sum_{\beta\oplus\gamma=n} V_{n\beta} q_\gamma = 0,$$

$$q_\alpha + \sum_{\beta \oplus \gamma = n} V_{n\beta} q_\gamma = 0,$$

and the series
$$\sum_{n=1}^{\infty} \Lambda_n^{-1} \sum_{\alpha=n}^{\infty} \Lambda_\alpha \left(\Lambda_\alpha - \Lambda_n\right) \left|V_{n\alpha}\right|$$

converges. We can easily see that at the points $\lambda = \mp \frac{\Lambda_n}{2} \left(\mp \frac{\Lambda_n}{2\beta}\right)$, $n \in N$ there can be simple poles of the function $f\left(x, \lambda\right)$.

Remark 1. *If* $\lambda \neq -\frac{\Lambda_n}{2}$ *and* $\operatorname{Im} \lambda < 0$, *then* $f_1^+\left(x, \lambda\right) \in L_2\left(-\infty, 0\right)$.

Remark 2. *If* $\lambda \neq -\frac{\Lambda_n}{2\beta}$ *and* $\operatorname{Im} \lambda > 0$, *then* $f_2^+\left(x, \lambda\right) \in L_2\left(0, \infty\right)$. *Taking into account that the potential* $q\left(x\right)$ *can be extended to the upper semi–plane as an analytic function, we find*

$$W\left[f_1^+\left(x,\lambda\right), f_1^-\left(x,\lambda\right)\right] = -2i\lambda \quad for \;\; \lambda \neq 0, \pm \frac{\Lambda_n}{2},$$

$$W\left[f_2^+\left(x,\lambda\right), f_2^-\left(x,\lambda\right)\right] = -2i\beta\lambda \quad for \;\; \lambda \neq 0, \pm \frac{\Lambda_n}{2\beta}.$$

Therefore, the functions $f_1^+\left(x,\lambda\right), f_1^-\left(x,\lambda\right) \left(f_2^+\left(x,\lambda\right), f_2^-\left(x,\lambda\right)\right)$ *are linearly independent solutions of the equation (1) for* $\lambda \neq 0, \pm\frac{\Lambda_n}{2}, \pm\frac{\Lambda_n}{2\beta}$. *Since*

$$W\left[f_n^\pm\left(x\right), f^\mp\left(x, \mp\frac{\Lambda_n}{2\beta}\right)\right] = 0,$$

we obtain
$$f_n^\pm\left(x\right) = S_n^\pm f^\mp\left(x, \mp\frac{\Lambda_n}{2\beta}\right). \tag{7}$$

If we compare these relations, we will see that

$$S_n^\pm = V_{nn}^\pm.$$

By simple calculations, we can obtain the following relation from (7) for the derivatives of the considered functions:

$$f_n^{\pm\prime}\left(x\right) = S_n^\pm f^{\mp\prime}\left(x, \mp\frac{\Lambda_n}{2\beta}\right).$$

Using linearly independent solutions of (1) in the intervals $(-\infty, 0)$ and $(0, \infty)$, we can express the general solution of (1) as follows:

$$\begin{cases} y_-\left(x,\lambda\right) = A_- f_1^+\left(x,\lambda\right) + B_- f_1^-\left(x,\lambda\right), & x < 0, \\ y_+\left(x,\lambda\right) = A_+ f_2^+\left(x,\lambda\right) + B_+ f_2^-\left(x,\lambda\right), & x > 0, \end{cases}$$

where $A_{\pm}$ and $B_{\pm}$ are constant coefficients depending on λ.

Using impulsive condition (2), we easily get the following system of linear equations:

$$\begin{cases} A_+ f_2^+(0,\lambda) + B_+ f_2^-(0,\lambda) = \alpha_1 \left(A_- f_1^+(0,\lambda) + B_- f_1^-(0,\lambda)\right) + \\ +\alpha_2 \left(A_- f_1^{+\prime}(0,\lambda) + B_- f_1^{-\prime}(0,\lambda)\right) \\ A_+ f_2^{+\prime}(0,\lambda) + B_+ f_2^{-\prime}(0,\lambda) = \alpha_3 \left(A_- f_1^+(0,\lambda) + B_- f_1^-(0,\lambda)\right) + \\ +\alpha_4 \left(A_- f_1^{+\prime}(0,\lambda) + B_- f_1^{-\prime}(0,\lambda)\right). \end{cases} \tag{8}$$

By solving (8) and after making simplifications, we get

$$\begin{gathered} A_+ = \frac{\left[f_1^+(0,\lambda)\left(\alpha_1 f_2^{-\prime}(0,\lambda) - \alpha_3 f_2^-(0,\lambda)\right) + f_1^{+\prime}(0,\lambda)\left(\alpha_2 f_2^{-\prime}(0,\lambda) - \alpha_4 f_2^-(0,\lambda)\right)\right] A_-}{f_2^+(0,\lambda) f_2^{-\prime}(0,\lambda) - f_2^{+\prime}(0,\lambda) f_2^-(0,\lambda)} + \\ + \frac{\left[f_1^-(0,\lambda)\left(\alpha_1 f_2^{-\prime}(0,\lambda) - \alpha_3 f_2^-(0,\lambda)\right) + f_1^{-\prime}(0,\lambda)\left(\alpha_2 f_2^{-\prime}(0,\lambda) - \alpha_4 f_2^-(0,\lambda)\right)\right] B_-}{f_2^+(0,\lambda) f_2^{-\prime}(0,\lambda) - f_2^{+\prime}(0,\lambda) f_2^-(0,\lambda)}, \end{gathered} \tag{9}$$

$$\begin{gathered} B_+ = \frac{\left[f_1^+(0,\lambda)\left(\alpha_1 f_2^{+\prime}(0,\lambda) - \alpha_3 f_2^+(0,\lambda)\right) + f_1^{+\prime}(0,\lambda)\left(\alpha_2 f_2^{+\prime}(0,\lambda) - \alpha_4 f_2^+(0,\lambda)\right)\right] A_-}{f_2^-(0,\lambda) f_2^{+\prime}(0,\lambda) - f_2^{-\prime}(0,\lambda) f_2^+(0,\lambda)} + \\ + \frac{\left[f_1^-(0,\lambda)\left(\alpha_1 f_2^{+\prime}(0,\lambda) - \alpha_3 f_2^+(0,\lambda)\right) + f_1^{-\prime}(0,\lambda)\left(\alpha_2 f_2^{+\prime}(0,\lambda) - \alpha_4 f_2^+(0,\lambda)\right)\right] B_-}{f_2^-(0,\lambda) f_2^{+\prime}(0,\lambda) - f_2^{-\prime}(0,\lambda) f_2^+(0,\lambda)}. \end{gathered} \tag{10}$$

If we rewrite (9)–(10) in matrix form, then we get

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \frac{1}{-2i\beta\lambda} \begin{bmatrix} f_2^{-\prime}(0,\lambda) & -f_2^-(0,\lambda) \\ -f_2^{+\prime}(0,\lambda) & f_2^+(0,\lambda) \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \begin{bmatrix} f_1^+(0,\lambda) & f_1^-(0,\lambda) \\ f_1^{+\prime}(0,\lambda) & f_1^{-\prime}(0,\lambda) \end{bmatrix} \begin{bmatrix} A_- \\ B_- \end{bmatrix}.$$

From the impulsive condition (2), we have a transfer matrix M satisfying

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = M \begin{bmatrix} A_- \\ B_- \end{bmatrix},$$

where

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = N^{-1} B D \tag{11}$$

with

$$D := \begin{bmatrix} f_1^+(0,\lambda) & f_1^-(0,\lambda) \\ f_1^{+\prime}(0,\lambda) & f_1^{-\prime}(0,\lambda) \end{bmatrix},$$

and

$$N := \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},$$

with

$$N_{11} = \left(1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{\Lambda_n + 2\lambda}\right),$$

$$N_{12}=\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n-2\lambda}\right),$$

$$N_{21}=i\lambda\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n+2\lambda}\right)+\left(\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{i\Lambda_\alpha V_{n\alpha}}{\Lambda_n+2\lambda}\right),$$

$$N_{22}=-i\lambda\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n-2\lambda}\right)+\left(\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{i\Lambda_\alpha V_{n\alpha}}{\Lambda_n-2\lambda}\right).$$

We can easily find that $\det N=-2i\beta\lambda$, and $\det N^{-1}=\frac{1}{-2i\beta\lambda}$. Then we obtain

$$M_{22}(\lambda)=\frac{i}{2\beta\lambda}\left\{-f_2^{+\prime}(0,\lambda)\left[\alpha_1 f_1^-(0,\lambda)+\alpha_2 f_1^{-\prime}(0,\lambda)\right]+\right.$$
$$\left.+f_2^+(0,\lambda)\left[\alpha_3 f_1^-(0,\lambda)+\alpha_4 f_1^{-\prime}(0,\lambda)\right]\right\},$$

$$M_{12}(\lambda)=\frac{i}{2\beta\lambda}\left\{f_2^{-\prime}(0,\lambda)\left[\alpha_1 f_1^-(0,\lambda)+\alpha_2 f_1^{-\prime}(0,\lambda)\right]-\right.$$
$$\left.-f_2^-(0,\lambda)\left[\alpha_3 f_1^-(0,\lambda)+\alpha_4 f_1^{-\prime}(0,\lambda)\right]\right\},$$

$$M_{21}(\lambda)=\frac{i}{2\beta\lambda}\left\{-f_2^{+\prime}(0,\lambda)\left[\alpha_1 f_1^+(0,\lambda)+\alpha_2 f_1^{+\prime}(0,\lambda)\right]+\right.$$
$$\left.+f_2^+(0,\lambda)\left[\alpha_3 f_1^+(0,\lambda)+\alpha_4 f_1^{+\prime}(0,\lambda)\right]\right\},$$

$$M_{11}(\lambda)=\frac{i}{2\beta\lambda}\left\{f_2^{-\prime}(0,\lambda)\left[\alpha_1 f_1^+(0,\lambda)+\alpha_2 f_1^{+\prime}(0,\lambda)\right]-\right.$$
$$\left.-f_2^-(0,\lambda)\left[\alpha_3 f_1^+(0,\lambda)+\alpha_4 f_1^{+\prime}(0,\lambda)\right]\right\}.$$

Now let us consider two solutions to the problem (1)-(6):

$$F(x,\lambda)=\begin{cases}A_-^+f_1^+(x,\lambda)+B_-^+f_1^-(x,\lambda), & -\infty<x<0,\\ A_+^+f_2^+(x,\lambda)+B_+^+f_2^-(x,\lambda), & 0<x<\infty,\end{cases}$$

and

$$G(x,\lambda)=\begin{cases}A_-^-f_1^+(x,\lambda)+B_-^-f_1^-(x,\lambda), & -\infty<x<0,\\ A_+^-f_2^+(x,\lambda)+B_+^-f_2^-(x,\lambda)\ , & 0<x<\infty.\end{cases}$$

From the asymptotic relations

$$\lim_{\operatorname{Im}x\to\infty}f_2^+(x,\lambda)\,e^{\mp i\beta\lambda x}=1\qquad for\quad \pm\operatorname{Im}\lambda>0,$$

$$\lim_{\operatorname{Im} x\to-\infty} f_1^{\pm}(x,\lambda)\,e^{\mp i\lambda x}=1 \qquad for \quad \pm\operatorname{Im}\lambda>0,$$

we find

$$A_+^+=1,\ B_+^+=0,\ A_-^-=0,\ B_-^-=1.$$

Next, using the equation (11), we obtain the following matrix equations:

$$\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}A_-^+\\ B_-^+\end{bmatrix},$$

$$\begin{bmatrix}A_+^-\\ B_+^-\end{bmatrix}=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}A_-^-\\ B_-^-\end{bmatrix}.$$

By solving these matrix equations, we obtain

$$A_-^+=\frac{M_{22}(\lambda)}{\det M},B_-^+=-\frac{M_{21}(\lambda)}{\det M},A_+^-=M_{12}(\lambda),B_+^-=M_{22}(\lambda).$$

Then for the solutions $F(x,\lambda)$ and $G(x,\lambda)$ we have the following relations:

$$F(x,\lambda)=\begin{cases}\frac{M_{22}(\lambda)}{\det M}f_1^+(x,\lambda)-\frac{M_{21}(\lambda)}{\det M}f_1^-(x,\lambda), & -\infty<x<0,\\ f_2^+(x,\lambda) & ,\ \ 0<x<\infty,\end{cases}$$

and

$$G(x,\lambda)=\begin{cases}f_1^+(x,\lambda), & -\infty<x<0,\\ M_{12}(\lambda)f_2^+(x,\lambda)+M_{22}(\lambda)f_2^-(x,\lambda), & 0<x<\infty.\end{cases}$$

Lemma 1. *The Wronskian of the solutions $F(x,\lambda)$ and $G(x,\lambda)$ is determined as follows:*

$$W[F(x,\lambda),G(x,\lambda)]=\frac{M_{22}(\lambda)}{\det M} \qquad on \quad -\infty<x<0,$$

$$W[F(x,\lambda),G(x,\lambda)]=-2i\lambda M_{22}(\lambda) \qquad on \quad 0<x<\infty.$$

3. Spectrum of the operator L

To study the spectrum of equations (1)-(4), we have to construct its resolvent. Let us consider the nonhomogeneous differential equation

$$-y''+q(x)y=\lambda^2\rho(x)y-f(x),\ \ x\in(-\infty,0)\cup(0,\infty) \qquad (12)$$

together with the conditions (2)–(6).

We can represent the general solution of a homogeneous differential equation corresponding to equation (12) in the form

$$U(x,\lambda)=\begin{cases} C_1 f_1^+(x,\lambda)\ +D_1 f_1^-(x,\lambda) & for\ \ -\infty<x<0, \\ C_2 f_2^+(x,\lambda)\ +D_2 f_2^-(x,\lambda) & for\ \ 0<x<\infty, \end{cases}$$

where C_1, D_1, C_2 and D_2 are arbitrary constants.

By applying the standard method of variation of parameters, we will search the general solution of the non-homogeneous linear differential equation ((12) in the form

$$U(x,\lambda)=\begin{cases} C_1(x,\lambda) f_1^+(x,\lambda)\ +D_1(x,\lambda) f_1^-(x,\lambda) & for\ \ -\infty<x<0, \\ C_2(x,\lambda) f_2^+(x,\lambda)\ +D_2(x,\lambda) f_2^-(x,\lambda) & for\ \ 0<x<\infty, \end{cases} \tag{13}$$

where the functions $C_1(x,\lambda)$, $D_1(x,\lambda)$ and $C_2(x,\lambda)$, $D_2(x,\lambda)$ satisfy the linear system of equations

$$\begin{cases} C_1'(x,\lambda) f_1^+(x,\lambda)+D_1'(x,\lambda) f_1^-(x,\lambda)=0 \\ C_1'(x,\lambda) f_1^{+\prime}(x,\lambda)+D_1'(x,\lambda) f_1^{-\prime}(x,\lambda)=f(x) \end{cases} \tag{14}$$

and

$$\begin{cases} C_2'(x,\lambda) f_2^+(x,\lambda)+D_2'(x,\lambda) f_2^-(x,\lambda)=0 \\ C_2'(x,\lambda) f_2^{+\prime}(x,\lambda)+D_2'(x,\lambda) f_2^{-\prime}(x,\lambda)=f(x), \end{cases} \tag{15}$$

respectively. Since the Wronskians of the solutions are

$$w_1(\lambda)=\left[f_1^+(x,\lambda), f_1^-(x,\lambda)\right]=-2i\lambda,$$

and

$$w_2(\lambda)=\left[f_2^+(x,\lambda), f_2^-(x,\lambda)\right]=-2i\beta\lambda,$$

each of the linear systems of equations (14) and (15) has a unique solution. These solutions can be expressed as

$$C_1{}'(x,\lambda)=-\frac{1}{w_1(\lambda)} f_1^-(x,\lambda) f(x), \tag{16}$$

$$D_1{}'(x,\lambda)=\frac{1}{w_1(\lambda)} f_1^+(x,\lambda) f(x) \tag{17}$$

for $x\in(-\infty,0)$ and

$$C_2{}'(x,\lambda)=-\frac{1}{w_2(\lambda)} f_2^-(x,\lambda) f(x), \tag{18}$$

$$D_2{}'(x,\lambda)=\frac{1}{w_2(\lambda)} f_2^+(x,\lambda) f(x) \tag{19}$$

for $x \in (0, \infty)$, respectively. From equations (16)–(19), we obtain the following relations:

$$C_1(x,\lambda) = -\frac{1}{w_1(\lambda)} \int\limits_{-\infty}^{x} f_1^-(t,\lambda) f(t)\,dt + C_1, \qquad x \in (-\infty, 0),$$

$$D_1(x,\lambda) = -\frac{1}{w_1(\lambda)} \int\limits_{x}^{0} f_1^+(t,\lambda) f(t)\,dt + D_1, \qquad x \in (-\infty, 0),$$

$$C_2(x,\lambda) = -\frac{1}{w_2(\lambda)} \int\limits_{0}^{x} f_2^-(t,\lambda) f(t)\,dt + C_2, \qquad x \in (0, \infty),$$

$$D_2(x,\lambda) = -\frac{1}{w_2(\lambda)} \int\limits_{\infty}^{x} f_2^+(t,\lambda) f(t)\,dt + D_2, \qquad x \in (0, \infty),$$

where C_1, D_1, C_2 and D_2 are arbitrary constants. Substituting the above equations in (13), the general solution of a non-homogeneous linear differential equation (12) is obtained as

$$\begin{gathered} U_-(x,\lambda) = -\tfrac{f_1^+(x,\lambda)}{w_1(\lambda)} \int\limits_{-\infty}^{x} f_1^-(t,\lambda) f(t)\,dt + C_1 f_1^+(x,\lambda) - \\ -\tfrac{f_1^-(x,\lambda)}{w_1(\lambda)} \int\limits_{x}^{0} f_1^+(t,\lambda) f(t)\,dt + D_1 f_1^-(x,\lambda) \end{gathered} \tag{20}$$

for $-\infty < x < 0$ and

$$\begin{gathered} U_+(x,\lambda) = -\tfrac{f_2^+(x,\lambda)}{w_2(\lambda)} \int\limits_{0}^{x} f_2^-(t,\lambda) f(t)\,dt + C_2 f_2^+(x,\lambda) - \\ -\tfrac{f_2^-(x,\lambda)}{w_2(\lambda)} \int\limits_{x}^{\infty} f_2^+(t,\lambda) f(t)\,dt + D_2 f_2^-(x,\lambda) \end{gathered} \tag{21}$$

for $0 < x < \infty$.

Since $U_-(x,\lambda) \in L_2(-\infty, 0)$ and $U_+(x,\lambda) \in L_2(0, \infty)$, we have $C_1 = 0$ and $D_2 = 0$.

Now by using the impulsive condition (2), let's find C_2 and D_1:

$$\begin{gathered} -\tfrac{f_2^-(0,\lambda)}{w_2(\lambda)} \int\limits_{0}^{\infty} f_2^+(t,\lambda) f(t)\,dt + C_2 f_2^+(0,\lambda) = \\ = \alpha_1 \left[-\tfrac{f_1^+(0,\lambda)}{w_1(\lambda)} \int\limits_{-\infty}^{0} f_1^-(t,\lambda) f(t)\,dt + D_1 f_1^-(0,\lambda) \right] + \\ \alpha_2 \left[-\tfrac{f_1^{+\prime}(0,\lambda)}{w_1(\lambda)} \int\limits_{-\infty}^{0} f_1^-(t,\lambda) f(t)\,dt + D_1 f_1^{-\prime}(0,\lambda) \right] \end{gathered}$$

$$
\begin{aligned}
&-\tfrac{f_2^{-\prime}(0,\lambda)}{w_2(\lambda)}\int\limits_0^{\infty} f_2^+(t,\lambda) f(t)\,dt + C_2 f_2^{+\prime}(0,\lambda) = \\
&= \alpha_3 \left[-\tfrac{f_1^+(0,\lambda)}{w_1(\lambda)}\int\limits_{-\infty}^{0} f_1^-(t,\lambda)\, f(t)\,dt + D_1 f_1^-(0,\lambda)\right] + \\
&+\alpha_4 \left[-\tfrac{f_1^{+\prime}(0,\lambda)}{w_1(\lambda)}\int\limits_{-\infty}^{0} f_1^-(t,\lambda)\, f(t)\,dt + D_1 f_1^{-\prime}(0,\lambda)\right].
\end{aligned}
$$

By solving the system of equations above, we obtain the following result:

$$
D_1 = \frac{M_{21}(\lambda)\int\limits_{-\infty}^{0} f_1^-(t,\lambda)\, f(t)\,dt - \frac{i}{2\beta\lambda}\int\limits_0^{\infty} f_2^+(t,\lambda)\, f(t)\,dt}{M_{22}(\lambda)},
$$

$$
C_2 = \frac{-M_{12}(\lambda)\int\limits_0^{\infty} f_2^+(t,\lambda)\, f(t)\,dt - \frac{i}{2\beta\lambda}\int\limits_{-\infty}^{0} f_1^-(t,\lambda)\, f(t)\,dt}{M_{22}(\lambda)}.
$$

Finally, by substituting the coefficients C_i and D_i $(i = 1,2)$ in (20) and (21), we obtain the following formula for the resolvent $U(x,\lambda)$:

$$
U(x,\lambda) = \begin{cases}
-\frac{f_1^+(x,\lambda)}{w_1(\lambda)}\int_{-\infty}^{x} f_1^-(t,\lambda)f(t)dt - \frac{f_1^-(x,\lambda)}{w_1(\lambda)}\int_x^0 f_1^+(t,\lambda)f(t)dt + \\
+\frac{M_{21}(\lambda)}{M_{22}(\lambda)} f_1^-(x,\lambda)\int_{-\infty}^{0} f_1^-(t,\lambda)f(t)dt - \\
-\frac{f_1^-(x,\lambda)}{M_{22}(\lambda)}\frac{i}{2\beta\lambda}\int_0^{\infty} f_2^+(t,\lambda)f(t)dt, \quad x \in (-\infty,0), \\
-\frac{f_2^+(x,\lambda)}{w_2(\lambda)}\int_0^{x} f_2^-(t,\lambda)f(t)dt - \frac{f_2^-(x,\lambda)}{w_2(\lambda)}\int_x^{\infty} f_2^+(t,\lambda)f(t)dt \\
-\frac{M_{12}(\lambda)}{M_{22}(\lambda)} f_2^+(x,\lambda)\int_0^{\infty} f_2^+(t,\lambda)f(t)dt - \\
-\frac{f_2^+(x,\lambda)}{M_{22}(\lambda)}\frac{i}{2\beta\lambda}\int_{-\infty}^{0} f_1^-(t,\lambda)f(t)dt, \quad x \in (0,\infty).
\end{cases}
$$

Furthermore, by using the representations

$$
f^+(x,\lambda) = \begin{cases} f_1^+(x,\lambda) & for\ x \in (-\infty,0), \\ f_2^+(x,\lambda) & for\ x \in (0,\infty), \end{cases}
$$

$$
f^-(x,\lambda) = \begin{cases} f_1^-(x,\lambda) & for\ x \in (-\infty,0), \\ f_2^-(x,\lambda) & for\ x \in (0,\infty), \end{cases}
$$

this formula can be rewritten in the form

$$U(x,\lambda) = -\tfrac{i f^{+}(x,\lambda)}{2\beta\lambda M_{22}(\lambda)} \int\limits_{-\infty}^{x} f^{-}(t,\lambda) f(t)\,dt - \tfrac{i f^{-}(x,\lambda)}{2\beta\lambda M_{22}(\lambda)} \int\limits_{x}^{\infty} f^{+}(t,\lambda) f(t)\,dt +$$
$$+\begin{cases} \tfrac{M_{21}(\lambda)}{M_{22}(\lambda)} f_1^{-}(x,\lambda) \int\limits_{-\infty}^{0} f_1^{-}(t,\lambda)\, f(t)\,dt & x\in(-\infty,0), \\ -\tfrac{M_{12}(\lambda)}{M_{22}(\lambda)} f_2^{+}(x,\lambda) \int\limits_{0}^{\infty} f_2^{+}(t,\lambda)\, f(t)\,dt & x\in(0,\infty). \end{cases} \tag{22}$$

Thus, the resolvent of the boundary value impulsive problem is obtained.

Let us denote

$$G(x,t,\lambda) = \begin{cases} -\tfrac{i}{2\beta\lambda M_{22}(\lambda)} f^{+}(x,\lambda)\, f^{-}(t,\lambda), & t<x, \quad x\neq 0,\ t\neq 0, \\ -\tfrac{i}{2\beta\lambda M_{22}(\lambda)} f^{-}(x,\lambda)\, f^{+}(t,\lambda), & x<t, \quad x\neq 0,\ t\neq 0. \end{cases}$$

Then (22) can be rewritten in the following form:

$$U(x,\lambda) = \int\limits_{-\infty}^{\infty} G(x,t,\lambda) f(t)\,dt +$$
$$+\begin{cases} \tfrac{M_{21}(\lambda)}{M_{22}(\lambda)} f_1^{-}(x,\lambda) \int\limits_{-\infty}^{0} f_1^{-}(t,\lambda)\, f(t)\,dt, & x\in(-\infty,0), \\ -\tfrac{M_{12}(\lambda)}{M_{22}(\lambda)} f_2^{+}(x,\lambda) \int\limits_{0}^{\infty} f_2^{+}(t,\lambda)\, f(t)\,dt, & x\in(0,\infty). \end{cases}$$

Theorem 2. *The eigenvalues of the problem* (1)–(6) *are the zeros of the function* $M_{22}(\lambda)$.

Proof. Let us show that the eigenvalues of the impulsive operator L are the zeros of $M_{22}(\lambda)$. Suppose $M_{22}(\lambda) = 0$. In this case, $W[F(x,\lambda_0), G(x,\lambda_0)] = 0$, which means, $F(x,\lambda_0)$ and $G(x,\lambda_0)$ are linearly dependent. Consequently, when $k\neq 0$, we have

$$F(x,\lambda_0) = kG(x,\lambda_0). \tag{23}$$

If we take into account that the solutions $F(x,\lambda)$ correspond to the solutions of equation (1), and solutions $G(x,\lambda)$ correspond to those of (1)–(6), from (23) it follows that the solution $F(x,\lambda_0)$ corresponds to the solutions of the problem (1)–(6). It means that $F(x,\lambda)$ is an eigenfunction.

Let's now consider the case where λ_0 represents an eigenvalue and $\eta(x,\lambda_0)$ is a corresponding eigenfunction, provided that $M_{22}(\lambda_0)\neq 0$.

Under these conditions, the functions $F(x,\lambda_0)$ and $G(x,\lambda_0)$ are assumed to be linearly independent and

$$\eta\left(x,\lambda_0\right)=\begin{cases} c_1\left[\frac{M_{22}}{\det M}f_1^+\left(x,\lambda\right)-\frac{M_{21}}{\det M}f_1^-\left(x,\lambda\right)\right]+c_2f_1^+\left(x,\lambda\right), & -\infty<x<0, \\ c_3f_2^+\left(x,\lambda\right)+c_4\left[M_{12}f_2^+\left(x,\lambda\right)+M_{22}f_2^-\left(x,\lambda\right)\right], & 0<x<\infty, \end{cases}$$

where one of the constants c_1, c_2, c_3 and c_4 is different from zero.

Taking into account that $\eta\left(x,\lambda_0\right)$ satisfies the impulse condition

$$\begin{bmatrix}\eta\left(0^+\right)\\ \eta'\left(0^-\right)\end{bmatrix}=B\begin{bmatrix}\eta\left(0^-\right)\\ \eta'\left(0^-\right)\end{bmatrix},$$

and an asymptotic condition

$$\lim_{\operatorname{Im}x\to\infty}\eta\left(x,\lambda_0\right)e^{-i\lambda_0x}=1,$$

we can find that $c_1=c_4=0$. Then, we have

$$\eta\left(x,\lambda_0\right)=\begin{cases} c_2f_1^+\left(x,\lambda_0\right), & -\infty<x<0, \\ c_3f_2^+\left(x,\lambda_0\right), & 0<x<\infty. \end{cases}$$

Now to find the constants c_2 and c_3, we use again the impulse condition in the following form:

$$c_3\begin{bmatrix}f_2^+\\ f_2^{+\prime}\end{bmatrix}=c_2\begin{bmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{bmatrix}\begin{bmatrix}f_1^+\\ f_1^{+\prime}\end{bmatrix}$$

or

$$\begin{array}{l} c_3f_2^+-c_2\left(\alpha_1f_1^++\alpha_2f_1^{+\prime}\right)=0, \\ c_3f_2^{+\prime}-c_2\left(\alpha_3f_1^++\alpha_4f_1^{+\prime}\right)=0. \end{array}$$

Since the main determinant $M_{22}\left(\lambda\right)\neq 0$, the system has only trivial solutions $c_2=c_3=0$. So all constants c_1,c_2,c_3 and c_4 are equal to zero. This contradicts the fact that $\eta\left(x,\lambda_0\right)$ is an eigenfunction. ◀

Theorem 3. *The spectrum of the operator consists of a continuous spectrum filling the positive half–axis* $[0,\infty)$ *on which there may exist spectral singularities at the points* $\left(\frac{\Lambda_n}{2}\right)^2$.

Proof. We denote the spectrum, point spectrum, residual spectrum, and continuous spectrum of L as $\sigma\left(L\right),\sigma_p\left(L\right),\sigma_r\left(L\right)$ and $\sigma_c\left(L\right)$, respectively.

In the case where $\operatorname{Im}\lambda=0$ and $\lambda\neq\frac{\Lambda_n}{2}$, $n\in N$, the fundamental set of solutions for equation (1) consists of the functions $f_2^+\left(x,\lambda\right)$ and $f_2^-\left(x,\lambda\right)$.

$$\begin{array}{l} \eta\left(x,\lambda\right)=K_1e^{i\beta\operatorname{Re}\lambda x}\left(1+\sum\limits_{n=1}^{\infty}\sum\limits_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n+2\beta\lambda}e^{i\Lambda_\alpha x}\right)+ \\ +K_2e^{-i\beta\operatorname{Re}\lambda x}\left(1+\sum\limits_{n=1}^{\infty}\sum\limits_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n-2\beta\lambda}e^{i\Lambda_\alpha x}\right). \end{array}$$

So, $\eta(x,\lambda) \notin L_2[0,\infty)$, since the principal parts of the solutions are almost periodic, which indicates that the operator does not have any purely real eigenvalues.

To demonstrate that the residual spectrum of the operator L is empty, we will investigate the function $g(x,\lambda) \in L_2[0,\infty)$ which is a solution to the adjoint equation $L^*(\lambda) = 0$. Then it satisfies

$$-g''(x,\lambda) + q(x)\,g(x,\lambda) = \lambda^2 g(x,\lambda). \tag{24}$$

Given that the equation (24) aligns with the form of equation (1), it follows that either $\sigma_p(L^*) = 0$ or $\sigma_r(L) = 0$.

This indicates that the spectrum of the operator $\sigma(L) = \sigma_c(L)$, and L^{-1} is defined within a dense subset of $L_2[0,\infty)$ for $\lambda \in C$.

Using a conventional approach, we can demonstrate that the kernel $G(x,t,\lambda)$ of the operator L^{-1} is bounded for $\lambda \notin \{\operatorname{Im}\lambda = 0\}$.

Conversely, the values $\lambda = \frac{\Lambda_n}{2}$, $n \in N$, can only represent simple poles for L^{-1}.

Since the operator L doesn't have any eigenvalues, it follows that there are no singularities at these specific points. Hence, the set where $\operatorname{Im}\lambda = 0$ encompasses a continuous spectrum of the operator L.

We conclude that the continuous spectrum spans the half–axis, and $M_{22}(\lambda) \neq 0$ when $\operatorname{Im}\lambda = 0$.

Thus, as $x \to \infty$ for $G(x,\lambda)$, we arrive at the following asymptotic expressions:

$$G(x,\lambda) = M_{12}(\lambda)\, f_2^+(x,\lambda) + M_{22}(\lambda)\, f_2^-(x,\lambda), \quad x \to \infty.$$

In this equation, by isolating $M_{22}(\lambda)$ on both sides, we arrive at the following problem:

$$U^+(x,\lambda) = \frac{M_{12}(\lambda)}{M_{22}(\lambda)} f_2^+(x,\lambda) + f_2^-(x,\lambda),\ \operatorname{Im}\lambda = 0.$$

The function $U^+(x,\lambda)$ is designated as an eigenfunction related to the problem at hand. ◄

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