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An Introduction to Newly Defined Quaternions: Split-like Quaternions

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Abstract. In this paper, we define split-like quaternions such that $p = p_0 + p_1 i + p_2 j + p_3 k$, where $i^2 = j^2 = -1$, $k^2 = 1$ and $p_0, p_1, p_2, p_3 \in \mathbb{R}$. It is the purpose of this paper to describe a new non-commutative and non-associative quaternion algebra which is isomorphic to the four-dimensional Lorentz space $\mathbb{R}^4_1$. In addition, we define the inner product, characteristic of split-like quaternions, cross product and we discuss their properties. Furthermore, we give the matrix representations of split-like quaternions. Also, Cauchy-Schwarz inequality and triangle inequality provided by the inner product are demonstrated. Finally, we present the polar forms of split-like quaternions by giving the angle between two split-like quaternions and give De Moivre's formulas for split-like quaternions.

Key Words and Phrases: split-like quaternions, split quaternions, De Moivre's formula, Lorentzian geometry

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1. Introduction

The pseudo-Euclidean space of index ν, denoted by $\mathbb{R}^n_\nu$, is the inner product space $\mathbb{R}^n$ equipped with the inner product given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \ldots + x_{n-\nu} y_{n-\nu} - x_{n-\nu+1} y_{n-\nu+1} - \ldots - x_n y_n,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, [1]. For $\nu = 1$, the space $\mathbb{R}^n_1$ is called a Lorentz-Minkowski space. In the literature, there is another definition of the pseudo-Euclidean space equipped with the inner product [2]

$$\mathbf{x} \cdot \mathbf{y} = -x_1 y_1 - \ldots - x_\nu y_\nu + x_{\nu+1} y_{\nu+1} + \ldots + x_n y_n.$$

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To distinguish them, we use the metric representations $(+, \ldots, -)$ for the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ and $(-, \ldots, +)$ for the inner product $\mathbf{x} \cdot \mathbf{y}$.

William R. Hamilton discovered the real quaternions in 1843, which can be represented as follows [3]:

$$\mathbb{H} = \{p = p_0 + p_1 i + p_2 j + p_3 k \,:\, p_0, p_1, p_2, p_3 \in \mathbb{R}\},$$

where

$$i^2 = j^2 = k^2 = ijk = -1 \quad \text{and} \quad ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The set of real quaternions is a non-commutative and associative division algebra which has no zero divisor and is isomorphic to $\mathbb{R}^4$.

In 1849, James Cockle introduced the set of split quaternions, which can be represented as follows [4]:

$$\mathbb{H}_S = \{p = p_0 + p_1 i + p_2 j + p_3 k \,:\, p_0, p_1, p_2, p_3 \in \mathbb{R}\},$$

where

$$i^2 = -1, j^2 = k^2 = ijk = 1 \quad \text{and} \quad ij = -ji = k, jk = -kj = -i, ki = -ik = j.$$

The set of split quaternions is a four-dimensional non-commutative associative algebra and has zero divisors, nilpotent and idempotent elements.

For all $p = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}_S$, the conjugate of the split quaternion p is defined as $\overline{p} = p_0 - p_1 i - p_2 j - p_3 k$ and the characteristic function of the split quaternion is defined as $\mathcal{I}_p = p\overline{p} = p_0^2 + p_1^2 - p_2^2 - p_3^2$. Moreover, $p \in \mathbb{H}_S$ is called timelike, spacelike or lightlike, where $\mathcal{I}_p > 0$, $\mathcal{I}_p < 0$ and $\mathcal{I}_p = 0$, respectively. Following [5, 6, 7, 8], we can see that the set of split quaternions is isomorphic to $\mathbb{R}^4_2$ with metric $(-, -, +, +)$ and the set of pure split quaternions is isomorphic to $\mathbb{R}^3_1$ with metric $(-, +, +)$.

The inner product space $\mathbb{R}^4_2$ with metric $(-, -, +, +)$ has some disadvantages. One such disadvantage, or limitation, is that the angle formed by the split quaternion with the negative real axis can be determined, while the angle formed with the positive real axis remains indeterminate and conflicts with the characterization in the Lorentz space.

When we consider the inner product space $\mathbb{R}^4_2$ with metric $(+, +, -, -)$, the mentioned disadvantages are avoided, but this time the set of pure split quaternions becomes isomorphic to the inner product space $\mathbb{R}^3_2$ with metric $(+, -, -)$. However, since it is very difficult to work in these spaces $\mathbb{R}^4_2$ and $\mathbb{R}^3_2$, the spaces that are isomorphic to them are also challenging to work with.

There is a need to define a new space that i) gives opportunity to determine the angle formed with the positive real axis, mentioned above as a limitation in the literature, ii) is compatible with the Lorentz space, and iii) is isomorphic to $\mathbb{R}_1^4$ with metric $(+,+,+,-)$ and $\mathbb{R}_1^3$ with metric $(+,+,-)$, on which there exist many studies until today. The elements of this space are described in the second section and are called split-like quaternions. Moreover, we will define the sum and the product of two split-like quaternions, the conjugate of split-like quaternion. We consider some properties about them and matrix representations of split-like quaternions.

There are some number systems in the literature that at first glance seem similar to the split-like quaternions. For example, Segre quaternions defined by the set [9]

$$\begin{aligned}\{q = q_0 + q_1 i + q_2 j + q_3 k \ : \ & q_0, q_1, q_2, q_3 \in \mathbb{R},\ i^2 = k^2 = -1,\ j^2 = 1, \\ & ij = k = ji,\ jk = i = kj,\ ki = -j = ik\},\end{aligned}$$

bicomplex numbers defined by the set [10]

$$\{q = q_0 + q_1 j \ : \ q_0, q_1 \in \mathbb{C},\ i^2 = j^2 = -1 \text{ and } ij = ji\}$$

and hypercomplex numbers defined by the set [11]

$$\begin{aligned}\{q = q_0 + q_1 i + q_2 j + q_3 k \ : \ & q_0, q_1, q_2, q_3 \in \mathbb{R},\ i^2 = j^2 = -1,\ k^2 = 1, \\ & ij = k = ji, jk = -i = kj,\ ki = -j = ik\}.\end{aligned}$$

Since the sets of these numbers are communicative and associative, these systems are different from the set of split-like quaternions.

2. Split-like Quaternions

The set of split-like quaternions is defined as

$$\mathbb{H}_{SL}^{4,1} = \{p = p_0 + p_1 i + p_2 j + p_3 k \ : \ p_0, p_1, p_2, p_3 \in \mathbb{R}\},$$

where $i^2 = j^2 = -1,\ k^2 = ijk = 1, ij = -ji = k, jk = -kj = -i, ki = -ik = -j$. From this point onward, we will use the notation $\mathbb{H}_{SL}$ instead of $\mathbb{H}_{SL}^{4,1}$ for clarity. Alternatively, split-like quaternions can be written as $p = S_p + V_p$, where $S_p = p_0$ and $V_p = p_1 i + p_2 j + p_3 k$. S_p is called scalar part of p and V_p is called vector part of p that is also denoted by $\mathbf{p}$. Additionally, $p = q$ if and only if $p_0 = q_0$, $p_1 = q_1$, $p_2 = q_2$, $p_3 = q_3$ or $S_p = S_q$ and $\mathbf{p} = \mathbf{q}$.

Definition 1. *Let p, $q \in \mathbb{H}_{SL}$.* ***The sum of two split-like quaternions*** *is defined by*

$$\begin{aligned} p+q &:= p_0+q_0+(p_1+q_1)i+(p_2+q_2)j+(p_3+q_3)k \\ &= (S_p+S_q)+(\mathbf{p}+\mathbf{q}) = S_{p+q}+V_{p+q}. \end{aligned}$$

It can be shown that $(\mathbb{H}_{SL},+)$ is an Abelian group with unit element $0_{\mathbb{H}_{SL}} = 0+0i+0j+0k \in \mathbb{H}_{SL}$. ***The multiplication by a real scalar*** λ ***of a split-like quaternion*** p is defined by $\lambda p = \lambda p_0 + \lambda p_1 i + \lambda p_2 j + \lambda p_3 k$.

Corollary 1. $\mathbb{H}_{SL}$ *is a four-dimensional real vector space. It can be written as* $\mathbb{H}_{SL} = \mathrm{Span}_{\mathbb{R}}\{1,i,j,k\}$ *and the system* $\{1,i,j,k\}$ *is the basis of* $\mathbb{H}_{SL}$.

Definition 2. *Let p, $q \in \mathbb{H}_{SL}$.* ***The product of two split-like quaternions*** *is defined by*

$$\begin{aligned} p\times q &= p_0q_0-p_1q_1-p_2q_2+p_3q_3+(p_0q_1+p_1q_0-p_2q_3+p_3q_2)i \\ &\quad +(p_0q_2+p_1q_3+p_2q_0-p_3q_1)j+(p_0q_3+p_1q_2-p_2q_1+p_3q_0)k \\ &= S_pS_q-\langle\mathbf{p},\mathbf{q}\rangle_{\mathbb{R}^3_1}+S_p\mathbf{q}+S_q\mathbf{p}-\mathbf{p}\wedge_{\mathbb{R}^3_1}\mathbf{q}, \end{aligned}$$

where $\langle\mathbf{p},\mathbf{q}\rangle_{\mathbb{R}^3_1} = p_1q_1+p_2q_2-p_3q_3$ *and*

$$\mathbf{p}\wedge_{\mathbb{R}^3_1}\mathbf{q} = \begin{vmatrix} i & j & -k \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = (p_2q_3-p_3q_2)i+(p_3q_1-p_1q_3)j+(p_2q_1-p_1q_2)k.$$

Proposition 1. *The product operation for split-like quaternions has the following properties:*

1. $p\times q \neq q\times p$,
2. *There exists a unique* $1_{\mathbb{H}_{SL}} = 1+0i+0j+0k \in \mathbb{H}_{SL}$ *such that* $p\times 1_{\mathbb{H}_{SL}} = 1_{\mathbb{H}_{SL}}\times p = p$,
3. $p\times(q+r) = (p\times q)+(p\times r)$ *and* $(p+q)\times r = (p\times r)+(q\times r)$,
4. $p\times(q\times r) \neq (p\times q)\times r$,
5. *The product of three split-like quaternions can be written as*

$$\begin{aligned} (p\times q)\times r &= S_pS_qS_r - S_p\langle\mathbf{q},\mathbf{r}\rangle_{\mathbb{R}^3_1} - S_q\langle\mathbf{p},\mathbf{r}\rangle_{\mathbb{R}^3_1} - S_r\langle\mathbf{p},\mathbf{q}\rangle_{\mathbb{R}^3_1} + \langle\mathbf{p}\wedge_{\mathbb{R}^3_1}\mathbf{q},\mathbf{r}\rangle_{\mathbb{R}^3_1} \\ &\quad + \Big(S_pS_q\mathbf{r}+S_rS_p\mathbf{q}+S_rS_q\mathbf{p}-\langle\mathbf{p},\mathbf{q}\rangle_{\mathbb{R}^3_1}\mathbf{r}-S_q(\mathbf{p}\wedge_{\mathbb{R}^3_1}\mathbf{r}) \\ &\quad -S_p(\mathbf{q}\wedge_{\mathbb{R}^3_1}\mathbf{r})-S_r(\mathbf{p}\wedge_{\mathbb{R}^3_1}\mathbf{q})+(\mathbf{p}\wedge_{\mathbb{R}^3_1}\mathbf{q})\wedge_{\mathbb{R}^3_1}\mathbf{r}\Big). \end{aligned}$$

Corollary 2. *The set of split-like quaternions* $\mathbb{H}_{SL} = \mathrm{Sp}\{1, i, j, k\}$ *is a* ***non-associative unital algebra*** *with split-like quaternion product. In addition,* $(\mathbb{H}_{SL}, +, \times)$ *is not a ring.*

Note 1. *The product* $p \times q$ *can be denoted by* pq *for the sake of brevity.*

Theorem 2. *The set of split-like quaternions* $\mathbb{H}_{SL}$ *has*

1. *zero divisors,*
2. *nilpotent elements,*
3. *idempotent elements.*

Proof.

1. Let $0 \neq p = p_0 + p_1 i + p_2 j + p_3 k, q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}_{SL}$ and $pq = 0$. According to the equality of two split-like quaternions, the system of equations

$$\begin{aligned} p_0q_0 - p_1q_1 - p_2q_2 + p_3q_3 &= 0 \\ p_0q_1 + p_1q_0 - p_2q_3 + p_3q_2 &= 0 \\ p_0q_2 + p_1q_3 + p_2q_0 - p_3q_1 &= 0 \\ p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0 &= 0 \end{aligned} \tag{1}$$

is obtained. If $p_0^2 + p_3^2 = p_1^2 + p_2^2$ or $p_0^2 + p_1^2 + p_2^2 = p_3^2$ are provided, the system of equations (1) has a non-zero solution. Consequently, there exists non-zero split-like quaternion q when $pq = 0$.

2. Let p be non-zero and $p^2 = 0$. Then it can be written

$$p^2 = p_0^2 - p_1^2 - p_2^2 + p_3^2 + 2p_0p_1 i + 2p_0p_2 j + 2p_0p_3 k = 0.$$

Thus, there exists a nilpotent element p in the set of split-like quaternions $\mathbb{H}_{SL}$ such that $p_0 = 0$ and $p_3^2 = p_2^2 + p_1^2$.

3. Let $p^2 = p$. Then it can be written

$$p^2 = p_0^2 - p_1^2 - p_2^2 + p_3^2 + 2p_0p_1 i + 2p_0p_2 j + 2p_0p_3 k = p_0 + p_1 i + p_2 j + p_3 k = p.$$

Thus, there exists an idempotent element p in the set of split-like quaternions $\mathbb{H}_{SL}$ such that $p_0 = \frac{1}{2}$ and $p_3^2 - p_2^2 - p_1^2 = \frac{1}{4}$. ◄

Example 1. *The split-like quaternions* $q = i + k$ *and* $p = 1 + i + j + k$ *are zero-divisors since* $pq = 0$. *The split-like quaternion* $r = j + k$ *is a nilpotent element since* $r^2 = 0$. *The split-like quaternion* $s = \frac{1}{2} + \frac{1}{2} i + \frac{1}{2} j + \frac{\sqrt{3}}{2} k$ *is an idempotent element since* $s^2 = s$.

Definition 3. *Let* $p = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}_{SL}$. *Then the* ***conjugate*** *of* p *is defined by* $\overline{p} := S_p - \mathbf{p} = p_0 - p_1 i - p_2 j - p_3 k$.

Theorem 3. *Let* $p = p_0 + \mathbf{p} = p_0 + p_1 i + p_2 j + p_3 k$, $q = q_0 + \mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}_{SL}$ *and* $\lambda \in \mathbb{R}$. *Then conjugate of split-like quaternions has the following properties:*

1. $\overline{\overline{p}} = p$,
2. $\overline{\lambda p} = \lambda \overline{p}$,
3. $\overline{p+q} = \overline{p} + \overline{q}$,
4. $\overline{pq} = \overline{q}\ \overline{p}$,
5. $\mathbf{pq} = \overline{\mathbf{qp}}$.
6. $p\overline{p} = p_0^2 + p_1^2 + p_2^2 - p_3^2$.
7. $(qp)\overline{p} \neq q(p\overline{p})$

Proof. Below, we provide the proof of statement (5). The other statements of the theorem can be proved in a similar way.

5. Let $p = p_0 + p_1 i + p_2 j + p_3 k, q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}_{SL}$. Then,

$$\mathbf{pq} = -p_1q_1 - p_2q_2 + p_3q_3 + (p_3q_2 - p_2q_3)i + (p_1q_3 - p_3q_1)j + (p_1q_2 - p_2q_1)k$$

and

$$\overline{\mathbf{qp}} = -p_1q_1 - p_2q_2 + p_3q_3 - (q_3p_2 - q_2p_3)i - (q_1p_3 - q_3p_1)j - (q_1p_2 - q_2p_1)k.$$

Thus, the equality is proved. ◀

Example 2. *Let* $p = 1 + i + j + k$ *and* $q = i - j + k$ *be split-like quaternions. We get* $(qp)\overline{p} = -2i - 2j - 2k$ *and* $q(p\overline{p}) = 2i - 2j + 2k$. *As a result,* $(qp)\overline{p} \neq q(p\overline{p})$.

2.1. Matrix Representations of Split-like Quaternions

Theorem 4. *Let* $p = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}_{SL}$. *Then*

$$\begin{aligned} R_p : \mathbb{H}_{SL} &\to \mathbb{H}_{SL} \\ q &\mapsto R_p(q) = qp \end{aligned} \quad \text{and} \quad \begin{aligned} L_p : \mathbb{H}_{SL} &\to \mathbb{H}_{SL} \\ q &\mapsto L_p(q) = pq \end{aligned}$$

are linear transformations.

Corollary 3. *The set of right real matrix representations of split-like quaternions is defined by*

$$S_4^r(\mathbb{R}) := \left\{ R_p = \begin{bmatrix} p_0 & -p_1 & -p_2 & p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{bmatrix} : p = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}_{SL} \right\}.$$

Similarly, the set of left real matrix representations of split-like quaternions is defined by

$$S_4^l(\mathbb{R}) := \left\{ L_p = \begin{bmatrix} p_0 & -p_1 & -p_2 & p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} : p = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}_{SL} \right\}.$$

Theorem 5. *Let $p, q \in \mathbb{H}_{SL}$ and $\lambda \in \mathbb{R}$. Then the following properties are satisfied:*

1. $R_p + R_q = R_{p+q} \quad (L_p + L_q = L_{p+q})$,
2. $R_p R_q \neq R_{pq}$ *and* $R_q R_p \neq R_{pq} \quad (L_p L_q \neq L_{pq}$ *and* $L_q L_p \neq L_{pq})$,
3. $\lambda R_p = R_{\lambda p} \quad (\lambda L_p = L_{\lambda p})$,
4. $R_1 = I_4 \quad (L_1 = I_4)$,
5. $qp = R_p\, q \quad (pq = L_p\, q)$.

Corollary 4. *Let $p \in \mathbb{H}_{SL}$. Then* $\det R_p = \det L_p$.

2.2. Inner Products

Theorem 6. *Let $p = p_0 + p_1 i + p_2 j + p_3 k$, $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}_{SL}$. The function h defined by $h(p,q) = \frac{1}{2}[p\overline{q} + q\overline{p}] = p_0q_0 + p_1q_1 + p_2q_2 - p_3q_3$ is a real-valued indefinite inner product. Furthermore, $h(p,q) = S_{p\overline{q}}$.*

Note 7. *The inner product h can be written as $h(p,q) = \langle p, q \rangle_{\mathbb{R}_1^4}$ and $h(p,q) = \langle \tilde{p}, \tilde{q} \rangle_{\mathbb{R}^3} - p_3q_3$, where p and q are split-like quaternions and $\tilde{p} = (p_0, p_1, p_2)$, $\tilde{q} = (q_0, q_1, q_2) \in \mathbb{R}^3$.*

Each bilinear transformation corresponds to a matrix. Hence, the following theorem can be given:

Theorem 8. *The matrix corresponding to the bilinear transformation $h : \mathbb{H}_{SL} \times \mathbb{H}_{SL} \to \mathbb{R}$ is*

$$\mathrm{Id}_{4,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Therefore, $h(p,q) = p^T \,\mathrm{Id}_{4,1}\, q$ such that p, $q \in \mathbb{H}_{SL}$.

Remark 1. *The matrix* $\mathrm{Id}_{4,1}$ *shows that the bilinear transformation* h *is symmetric and non-degenerate.*

Definition 4. *Let* $p = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}_{SL}$. *The* ***characteristic function*** *of* p *is defined by* $I_p := p\overline{p} = p_0^2 + p_1^2 + p_2^2 - p_3^2$.

Definition 5. *Let* $p \in \mathbb{H}_{SL}$. *Then* p *is called a* ***spacelike*** *split-like quaternion if* $I_p > 0$ *or* $p = 0$, *a* ***timelike*** *split-like quaternion if* $I_p < 0$ *and a* ***lightlike*** *split-like quaternion if* $I_p = 0$.

Example 3. *Let* $p = 1 - i$, $q = i + k$, $r = j + 2k$ *be split-like quaternions. Then,* p *is spacelike since* $I_p = 2 > 0$, q *is lightlike since* $I_q = 0$ *and* r *is timelike since* $I_r = -3 < 0$.

Definition 6. *For an all non-zero and non-lightlike split-like quaternion* p, ***indicator*** *of* p *is defined by* $\epsilon_p = 1$, *where* $I_p > 0$, *and* $\epsilon_p = -1$, *where* $I_p < 0$.

Definition 7. *Let* $p = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}_{SL}$ *be a non-spacelike split-like quaternion. If* $p_3 > 0$ $(p_3 < 0)$, p *is called a* ***future-directed*** *or* ***positive*** *(**past-directed** or **negative**) split-like quaternion. In other words,* p *is called future-directed (positive) if* $h(p, k) < 0$ *and past-directed (negative) if* $h(p, k) > 0$.

Note 9. *Let* $0 \neq \lambda \in \mathbb{R}$ *and* $p \in \mathbb{H}_{SL}$. *Then* $I_{\lambda p} = (\lambda p)\overline{(\lambda p)} = \lambda^2 p\overline{p} = \lambda^2 I_p$ *is satisfied. Therefore,* p *and* λp *have the same characteristic. If* p *is positive (negative) and* $\lambda \in \mathbb{R}$ *is positive, then* λp *is also a positive (negative) split-like quaternion.*

Theorem 10. *Let* $p, q \in \mathbb{H}_{SL}$ *be two positive (negative) non-zero and non-spacelike split-like quaternions. Then,*

$$h(p, q) = \langle p, q \rangle_{\mathbb{R}_1^4} \leq 0$$

with equality if and only if p *and* q *are linearly dependent lightlike split-like quaternions.*

Proof. Let $p = \tilde{p} + p_3 k,\ q = \tilde{q} + q_3 k \in \mathbb{H}_{SL}$.

1. If p, q are positive, then $p_3 \geq \|\tilde{p}\|_{\mathbb{R}^3}$, $q_3 \geq \|\tilde{q}\|_{\mathbb{R}^3}$.
2. If p, q are negative, then $-p_3 \geq \|\tilde{p}\|_{\mathbb{R}^3}$, $-q_3 \geq \|\tilde{q}\|_{\mathbb{R}^3}$.

By using Cauchy-Schwarz inequality in Euclidean space, we get

$$p_3 q_3 \geq \|\tilde{p}\|_{\mathbb{R}^3} \|\tilde{q}\|_{\mathbb{R}^3} \geq \langle \tilde{p}, \tilde{q} \rangle_{\mathbb{R}^3}.$$

Then, $0 \geq \langle \tilde{p}, \tilde{q} \rangle_{\mathbb{R}^3} - p_3 q_3 = h(p,q) = \langle p, q \rangle_{\mathbb{R}^4_1}$.

If $p,\, q \in \mathbb{H}_{SL}$ are linearly dependent lightlike split-like quaternions, then we can write $q = \lambda p$ and obtain $h(p,q) = \langle p, \lambda p \rangle_{\mathbb{R}^4_1} = \lambda I_p = 0$ for $\lambda \in \mathbb{R}$.

Let $p,\, q \in \mathbb{H}_{SL}$ be two positive (negative) lightlike split-like quaternions. If $h(p,q) = 0$, then, in view of the fact that $\|\tilde{p}\|_{\mathbb{R}^3} = p_3$ ($\|\tilde{p}\|_{\mathbb{R}^3} = -p_3$) and $\|\tilde{q}\|_{\mathbb{R}^3} = q_3$ ($\|\tilde{q}\|_{\mathbb{R}^3} = -q_3$), we obtain

$$h(p,q) = \langle \tilde{p}, \tilde{q} \rangle_{\mathbb{R}^3} - p_3 q_3 = 0 \Rightarrow \langle \tilde{p}, \tilde{q} \rangle_{\mathbb{R}^3} = p_3 q_3 = \|\tilde{p}\|_{\mathbb{R}^3} \|\tilde{q}\|_{\mathbb{R}^3}.$$

In accordance with Euclidean Cauchy-Schwarz property, $\tilde{p}$ and $\tilde{q}$ are linearly dependent, i.e., $\tilde{p} = \lambda \tilde{q}$ for $\lambda \in \mathbb{R}$. Then, $\|\tilde{p}\|_{\mathbb{R}^3} = \pm p_3 = \lambda \|\tilde{q}\|_{\mathbb{R}^3} = \pm \lambda q_3$ and we obtain $p = \lambda q$. ◄

Theorem 11. *Let $p = p_0 + p_1 i + p_2 j + p_3 k$ be a split-like quaternion.*

1. *If p is spacelike, then $\mathbf{p}$ can be spacelike or timelike or lightlike.*
2. *If p is timelike, then $\mathbf{p}$ can only be timelike.*
3. *If p is lightlike, then $\mathbf{p}$ can be timelike, where $p_0 \neq 0$, or lightlike, where $p_0 = 0$.*

Proof.

2. Let p be timelike, so it can be written

$$I_p = p\bar{p} = (p_0 + \mathbf{p})(p_0 - \mathbf{p}) = p_0^2 - \mathbf{p}^2 = p_0^2 + I_{\mathbf{p}} < 0.$$

Then, we get $I_{\mathbf{p}} < -p_0^2$. Thus, $\mathbf{p}$ can only be timelike.

The other statements of the theorem can be proved similarly. ◄

2.3. The Characteristic of $p + q$

We will discuss the characteristic of $p + q$ with respect to $h(p,q)$ and the characteristics of q and p, where q and p are two split-like quaternions. Let $p,\, q \in \mathbb{H}_{SL}$. The characteristic of $p + q$ can be written as

$$I_{p+q} = I_p + I_q + 2h(p,q).$$

We have to restrict characteristics of p and q or $h(p,q)$ to interpret the characteristic of $p + q$. We will examine it in two parts.

Theorem 12. *Let p, $q \in \mathbb{H}_{SL}$. The characteristic of $p+q$ can be examined as follows:*

1. *Assume that p and q have the same characteristics.*
   - i. *Let p and q be spacelike.*
     - (a) *$p+q$ is spacelike if and only if $h(p,q) \geq 0$ or $h(p,q) < 0$ and $|h(p,q)| < \frac{1}{2}(I_p + I_q)$.*
     - (b) *$p+q$ is timelike if and only if $h(p,q) < 0$ and $|h(p,q)| > \frac{1}{2}(I_p + I_q)$.*
     - (c) *$p+q$ is lightlike if and only if $h(p,q) < 0$ and $|h(p,q)| = \frac{1}{2}(I_p + I_q)$.*
   - ii. *Let p and q be timelike.*
     - (a) *$p+q$ is spacelike if and only if $h(p,q) > \frac{1}{2}|I_p + I_q| > 0$.*
     - (b) *$p+q$ is timelike if and only if $h(p,q) < \frac{1}{2}|I_p + I_q|$.*
     - (c) *$p+q$ is lightlike if and only if $h(p,q) = \frac{1}{2}|I_p + I_q| > 0$.*
   - iii. *Let p and q be lightlike.*
     - (a) *$p+q$ is spacelike if and only if $h(p,q) > 0$.*
     - (b) *$p+q$ is timelike if and only if $h(p,q) < 0$.*
     - (c) *$p+q$ is lightlike if and only if $h(p,q) = 0$.*

2. *Assume that p and q have different characteristics.*
   - i. *Let p be spacelike and q be timelike.*
     - (a) *$p+q$ is spacelike if and only if*
       - ⋆ *$h(p,q) < 0$ and $|h(p,q)| < \frac{1}{2}|I_p + I_q|$*
       - ⋆ *$h(p,q) > \frac{1}{2}|I_p + I_q| \geq 0$, where $I_p \leq |I_q|$*
       - ⋆ *$h(p,q) > 0$, where $I_p = |I_q|$*
       - ⋆ *$h(p,q) = 0$, where $I_p > |I_q|$*
     - (b) *$p+q$ is timelike if and only if*
       - ⋆ *$h(p,q) < \frac{1}{2}|I_p + I_q|$, where $I_p < |I_q|$*
       - ⋆ *$h(p,q) < 0$, where $I_p \geq |I_q|$*
       - ⋆ *$h(p,q) = 0$, where $I_p < |I_q|$*
     - (c) *$p+q$ is lightlike if and only if*
       - ⋆ *$h(p,q) < 0$ and $|h(p,q)| = \frac{1}{2}|I_p + I_q|$, where $I_p > |I_q|$*
       - ⋆ *$h(p,q) = \frac{1}{2}|I_p + I_q| > 0$, where $I_p < |I_q|$*
       - ⋆ *$h(p,q) = 0$, where $I_p = |I_q|$*
   - ii. *Let p be spacelike and q be lightlike.*

   - *(a) $p+q$ is spacelike if and only if $h(p,q) \geq 0$ or, $h(p,q) < 0$ and $|h(p,q)| < \frac{1}{2}I_p$.*
   - *(b) $p+q$ is timelike if and only if $h(p,q) < 0$ and $|h(p,q)| > \frac{1}{2}I_p$.*
   - *(c) $p+q$ is lightlike if and only if $h(p,q) < 0$ and $|h(p,q)| = \frac{1}{2}I_p$.*

- *iii. Let p be timelike and q be lightlike.*
   - *(a) $p+q$ is spacelike if and only if $h(p,q) > \frac{1}{2}|I_p| > 0$.*
   - *(b) $p+q$ is timelike if and only if $h(p,q) < \frac{1}{2}|I_p|$.*
   - *(c) $p+q$ is lightlike if and only if $h(p,q) = \frac{1}{2}|I_p| > 0$.*

Corollary 5. *If p and q have the same characteristics and both of them are positive (negative), then the following statements are satisfied:*

1. *Let p and q be spacelike. Then the statements in Theorem 12/1-i are valid.*

2. *Let p and q be timelike. $p+q$ is timelike if and only if $h(p,q) < 0$. Also, $p+q$ can not be spacelike and lightlike.*

Corollary 6. *If p and q have different characteristics and both of them are positive (negative), then the following statements are satisfied:*

1. *Let p be spacelike and q be timelike. Then the statements in Theorem 12/2-i are valid.*

2. *Let p be spacelike and q be lightlike. Then the statements in Theorem 12/2-ii are valid.*

3. *Let p be timelike and q be lightlike. $p+q$ is timelike if and only if $h(p,q) < 0$. Also, $p+q$ can not be spacelike and lightlike.*

Example 4. *Let $p = 1+i-2k$ and $q = -1-i+j+4k$ be the given two timelike split-like quaternions. Then, $p+q = j+2k$ is timelike, since $I_{p+q} = -3 < 0$. Moreover, it can be seen that $0 < h(p,q) < \frac{1}{2}|I_p + I_q| = \frac{15}{2}$, since $I_p = -2$, $I_q = -13$ and $h(p,q) = 6$. Let $r = 1+i+j+k$ and $s = i+j-\sqrt{2}k$. Here r is spacelike and s is lightlike. Then, $r+s = 1+2i+2j+(1-\sqrt{2})k$ is also spacelike and $h(p,q) = 2+\sqrt{2} > 0$.*

Remark 2. *We can not interpret the characteristic of pq because of the fact that multiplication of two split-like quaternions is not associative:*

$$I_{pq} = (pq)\overline{(pq)} = (pq)(\overline{q}\,\overline{p}) \neq p(q\overline{q})\overline{p}.$$

Hereby, a connection could not be established between the characteristics of p and q and the characteristic of pq.

Theorem 13. *Let* p, $q \in \mathbb{H}_{SL}$ *be positive (negative) non-zero and non-spacelike split-like quaternions. Then,*

1. *$p+q$ is a positive (negative) non-zero and non-spacelike split-like quaternion,*
2. *$p+q$ is a lightlike split-like quaternion if and only if p and q are linearly dependent lightlike split-like quaternions.*

Proof. In accordance with Theorem 10, for $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$ non-spacelike split-like quaternions, we have $h(p,p) = I_p \leq 0$ and $h(q,q) = I_q \leq 0$.

1. If p, q are positive (negative), i.e., $p_3 > 0,\ q_3 > 0$ ($p_3 < 0,\ q_3 < 0$) then $p_3 + q_3 > 0$ ($p_3 + q_3 < 0$) and $p+q$ is a positive (negative) split-like quaternion. Since $I_{p+q} = h(p+q, p+q) = I_p + I_q + 2h(p,q)$, by using Theorem 10 we obtain $I_{p+q} \leq 0$ and $p+q$ is a non-spacelike split-like quaternion.
2. $(\Rightarrow)$: If $p+q$ lightlike, then $I_{p+q} = 0$. For p, $q \in \mathbb{H}_{SL}$ non-spacelike split-like quaternions, $I_p \leq 0$ and $I_q \leq 0$. In accordance with Theorem 10, $h(p,q) \leq 0$. For $I_{p+q} = I_p + I_q + 2h(p,q) = 0$ there exists only one solution $I_p = 0$, $I_q = 0$ and $h(p,q) = 0$. Moreover, p and q are linearly dependent lightlike split-like quaternions in accordance with Theorem 10.
   $(\Leftarrow)$: If p and q are linearly dependence lightlike split-like quaternions, then $q = \lambda p$, where $\lambda \in \mathbb{R}$. Therefore, $I_{p+q} = I_p + \lambda^2 I_p + 2\lambda I_p = 0$ and then $p+q$ is a lightlike split-like quaternion. ◄

2.4. Inverse and Division

Definition 8. *Let p be a non-zero and non-lightlike split-like quaternion. Then, there exists only one non-zero $p^{-1} \in \mathbb{H}_{SL}$ such that $pp^{-1} = p^{-1}p = 1$ and it is called the* ***inverse*** *of p. In addition, the inverse of p is found as $p^{-1} = \frac{\overline{p}}{I_p}$.*

Theorem 14. *Let p be a non-zero and non-lightlike split-like quaternion. If $p_0^2 + p_3^2 \neq p_1^2 + p_2^2$, then the inverse of p is the first column vector of matrices R_p^{-1} and L_p^{-1}.*

Proof. By using the property 5 in Theorem 5, we can write

$$\underbrace{\begin{bmatrix} p_0 & -p_1 & -p_2 & p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{bmatrix}}_{R_p} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for $p^{-1}p = 1$, and

$$\underbrace{\begin{bmatrix} p_0 & -p_1 & -p_2 & p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix}}_{L_p} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for $pp^{-1} = 1$. Then, we obtain $\det R_p = \det L_p = I_p \left(p_0^2 - p_1^2 - p_2^2 + p_3^2\right)$. Since $I_p \neq 0$ and $p_0^2 + p_3^2 \neq p_1^2 + p_2^2$, there are inverse matrices of R_p and L_p. Moreover, we obtain the first columns of the inverse matrices of R_p and L_p as

$$\frac{1}{I_p \left(p_0^2 - p_1^2 - p_2^2 + p_3^2\right)} \begin{bmatrix} p_0 \left(p_0^2 - p_1^2 - p_2^2 + p_3^2\right) \\ -p_1 \left(p_0^2 - p_1^2 - p_2^2 + p_3^2\right) \\ -p_2 \left(p_0^2 - p_1^2 - p_2^2 + p_3^2\right) \\ -p_3 \left(p_0^2 - p_1^2 - p_2^2 + p_3^2\right) \end{bmatrix} = \frac{1}{I_p} \begin{bmatrix} p_0 \\ -p_1 \\ -p_2 \\ -p_3 \end{bmatrix}.$$

Then, the desired result is obtained. ◀

Example 5. *Let $p = 1 + i + j + 2k$ and $q = 1 + i + j + k$ be two split-like quaternions. The inverse of p is $p^{-1} = -1 + i + j + 2k$ and the inverse of q is*

$$p^{-1} = \frac{\overline{p}}{I_p} = \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k.$$

Theorem 15. *Let $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$ be non-zero and non-lightlike split-like quaternions such that pq is non-zero and non-lightlike. Then, generally, $(pq)^{-1} \neq p^{-1}q^{-1}$ or $(pq)^{-1} \neq q^{-1}p^{-1}$.*

Example 6. *Let $p = 1 + i - k$ and $q = 2 - i + j$, then $pq = 3 - k$. Hence we get $(pq)^{-1} = \frac{3+k}{8}$, $p^{-1}q^{-1} = \frac{3-2i-2j+3k}{6}$ and $q^{-1}p^{-1} = \frac{3-k}{6}$. Therefore, $(pq)^{-1} \neq q^{-1}p^{-1}$ or $(pq)^{-1} \neq p^{-1}q^{-1}$.*

Definition 9. *Let $p, q \in \mathbb{H}_{SL}$ be non-zero and non-lightlike split-like quaternions. The **left division** and **right division** of two split-like quaternions is defined as, respectively, $q^{-1}p$ and pq^{-1}.*

Theorem 16. *Let p, q, r and s be non-zero and non-lightlike split-like quaternions such that qs is non-zero and non-lightlike and $\lambda \in \mathbb{R}$. Then, the following properties are satisfied:*

1. $(p + r)q^{-1} = pq^{-1} + rq^{-1}$,
2. $(\lambda p)q^{-1} = \lambda(pq^{-1})$,

3. $(pr)q^{-1} \neq \left(pq^{-1}\right) r$,

4. $\left(pq^{-1}\right)\left(rs^{-1}\right) \neq pr(qs)^{-1}$,

5. $\left(pq^{-1}\right)\left(qr^{-1}\right) \neq pr^{-1}$.

Proof.

5. Let p, q, $r \neq 0_{\mathbb{H}_{SL}}$ and $\lambda \in \mathbb{R}$. If q and r are non-lightlike, then

$$\left(pq^{-1}\right)\left(qr^{-1}\right) \neq p\left(q^{-1}q\right) r^{-1} = pr^{-1}.$$

The other statements can be proved in a similar way. ◀

2.5. Cross Product

Let $\mathbf{p} = p_1 i + p_2 j + p_3 k,\ \mathbf{q} = q_1 i + q_2 j + q_3 k \in \mathbb{H}_{SL}$ be given. The ***cross product*** of two pure split-like quaternions is defined by

$$\mathbf{p} \wedge \mathbf{q} = \begin{vmatrix} i & j & -k \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}. \tag{2}$$

The cross product in equation (2) is equivalent to $\wedge_{\mathbb{R}^3_1}$. For $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{H}_{SL}$ and $\lambda, \mu \in \mathbb{R}$, the following properties are satisfied:

1. $\mathbf{p} \wedge \mathbf{q} = -(\mathbf{q} \wedge \mathbf{p})$,
2. $\mathbf{p} \wedge \mathbf{p} = 0$,
3. $(\lambda \mathbf{p} + \mu \mathbf{q}) \wedge \mathbf{r} = \lambda(\mathbf{p} \wedge \mathbf{r}) + \mu(\mathbf{q} \wedge \mathbf{r})$.

The ***triple product*** of pure split-like quaternions $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ is defined by

$$h(\mathbf{p} \wedge \mathbf{q}, \mathbf{r}) = \langle \mathbf{p} \wedge \mathbf{q}, \mathbf{r} \rangle_{\mathbb{R}^3_1} = \det(\mathbf{p}, \mathbf{q}, \mathbf{r}).$$

Theorem 17. *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be pure split-like quaternions. Then,* $h(\mathbf{p}, \mathbf{p} \wedge \mathbf{q}) = 0$ *and* $h(\mathbf{q}, \mathbf{p} \wedge \mathbf{q}) = 0$ *hold.*

The cross product of three split-like quaternions is defined by

$$p \wedge q \wedge r = \begin{vmatrix} 1 & i & j & -k \\ p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \\ r_0 & r_1 & r_2 & r_3 \end{vmatrix},$$

where $p = p_0 + p_1 i + p_2 j + p_3 k, q = q_0 + q_1 i + q_2 j + q_3 k, r = r_0 + r_1 i + r_2 j + r_3 k \in \mathbb{H}_{SL}$.

Theorem 18. *Let p, q and r be split-like quaternions. Then, $h(p, p\wedge q\wedge r)=0$, $h(q, p\wedge q\wedge r)=0$ and $h(r, p\wedge q\wedge r)=0$ hold.*

Proof. The cross product of $p=p_0+p_1i+p_2j+p_3k$, $q=q_0+q_1i+q_2j+q_3k$, $r=r_0+r_1i+r_2j+r_3k\in\mathbb{H}_{SL}$ is

$$\begin{aligned} p\wedge q\wedge r &= p_1q_2r_3-p_1q_3r_2-p_2q_1r_3+p_2q_3r_1+p_3q_1r_2-p_3q_2r_1\\ &\quad+\left(-p_0q_2r_3+p_0q_3r_2+p_2q_0r_3-p_2q_3r_0-p_3q_0r_2+p_3q_2r_0\right)i\\ &\quad+\left(p_0q_1r_3-p_0q_3r_1-p_1q_0r_3+p_1q_3r_0+p_3q_0r_1-p_3q_1r_0\right)j\\ &\quad+\left(p_0q_1r_2-p_0q_2r_1-p_1q_0r_2+p_1q_2r_0+p_2q_0r_1-p_2q_1r_0\right)k. \end{aligned}$$

Then, we get

$$\begin{aligned} h(p, p\wedge q\wedge r) &= p_0\left(p_1q_2r_3-p_1q_3r_2-p_2q_1r_3+p_2q_3r_1+p_3q_1r_2-p_3q_2r_1\right)\\ &\quad+p_1\left(-p_0q_2r_3+p_0q_3r_2+p_2q_0r_3-p_2q_3r_0-p_3q_0r_2+p_3q_2r_0\right)\\ &\quad+p_2\left(p_0q_1r_3-p_0q_3r_1-p_1q_0r_3+p_1q_3r_0+p_3q_0r_1-p_3q_1r_0\right)\\ &\quad-p_3\left(p_0q_1r_2-p_0q_2r_1-p_1q_0r_2+p_1q_2r_0+p_2q_0r_1-p_2q_1r_0\right)\\ &= p_0p_1q_2r_3-p_0p_1q_3r_2-p_0p_2q_1r_3+p_0p_2q_3r_1+p_0p_3q_1r_2-p_0p_3q_2r_1\\ &\quad-p_1p_0q_2r_3+p_1p_0q_3r_2+p_1p_2q_0r_3-p_1p_2q_3r_0-p_1p_3q_0r_2+p_1p_3q_2r_0\\ &\quad+p_2p_0q_1r_3-p_2p_0q_3r_1-p_2p_1q_0r_3+p_2p_1q_3r_0+p_2p_3q_0r_1-p_2p_3q_1r_0\\ &\quad-p_3p_0q_1r_2+p_3p_0q_2r_1+p_3p_1q_0r_2-p_3p_1q_2r_0-p_3p_2q_0r_1+p_3p_2q_1r_0\\ &= 0. \end{aligned}$$

The other equalities can be proved in a similar way. ◄

2.6. Norm, Cauchy-Schwarz Inequality and Triangle Inequality

Definition 10. *Let $p=p_0+p_1i+p_2j+p_3k\in\mathbb{H}_{SL}$. The* ***norm*** *of p is defined by $\|p\|=\sqrt{|h(p,p)|}$.*

The norm of p can be also written as

$$\|p\|=\sqrt{|p\overline{p}|}=\sqrt{|I_p|}=\sqrt{|p_0{}^2+p_1{}^2+p_2{}^2-p_3{}^2|}.$$

Theorem 19. *Let p, $q\in\mathbb{H}_{SL}$. Then the following statements are true:*

1. $\|pq\|\neq\|p\|\,\|q\|$ *(generally),*
2. $\|pq\|=\|qp\|$,
3. $\|\overline{p}\|=\|-p\|=\|p\|$.

Proof. Below, we provide the proof of statement (2). The other statements of the theorem can be proved easily.

2. Let p, $q \in \mathbb{H}_{SL}$. Since

$$\begin{aligned}\|pq\|^2 &= \big|(p_0q_0 - p_1q_1 - p_2q_2 + p_3q_3)^2 + (p_0q_1 + p_1q_0 - p_2q_3 + p_3q_2)^2 \\ &\quad +(p_0q_2 + p_1q_3 + p_2q_0 - p_3q_1)^2 - (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)^2\big| \\ &= \big|p_0^2(q_0^2 + q_1^2 + q_2^2 - q_3^2) + p_1^2(q_0^2 + q_1^2 - q_2^2 + q_3^2) + p_2^2(q_0^2 - q_1^2 + q_2^2 + q_3^2) \\ &\quad +p_3^2(-q_0^2 + q_1^2 + q_2^2 + q_3^2) - 4p_2p_3q_2q_3 + 4p_1q_1(p_2q_2 - p_3q_3)\big|\end{aligned}$$

and

$$\begin{aligned}\|qp\|^2 &= \big|(q_0p_0 - q_1p_1 - q_2p_2 + q_3p_3)^2 + (q_0p_1 + q_1p_0 - q_2p_3 + q_3p_2)^2 \\ &\quad +(q_0p_2 + q_1p_3 + q_2p_0 - q_3p_1)^2 - (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)^2\big| \\ &= \big|p_0^2(q_0^2 + q_1^2 + q_2^2 - q_3^2) + p_1^2(q_0^2 + q_1^2 - q_2^2 + q_3^2) + p_2^2(q_0^2 - q_1^2 + q_2^2 + q_3^2) \\ &\quad +p_3^2(-q_0^2 + q_1^2 + q_2^2 + q_3^2) - 4p_2p_3q_2q_3 + 4p_1q_1(p_2q_2 - p_3q_3)\big|,\end{aligned}$$

we obtain $\|pq\| = \|qp\|$. ◄

Note 20. *Let* $\tilde{p} = (p_0, p_1, p_2) \in \mathbb{R}^3$ *and* $p = \tilde{p} + p_3k \in \mathbb{H}_{SL}$. *Then* $\|p\|^2$ *can be written as* $\|p\|^2 = \Big|\, \|\tilde{p}\|_{\mathbb{R}^3}^2 - p_3^2\Big|$.

Theorem 21. *Let* $\tilde{p} = (p_0, p_1, p_2) \in \mathbb{R}^3$ *and* $p = \tilde{p} + p_3k \in \mathbb{H}_{SL}$. *Then*

1. $\|\tilde{p}\|_{\mathbb{R}^3} > |p_3|$ *if and only if* p *is spacelike,*
2. $\|\tilde{p}\|_{\mathbb{R}^3} < |p_3|$ *if and only if* p *is timelike,*
3. $\|\tilde{p}\|_{\mathbb{R}^3} = |p_3|$ *if and only if* p *is lightlike.*

Corollary 7. *The split-like quaternions space* $\mathbb{H}_{SL}$ *is isomorphic to the Lorentz-Minkowski space* $\mathbb{R}_1^4$. *The pure split-like quaternions space is isomorphic to* $\mathbb{R}_1^3$.

Corollary 8. *Let* $p \in \mathbb{H}_{SL}$. *Then, generally,* $\det R_p = \det L_p \neq \|p\|^4$.

Theorem 22 (Cauchy-Schwarz Inequality). *Let* p, $q \in \mathbb{H}_{SL}$ *be non-lightlike split-like quaternions. Then following statements are true:*

1. $|h(p, q)| \geq \|p\|\,\|q\|$ *if* p *and* q *are timelike. Moreover, equality holds if and only if* p *and* q *are linear dependent split-like quaternions.*

2. *The subspace*[†] $\mathrm{Sp}\{p,q\}$ *is spacelike if and only if* $|h(p,q)| \leq \|p\|\,\|q\|$, *where* p *and* q *are spacelike. Furthermore, equality holds if and only if* p, q *are linearly dependent split-like quaternions and the subspace* $\mathrm{Sp}\{p,q\}$ *is spacelike.*

3. *The subspace*[†] $\mathrm{Sp}\{p,q\}$ *is timelike if and only if* $|h(p,q)| > \|p\|\,\|q\|$, *where* p *and* q *are spacelike.*

Proof.

1. From Corollary 1.2.15 and 1.2.16 in [1], we can decompose $\mathbb{H}_{SL}$ as $\mathbb{H}_{SL} = \mathrm{Sp}\{q\} + q^{\perp}$. $\breve{q} \in q^{\perp}$ is spacelike split-like quaternion and $p = \lambda q + \breve{q}$ is timelike split-like quaternion, where $\lambda \in \mathbb{R}$. Then,

$$h(p,p) = h(\lambda q + \breve{q}, \lambda q + \breve{q}) = \lambda^2 h(q,q) + h(\breve{q},\breve{q})$$

or

$$\lambda^2 h(q,q) = h(p,p) - h(\breve{q},\breve{q}). \tag{3}$$

Also, using the fact that $h(q,\breve{q}) = 0$ and equation (3) in

$$h(q,p)^2 = h(q,\lambda q + \breve{q})^2 = \lambda^2 h(q,q)^2 + h(q,\breve{q})^2 + 2\lambda h(q,q)h(q,\breve{q}),$$

we get

$$h(q,p)^2 = (h(p,p) - h(\breve{q},\breve{q}))h(q,q). \tag{4}$$

Then, $I_p = h(p,p) < 0$, $I_q = h(q,q) < 0$ and $I_{\breve{q}} = h(\breve{q},\breve{q}) > 0$. The inequality $h(p,p) - h(\breve{q},\breve{q}) \leq h(p,p)$ leads to

$$h(p,q)^2 \geq h(p,p)h(q,q) = \|p\|^2\|q\|^2 > 0.$$

Therefore, it follows that $|h(p,q)| \geq \|p\|\,\|q\|$, as desired.

If p and q are parallel timelike, i.e., $p = \lambda q$, then $\breve{q} = 0$. Upon substituting $\breve{q} = 0$ into (4), we obtain

$$h(p,q)^2 = h(p,p)h(q,q) = \|p\|^2\|q\|^2.$$

Thus, the equation $|h(p,q)| = \|p\|\,\|q\|$ is satisfied.

[†] Let $V \neq \{0\}$ be a subspace of $\mathbb{H}_{SL}$. Then the subspace V is ***spacelike*** if $\langle\cdot,\cdot\rangle_V$ is positive definite, ***timelike*** if $\langle\cdot,\cdot\rangle_V$ is negative definite, or indefinite and non-degenerate and ***lightlike*** if $\langle\cdot,\cdot\rangle_V$ is degenerate [12].

2. $(\Rightarrow)$: Assume that $|h(p,q)| \leq \|p\|\,\|q\|$. For $0 \neq \lambda, \mu \in \mathbb{R}$, we have

$$\begin{aligned} I_{\lambda p+\mu q} &= (\lambda p + \mu q)\overline{(\lambda p + \mu q)} \\ &= (\lambda p\overline{\lambda p}) + (\lambda p\overline{\mu q}) + (\mu q\overline{\lambda p}) + (\mu q\overline{\mu q}) \\ &= \|\lambda p\|^2 + \|\mu q\|^2 + 2h(\lambda p, \mu q). \end{aligned}$$

$-\|p\|\,\|q\| \leq h(p,q) \leq \|p\|\,\|q\|$ gives us

$$\begin{aligned} h(\lambda p + \mu q, \lambda p + \mu q) = I_{\lambda p+\mu q} &\geq \|\lambda p\|^2 + \|\mu q\|^2 - 2\|\lambda p\|\,\|\mu q\| \\ &= (\|\lambda p\| - \|\mu q\|)^2 > 0. \end{aligned}$$

Therefore, $\langle , \rangle|_V$ is positive definite and $V \subset \mathbb{H}_{SL}$ which is the subspace $\mathrm{Sp}\{p,q\}$ is spacelike.

$(\Leftarrow)$: If $V \subset \mathbb{H}_{SL}$ which is the subspace $\mathrm{Sp}\{p,q\}$ is spacelike, then $\langle , \rangle|_V$ is positive definite and Euclidean Cauchy-Schwarz inequality is satisfied. Thus, $|h(p,q)| = \|p\|\,\|q\|$ if and only if p and q are linearly dependent.

3. For $0 \neq \lambda, \mu \in \mathbb{R}$, we have the quadratic equation

$$I_{\lambda p+\mu q} = \lambda^2\|p\|^2 + \mu^2\|q\|^2 + 2\lambda\mu h(p,q)$$

in the variable λ. So, $I_{\lambda p+\mu q} < 0$ if and only if

$$\Delta = b^2 - 4ac = 4\mu^2 h(p,q)^2 - 4\|p\|^2\mu^2\|q\|^2 > 0,$$

that is, $|h(p,q)| > \|p\|\,\|q\|$. As $I_{\lambda p+\mu q} = h(\lambda p + \mu q, \lambda p + \mu q) < 0$, $\langle , \rangle|_V$ is negative definite and $V \subset \mathbb{H}_{SL}$ which is the subspace $\mathrm{Sp}\{p,q\}$ is timelike. ◀

Theorem 23 (Triangle Inequality)**.** *Let p, $q \in \mathbb{H}_{SL}$. Then the following statements are true:*

1. $\|p+q\| \geq \|p\| + \|q\|$, *where p and q are positive (negative) timelike.*
2. *If p, q and the subspace* $\mathrm{Sp}\{p,q\}$ *are spacelike, then* $\|p+q\| \leq \|p\| + \|q\|$.
3. $\|p+q\| \leq \|p\| + \|q\|$, *where* $|h(p,q)| \leq \|p\|\,\|q\|$ *if p is spacelike and q is timelike.*

Proof.

1. If p and q are timelike, then $\|p\|^2 = -h(p,p)$ and $\|q\|^2 = -h(q,q)$. From Theorem 13, $p+q$ is timelike and then $h(p+q,p+q) < 0$. We have

   $$\begin{aligned}\|p+q\|^2 &= -h(p+q,p+q) = -h(p,p) - 2h(p,q) - h(q,q)\\ &= \|p\|^2 + \|q\|^2 - 2h(p,q)\end{aligned}$$

   and we know $\|p+q\|^2 > 0$. From Theorem 22 and Theorem 10, we get $-h(p,q) \geq \|p\| \, \|q\|$. Then it can be seen that

   $$\|p+q\|^2 \geq \|p\|^2 + \|q\|^2 + 2\|p\| \, \|q\| = (\|p\| + \|q\|)^2.$$

   Thus, the inequality $\|p+q\| \geq \|p\| + \|q\|$ is satisfied.

2. If the subspace $\mathrm{Sp}\{p,q\}$ is spacelike, then $p+q$ is spacelike. Hence

   $$\begin{aligned}\|p+q\|^2 &= h(p+q,p+q) = h(p,p) + 2h(p,q) + h(q,q)\\ &= \|p\|^2 + \|q\|^2 + 2h(p,q).\end{aligned}$$

   According to Theorem 22, we obtain

   $$\|p+q\|^2 \leq \|p\|^2 + \|q\|^2 + 2\|p\| \, \|q\| = (\|p\| + \|q\|)^2$$

   that leads to $\|p+q\| \leq \|p\| + \|q\|$, as desired.

3. If p is spacelike and q is timelike, then we get

   $$\begin{aligned}\|p+q\|^2 &= |h(p+q,p+q)|\\ &= |h(p,p) + 2h(p,q) + h(q,q)|\\ &= \big|\|p\|^2 - \|q\|^2 + 2h(p,q)\big|\\ &\leq |\|p\|^2| + |-\|q\|^2| + |2h(p,q)|\\ &= \|p\|^2 + \|q\|^2 + 2|h(p,q)|.\end{aligned}$$

   Consequently, $\|p+q\| \leq \|p\| + \|q\|$, where $|h(p,q)| \leq \|p\| \, \|q\|$. ◄

Note 24. *Assume that p and q are spacelike and the subspace $\mathrm{Sp}\{p,q\}$ is timelike. Then $p+q$ is timelike. We can write*

$$\begin{aligned}\|p+q\|^2 = -h(p+q,p+q) &= -h(p,p) - 2h(p,q) - h(q,q)\\ &= -\|p\|^2 - \|q\|^2 - 2h(p,q)\end{aligned}$$

by using the fact that h is bilinear. From Theorem 22, we have

$$\|p+q\|^2 > -\|p\|^2 - \|q\|^2 + 2\|p\| \, \|q\| = -(\|p\| - \|q\|)^2.$$

Theorem 25. *Let* p, q, $p-q \in \mathbb{H}_{SL}$ *be spacelike and the subspaces* $\mathrm{Sp}\{p, p-q\}$ *and* $\mathrm{Sp}\{q, p-q\}$ *be spacelike. Then*

$$\big|\|p\| - \|q\|\big| \leq \|p-q\|.$$

Proof. The theorem can be proved similarly to the proof of the corresponding theorem in Euclidean space. ◀

2.7. Angles Between Split-like Quaternions and Polar Forms

Theorem 26. *Let* p, $q \in \mathbb{H}_{SL}$.

1. *There exists only one* $\eta(p,q) \in [0,\infty)$ *such that* $h(p,q) = -\|p\|\|q\| \cosh \eta(p,q)$, *where* p *and* q *are positive (negative) timelike.*
2. *There exists only one* $\eta(p,q) \in [0,\pi]$ *such that* $h(p,q) = \|p\|\|q\| \cos \eta(p,q)$, *where* p *and* q *are spacelike and the subspace* $\mathrm{Sp}\{p,q\}$ *is spacelike.*
3. *There exists only one* $\eta(p,q) \in (0,\infty)$ *such that* $|h(p,q)| = \|p\|\|q\| \cosh \eta(p,q)$, *where* p *and* q *are spacelike and the subspace* $\mathrm{Sp}\{p,q\}$ *is timelike.*
4. *There exists only one* $\eta(p,q) \in \mathbb{R}$ *such that* $|h(p,q)| = \|q\|\|p\| \sinh \eta(p,q)$, *where* p *is spacelike and* q *is timelike.*

Proof. Let p, $q \in \mathbb{H}_{SL}$. We will use Theorem 22.

1. Assume that p and q are positive (negative) timelike. Then

   $$\frac{-h(p,q)}{\|p\|\,\|q\|} \geq 1.$$

   By using the fact that the function cosh is bijective, we get the consequence: there exists only one $\eta(p,q) \in [0,\infty)$ such that

   $$\frac{-h(p,q)}{\|p\|\,\|q\|} = \cosh \eta(p,q) \geq 1.$$

   Thus, $h(p,q) = -\|p\|\,\|q\| \cosh \eta(p,q)$.

2. Assume that p and q are spacelike and the subspace $\mathrm{Sp}\{p,q\}$ is spacelike. Then

   $$\frac{|h(p,q)|}{\|p\|\,\|q\|} \leq 1.$$

By using the fact that the function cos is bijective, we get this consequence: there exists one $\eta(p,q) \in [0,\pi]$ such that

$$\frac{|h(p,q)|}{\|p\|\,\|q\|} = |\cos\eta(p,q)| \leq 1.$$

Therefore, $h(p,q) = \|p\|\,\|q\| \cos\eta(p,q)$.

3. Assume that p and q are spacelike and the subspace $\mathrm{Sp}\{p,q\}$ is timelike. Then
$$\frac{|h(p,q)|}{\|p\|\,\|q\|} > 1.$$
There exists one $\eta(p,q) \in (0,\infty)$ such that $\frac{|h(p,q)|}{\|p\|\,\|q\|} = \cosh\eta(p,q) > 1$. Hence, $|h(p,q)| = \|p\|\,\|q\| \cosh\eta(p,q)$.

4. Assume that p is spacelike and q is timelike. Then
$$\begin{cases} \frac{|h(p,q)|}{\|p\|\,\|q\|} \leq 1 & \text{while} \quad |h(p,q)| \leq \|p\|\,\|q\|, \\ \frac{|h(p,q)|}{\|p\|\,\|q\|} \geq 1 & \text{while} \quad |h(p,q)| \geq \|p\|\,\|q\|. \end{cases}$$
There exists one $\eta(p,q) \in (-\infty,\infty)$ such that $\frac{|h(p,q)|}{\|p\|\,\|q\|} = \sinh\eta(p,q)$, since the function sinh is bijective. Therefore,
$$|h(p,q)| = \|p\|\,\|q\| \sinh\eta(p,q).$$

◄

Theorem 27 (Polar Form for Timelike Split-like Quaternions)**.** *Every timelike split-like quaternion* $p = p_0 + p_1 i + p_2 j + p_3 k$ *can be written in the form*

$$p = \|p\| \left(\sinh\theta + \vec{v}\cosh\theta\right),$$

where timelike unit vector $\vec{v} = \frac{\mathbf{p}}{\|\mathbf{p}\|}$ *and* $\sinh\theta = \frac{p_0}{\|p\|}$, $\cosh\theta = \frac{\|\mathbf{p}\|}{\|p\|}$.

Theorem 28 (De Moivre's Formula for Timelike Unit Split-like Quaternions)**.** *Let p be a timelike unit split-like quaternion. Then, we have*

$$p^n = \begin{cases} \sinh(n\theta) + \vec{v}\cosh(n\theta), & n \text{ is odd} \\ \cosh(n\theta) + \vec{v}\sinh(n\theta), & n \text{ is even} \end{cases}$$

for every $n \in \mathbb{Z}^+$.

Proof. The theorem can be proved by using mathematical induction and trigonometric equations. ◀

Theorem 29 (Polar Form for Spacelike Split-like Quaternions)**.** *Let $p = p_0 + \mathbf{p} \in \mathbb{H}_{SL}$ be a spacelike split-like quaternion.*

1. *If $\mathbf{p}$ is spacelike, then p can be written in the form*

$$p = \|p\| \left(\cos\theta + \vec{w}\sin\theta\right),$$

where spacelike unit vector $\vec{w} = \frac{\mathbf{p}}{\|\mathbf{p}\|}$ and $\cos\theta = \frac{p_0}{\|p\|}$, $\sin\theta = \frac{\|\mathbf{p}\|}{\|p\|}$.

2. *If $\mathbf{p}$ is timelike, then p can be written in the form*

$$p = \|p\| \left(\cosh\theta + \vec{u}\sinh\theta\right),$$

where timelike unit vector $\vec{u} = \frac{\mathbf{p}}{\|\mathbf{p}\|}$ and $\cosh\theta = \frac{p_0}{\|p\|}$, $\sinh\theta = \frac{\|\mathbf{p}\|}{\|p\|}$.

3. *If p is unit and $\mathbf{p}$ is lightlike, then p can be written in the form*

$$p = \pm 1 + \mathbf{p}.$$

Theorem 30 (De Moivre's Formula for Spacelike Unit Split-like Quaternions)**.** *Let $p = p_0 + \mathbf{p} \in \mathbb{H}_{SL}$ be a spacelike unit split-like quaternion.*

1. *If $\mathbf{p}$ is spacelike, then we have*

$$p^n = \cos(n\theta) + \vec{w}\sin(n\theta),$$

2. *If $\mathbf{p}$ is timelike, then we have*

$$p^n = \cosh(n\theta) + \vec{u}\sinh(n\theta),$$

3. *If $\mathbf{p}$ is lightlike and $p_0 = 1$, then we have*

$$p^n = 1 + n\mathbf{p},$$

4. *If $\mathbf{p}$ is lightlike and $p_0 = -1$, then we have*

$$p^n = \begin{cases} 1 - n\mathbf{p}, & n \text{ is even}, \\ -1 + n\mathbf{p}, & n \text{ is odd} \end{cases}$$

for every positive integer n.

Proof. By using mathematical induction and trigonometric equations, we can prove the statements (1),(2) and (4) of the theorem. Below, we provide the proof of statement (3).

3. Assume that $p = 1 + \mathbf{p}$ such that $\mathbf{p}^2 = 0$. By using mathematical induction,

   - ⋆ for $n = 1$ the result is trivial,
   - ⋆ suppose that $p^{n-1} = 1 + (n-1)\mathbf{p}$ for $n-1$,
   - ⋆ multiplying the equation by p, we get $p^n = pp^{n-1}$. Then we have

$$p^n = pp^{n-1} = (1+\mathbf{p})\left(1+(n-1)\mathbf{p}\right) = 1 + n\mathbf{p} + (n-1)\mathbf{p}^2 = 1 + n\mathbf{p}.$$

◀

Note 31. *Let $p = p_0 + p_1 i + p_2 j + p_3 k$ be a lightlike split-like quaternion with timelike vector part. Then,*

$$p = |p_0|\left(\frac{p_0}{|p_0|} + \epsilon\right),$$

where $\epsilon = \frac{\mathbf{p}}{\|\mathbf{p}\|}$ is a timelike unit vector.

Remark 3. *Let p be a lightlike split-like quaternion with a timelike vector part. Then, p can be written as*

$$p = \begin{cases} p_0(1+\epsilon), & p_0 > 0, \\ p_0(1-\epsilon), & p_0 < 0. \end{cases}$$

Theorem 32 (De Moivre's Formula for Lightlike Split-like Quaternions)**.** *Let p be a lightlike split-like quaternion with a timelike vector part and $n \in \mathbb{Z}^+$. Then, we have*

$$p^n = (2p_0)^{n-1} p.$$

Proof. Let $p_0 > 0$. Then, we can write $p = p_0(1+\epsilon)$ such that $\epsilon^2 = 1$. By using the mathematical induction,

- ⋆ for $n = 1$ the result is trivial,
- ⋆ suppose that $p^{n-1} = (2p_0)^{n-2} p$ for $n-1$,

⋆ by multiplying the equation by p, we get $p^n = pp^{n-1}$. By the induction hypothesis, we have

$$\begin{aligned} p^n = pp^{n-1} &= [p_0(1+\epsilon)]\left[(2p_0)^{n-2}p\right] = [p_0(1+\epsilon)]\left[(2p_0)^{n-2}p_0(1+\epsilon)\right] \\ &= (2p_0)^{n-2}p_0^2\,(2+2\epsilon) = (2p_0)^{n-1}\,[p_0\,(1+\epsilon)] \\ &= (2p_0)^{n-1}p. \end{aligned}$$

For $p_0 < 0$, the theorem can be proved in the same way. ◀

3. Conclusion

In this paper, we defined split-like quaternions and demonstrated that the set of split-like quaternions $\mathbb{H}_{SL}$ is a non-commutative and non-associative algebra. Moreover, it has been shown that this space is an $\mathbb{R}$-vector space. We introduced the inner product and vector product for split-like quaternions, providing their properties. Different types of matrices corresponding to split-like quaternions were given with their algebraic properties. The characteristics of split-like quaternions were defined, and the characteristics of the sum of two split-like quaternions were examined. The definitions of inverse and norm for split-like quaternions are given. Inequalities provided by the inner product, such as the Cauchy Schwarz and triangle inequalities, were demonstrated. The angle between two split-like quaternions was given and their polar forms were shown. Finally, it was obtained that any type of split-like quaternion satisfies De Moivre's formula. In future research, the differential geometry and matrix theory of split-like quaternions will be investigated.

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