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Commutator of the Maximal Function in Total Morrey Spaces for the Dunkl Operator on the Real Line

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Abstract. On the real line, the Dunkl operators D_ν are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In this paper, in the setting $\mathbb{R}$ we study the commutators of the maximal operator associated with the Dunkl operator $[b, M_\nu]$ in the total D_ν-Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$. We give necessary and sufficient conditions for the boundedness of the operator $[b, M_\nu]$ on total D_ν-Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ when b belongs to $BMO(\mathbb{R}, dm_\nu)$ spaces, whereby some new characterizations for certain subclasses of $BMO(\mathbb{R}, dm_\nu)$ spaces are obtained.

Key Words and Phrases: Maximal operator; total D_ν-Morrey space; Dunkl operator; commutator; BMO

2010 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

On the real line, the Dunkl operators Λ_ν are differential-difference operators introduced in 1989 by Dunkl [11]. For a real parameter $\nu \geq -1/2$, we consider the *Dunkl operator*, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x) - f(-x)}{2x}, \quad x \in \mathbb{R}.$$

Note that $D_{-1/2} = d/dx$.

Let $\nu > -1/2$ be a fixed number and m_ν be the *weighted Lebesgue measure* on $\mathbb{R}$, given by

$$dm_\nu(x) := \left(2^{\nu+1}\Gamma(\nu+1)\right)^{-1} |x|^{2\nu+1}\, dx, \quad x \in \mathbb{R}.$$

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For any $x \in \mathbb{R}$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R} : |y| \in]\max\{0, |x| - r\}, |x| + r[\ \}$ be a Dunkl-ball in $\mathbb{R}$. Then $B(0,r) =]-r, r[$ and $m_\nu B(0,r) = c_\nu\, r^{2\nu+2}$, where $c_\nu := \left[2^{\nu+1}\,(\nu+1)\,\Gamma(\nu+1)\right]^{-1}$.

The *maximal operator* M_ν associated with the Dunkl operator on the real line is given by

$$M_\nu f(x) := \sup_{r>0} \big(m_\nu(B(x,r))\big)^{-1} \int_{B(x,r)} |f(y)|\, dm_\nu(y), \quad x \in \mathbb{R},$$

and *sharp maximal operator* $M_\nu^\sharp$ associated with the Dunkl operator on the real line is given by

$$M_\nu^\sharp f(x) := \sup_{r>0} \big(m_\nu(B(x,r))\big)^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\, dm_\nu(y), \quad x \in \mathbb{R},$$

where $f_{B(x,r)} := \big(m_\nu(B(x,r))\big)^{-1} \int_{B(x,r)} f(y)\; dm_\nu(y)$. For a fixed $q \in (0,1)$, any suitable function h and $x \in \mathbb{R}$, let $M_{\nu,q}^\sharp h(x) = \big(M_\nu^\sharp\big(|h|^q\big)(x)\big)^{1/q}$ and $M_{\nu,q} h(x) = \big(M_\nu\big(|h|^q\big)(x)\big)^{1/q}$.

The *maximal commutator* $M_{b,\nu}$ associated with the Dunkl operator on the real line and with a locally integrable function $b \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$ is defined by

$$M_{b,\nu} f(x) := \sup_{r>0} \big(m_\nu(B(x,r))\big)^{-1} \int_{B(x,r)} |b(x) - b(y)|\, |f(y)|\, dm_\nu(y), \quad x \in \mathbb{R}.$$

We can define the (nonlinear) commutator of the maximal operator M_ν with a locally integrable function b by

$$[b, M_\nu](f)(x) = b(x) M_\nu(f)(x) - M_\nu(bf)(x).$$

For more details about the operators $M_{b,\nu}$ and $[b, M_\nu]$, we refer to [6, 22] and references therein.

It is well known that maximal and fractional maximal operators play an important role in harmonic analysis (see [34]). Also, the fractional maximal function and the fractional integral, associated with differential-difference Dunkl operators D_ν play an important role in Dunkl harmonic analysis, differentiation theory and PDE's. The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was developed in [4, 5, 24]. The Dunkl operator and Dunkl transform considered here are the rank-one case of the general Dunkl theory, which is associated with a finite reflection group acting on a Euclidean space. The Dunkl theory provides a useful framework for the study of multivariable analytic structures and has gained considerable interest in various fields of mathematics and physical applications (see, for example, [12]). The maximal function, the fractional integral and related topics associated with the Dunkl differential-difference operator have been research areas for many mathematicians such as C. Abdelkefi and M. Sifi [2], V.S. Guliyev and Y.Y. Mammadov [4, 5, 6], Y.Y. Mammadov [21], L. Kamoun [16], M.A. Mourou [25], F. Soltani [32, 33], K. Trimeche [35] and others. Moreover, the results on $L_\Phi(\mathbb{R}, dm_\nu)$-boundedness of fractional maximal operator and its commutators associated with D_ν were obtained in [6, 22].

It is well known that the maximal operator plays an important role in harmonic analysis (see [34]). Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In this paper, in the framework of this analysis in the setting $\mathbb{R}$, we study the boundedness of the maximal commutator $M_{b,\nu}$ and the commutator of the maximal operator $[b, M_\nu]$ on total D_ν-Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$, when b belongs to the space $BMO(\mathbb{R}, dm_\nu)$, by which some new characterizations of the space $BMO(\mathbb{R}, dm_\nu)$ are given.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries in the Dunkl setting on $\mathbb{R}$

Definition 1. *Let* $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. *We denote by* $L_{p,\lambda}(\mathbb{R}, dm_\nu)$ *the Morrey space [26] (*$\equiv$ D_ν*-Morrey space), by* $\widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ *the modified Morrey space [26] (*$\equiv$ *modified* D_ν*-Morrey space), and by* $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ *the total Morrey space [27, 28] (*$\equiv$ *total* D_ν*-Morrey space) associated with the Dunkl operator, the set of all classes of locally integrable functions* f*, with the finite norms*

$$\|f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} t^{-\frac{\lambda}{p}}\, \|f\|_{L_p(B(x,t),dm_\nu)},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} [t]_1^{-\frac{\lambda}{p}}\, \|f\|_{L_p(B(x,t),dm_\nu)},$$

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\, [1/t]_1^{\frac{\mu}{p}}\, \|f\|_{L_p(B(x,t),dm_\nu)},$$

respectively, see also [7, 9, 10, 29, 30].

Definition 2. *Let* $0 < p < \infty$, $\lambda \in \mathbb{R}$ *and* $\mu \in \mathbb{R}$. *We define the weak Morrey space* $L_{p,\lambda}(\mathbb{R}, dm_\nu)$ *[26] (*$\equiv$ *weak* D_ν*-Morrey space), the weak modified Morrey space* $\widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ *[26] (*$\equiv$ *weak modified* D_ν*-Morrey space), and the weak total*

Morrey space $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ [27] ($\equiv$ weak total D_ν-Morrey space) associated with the Dunkl operator, the set of all classes of locally integrable functions f, with the finite norms

$$\|f\|_{WL_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} t^{-\frac{\lambda}{p}} \, \|f\|_{WL_p(B(x,t),dm_\nu)},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} [t]_1^{-\frac{\lambda}{p}} \, \|f\|_{WL_p(B(x,t),dm_\nu)},$$

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, \|f\|_{WL_p(B(x,t),dm_\nu)},$$

respectively.

Lemma 1. [23, 28] *If $0 < p < \infty$, $0 \le \mu \le \lambda \le 2\nu + 2$, then*

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = L_{p,\lambda}(\mathbb{R}, dm_\nu) \cap L_{p,\mu}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \max\left\{\|f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R},dm_\nu)}\right\}.$$

Lemma 2. [23, 28] *If $0 < p < \infty$, $0 \le \mu \le \lambda \le 2\nu + 2$, then*

$$WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = WL_{p,\lambda}(\mathbb{R}, dm_\nu) \cap WL_{p,\mu}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \max\left\{\|f\|_{WL_{p,\lambda}(\mathbb{R},dm_\nu)}, \|f\|_{WL_{p,\mu}(\mathbb{R},dm_\nu)}\right\}.$$

Remark 1.. *If $0 < p < \infty$, and $\lambda > 2\nu + 2$ or $\mu < 0$, then*

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = \Theta(\mathbb{R}),$$

where $\Theta \equiv \Theta(\mathbb{R})$ is the set of all functions equivalent to 0 on $\mathbb{R}$.

Lemma 3. [23] *If $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le 2\nu + 2$ and $0 \le \mu_1 \le \mu_2 \le 2\nu + 2$, then*

$$L_{p,\lambda_1,\mu_1}(\mathbb{R}, dm_\nu) \subset\!\!\succ L_{p,\lambda_2,\mu_2}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R},dm_\nu)} \le \|f\|_{L_{p,\lambda_1,\mu_1}(\mathbb{R},dm_\nu)}.$$

Lemma 4. [23] *If $0 < p < \infty$, $0 \le \lambda \le 2\nu + 2$ and $0 \le \mu \le 2\nu + 2$, then*

$$L_{p,2\nu+2,\mu}(\mathbb{R}, dm_\nu) \subset\!\!\succ L_\infty(\mathbb{R}, dm_\nu) \subset\!\!\succ L_{p,\lambda,2\nu+2}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,2\nu+2}(\mathbb{R},dm_\nu)} \le c_\nu^{1/p} \|f\|_{L_\infty(\mathbb{R},dm_\nu)} \le \|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R},dm_\nu)}.$$

Lemma 5. *[23] If* $0 \le \lambda < 2\nu + 2$, $0 \le \mu < 2\nu + 2$, $0 \le \alpha < 2\nu + 2 - \lambda$ *and* $0 \le \beta < 2\nu + 2 - \mu$, *then for* $\frac{2\nu+2-\lambda}{\alpha} \le p \le \frac{2\nu+2-\mu}{\beta}$

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) \subset\!\!\succ L_{1,2\nu+2-\alpha,2\nu+2-\beta}(\mathbb{R}, dm_\nu)$$

and for $f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ *the inequality*

$$\|f\|_{L_{1,2\nu+2-\alpha,2\nu+2-\beta}(\mathbb{R},dm_\nu)} \le c_\nu^{1/p'} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}$$

is valid.

3. Maximal commutators $M_{b,\alpha,\nu}$ in total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$

In this section, we investigate the boundedness of the maximal commutator $M_{b,\nu}$ in total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

The following Guliyev type local estimates are valid (see also [3]).

Lemma 6. *Let* $1 \le p < \infty$ *and* $B(x,r)$ *be any Dunkl-ball in* $\mathbb{R}$. *If* $p > 1$, *then the inequality*

$$\|M_\nu f\|_{L_p(B(x,r),dm_\nu)} \lesssim r^{\frac{2\nu+2}{p}} \sup_{t>2r} t^{-\frac{2\nu+2}{p}} \|f\|_{L_p(B(x,t),dm_\nu)} \tag{1}$$

holds for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}, dm_\nu)$.

Moreover, if $p = 1$, *then the inequality*

$$\|M_\nu f\|_{WL_1(B(x,r),dm_\nu)} \lesssim r^{2\nu+2} \sup_{t>2r} t^{-2\nu-2} \|f\|_{L_1(B(x,t),dm_\nu)} \tag{2}$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}, dm_\nu)$.

Proof. Let $1 \le p < \infty$. For arbitrary Dunkl-ball $B = B(x,r)$, let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{^{\complement}(2B)}$.

$$\|M_\nu f\|_{L_p(B,dm_\nu)} \le \|M_\nu f_1\|_{L_p(B,dm_\nu)} + \|M_\nu f_2\|_{L_p(B,dm_\nu)}.$$

By the continuity of the operator $M_\nu : L_p(\mathbb{R}, dm_\nu) \to L_p(\mathbb{R}, dm_\nu)$ (see, for example, [26]), we have

$$\|M_\nu f_1\|_{L_p(B,dm_\nu)} \lesssim \|f\|_{L_p(2B,dm_\nu)}.$$

Let y be an arbitrary point from B. If $B(y,\tau) \cap {}^{\complement}(2B) \ne \emptyset$, then $\tau > r$. Indeed, if $z \in B(y,\tau) \cap {}^{\complement}(2B)$, then $\tau > |y - z| \ge |x - z| - |x - y| > 2r - r = r$.

On the other hand, $B(y,\tau)\cap {}^{\complement}(2B)\subset B(x,2\tau)$. Indeed, $z\in B(y,\tau)\cap {}^{\complement}(2B)$. Then we get $|x-z|\le |y-z|+|x-y|<\tau+r<2\tau$.

Hence,

$$\begin{aligned} M_\nu f_2(y) &= \sup_{\tau>0}\frac{1}{m_\nu(B(y,\tau))}\int_{B(y,\tau)\cap {}^{\complement}(2B)}|f(z)|\,dm_\nu(z) \\ &\le 2^{2\nu+2}\sup_{\tau>r}\frac{1}{m_\nu(B(x,2\tau))}\int_{B(x,2\tau)}|f(z)|\,dm_\nu(z) \\ &= 2^{2\nu+2}\sup_{\tau>2r}\frac{1}{m_\nu(B(x,\tau))}\int_{B(x,\tau)}|f(z)|\,dm_\nu(z). \end{aligned}$$

Therefore, for all $y\in B$ we have

$$M_\nu f_2(y)\le 2^{2\nu+2}\sup_{\tau>2r}\frac{1}{m_\nu(B(x,\tau))}\int_{B(x,\tau)}|f(z)|\,dm_\nu(z). \tag{3}$$

Applying Hölder's inequality, we get

$$M_\nu f_2(y)\lesssim \sup_{\tau>2r}\frac{1}{m_\nu(B(x,\tau))^{\frac{1}{p}}}\int_{B(x,\tau)}|f(z)|^p\,dm_\nu(z). \tag{4}$$

Thus,

$$\begin{aligned} \|M_\nu f\|_{L_p(B,dm_\nu)} &\lesssim \|f\|_{L_p(2B,dm_\nu)} \\ &\quad + m_\nu(B(x,\tau))^{\frac{1}{p}}\left(\sup_{\tau>2r}\frac{1}{m_\nu(B(x,\tau))}\int_{B(x,\tau)}|f(z)|\,dm_\nu(z)\right). \end{aligned}$$

Let $p=1$. It is obvious that for any ball $B=B(x,r)$

$$\|M_\nu f\|_{WL_p(B,dm_\nu)}\le \|M_\nu f_1\|_{WL_p(B,dm_\nu)}+\|M_\nu f_2\|_{WL_p(B,dm_\nu)}.$$

By the continuity of the operator $M_\nu : L_1(\mathbb{R},dm_\nu)\to WL_1(\mathbb{R},dm_\nu)$, we have

$$\|M_\nu f_1\|_{WL_1(B,dm_\nu)}\lesssim \|f\|_{L_1(2B,dm_\nu)}.$$

Then by (4) we get the inequality (2). ◄

The following result completely characterizes the boundedness of M_ν on total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)$.

Theorem 1. *1. If* $f\in L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)$, $0\le\lambda<2\nu+2$ *and* $0\le\mu<2\nu+2$, *then* $M_\nu f\in WL_{1,\lambda,\mu}(\mathbb{R},dm_\nu)$ *and*

$$\|M_\nu f\|_{WL_{1,\lambda,\mu}(\mathbb{R},dm_\nu)} \le C_{1,\lambda,\mu}\, \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)}, \tag{5}$$

where $C_{1,\lambda,\mu}$ *is independent of* f.

2. If $f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$, $1 < p < \infty$, $0 \le \lambda < 2\nu + 2$ *and* $0 \le \mu < 2\nu + 2$, *then* $M_\nu f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_\nu f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \le C_{p,\lambda,\mu}\, \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}, \tag{6}$$

where $C_{p,\lambda,\mu}$ *depends only on* p,λ,μ *and* n.

Proof. Let $p = 1$. From the inequality (2) we get

$$\begin{aligned}
&\|M_\nu f\|_{WL_{1,\lambda,\mu}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda}\, [1/t]_1^{\mu}\, \|M_\nu f\|_{WL_1(B(x,t),dm_\nu)} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda}\, [1/t]_1^{\mu}\, t^{2\nu+2} \sup_{\tau>2t} \tau^{-2\nu+2}\, \|f\|_{L_1(B(x,\tau))} \\
&\lesssim \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda}\, [1/t]_1^{\mu}\, t^{2\nu+2} \sup_{\tau>t} \tau^{-2\nu+2}\, [\tau]_1^{\lambda}\, [1/\tau]_1^{-\mu} \\
&= \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{2\nu+2-\lambda}\, [1/t]_1^{\mu-2\nu+2} \sup_{\tau>t} [\tau]_1^{\lambda-2\nu+2}\, [1/\tau]_1^{2\nu+2-\mu} \\
&= \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)},
\end{aligned}$$

which implies that the operator M_ν is bounded from $L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$ to $WL_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Let $1 < p < \infty$. From the inequality (1) we get

$$\begin{aligned}
&\|M_\nu f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\, [1/t]_1^{\frac{\mu}{p}}\, \|M_\nu f\|_{L_p(B(x,t),dm_\nu)} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\, [1/t]_1^{\frac{\mu}{p}}\, t^{\frac{2\nu+2}{den}} \sup_{\tau>2t} \tau^{-\frac{2\nu+2}{p}}\, \|f\|_{L_p(B(x,\tau))} \\
&\lesssim \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\, [1/t]_1^{\frac{\mu}{p}}\, t^{\frac{2\nu+2}{p}} \sup_{\tau>t} \tau^{-\frac{2\nu+2}{p}}\, [\tau]_1^{\frac{\lambda}{p}}\, [1/\tau]_1^{-\frac{\mu}{p}} \\
&= \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{\frac{2\nu+2-\lambda}{p}}\, [1/t]_1^{\frac{\mu-2\nu+2}{p}} \sup_{\tau>t} [\tau]_1^{\frac{\lambda-2\nu+2}{p}}\, [1/\tau]_1^{\frac{2\nu+2-\mu}{p}} \\
&= \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)},
\end{aligned}$$

which implies that the operator M_ν is bounded in $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$. ◀

From Theorem 1, in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 1. [*2*, *20*, *32*] *1. If* $f \in L_{1,\lambda}(\mathbb{R}, dm_\nu)$ *and* $0 \le \lambda < 2\nu + 2$, *then* $M_\nu f \in WL_{1,\lambda}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_\nu f\|_{WL_{1,\lambda}(\mathbb{R},dm_\nu)} \le C_{1,\lambda}\, \|f\|_{L_{1,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{1,\lambda}$ *is independent of* f.

2. If $f \in L_{p,\lambda}(\mathbb{R}, dm_\nu)$, $1 < p < \infty$ *and* $0 \le \lambda < 2\nu + 2$, *then* $M_\nu f \in L_{p,\lambda}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_\nu f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)} \le C_{p,\lambda} \, \|f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{p,\lambda}$ *depends only on* p, λ *and* n.

Corollary 2. *[21] 1. If* $f \in \widetilde{L}_{1,\lambda}(\mathbb{R}, dm_\nu)$ *and* $0 \le \lambda < 2\nu + 2$, *then* $M_\nu f \in W\widetilde{L}_{1,\lambda}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_\nu f\|_{W\widetilde{L}_{1,\lambda}(\mathbb{R},dm_\nu)} \le C_{1,\lambda} \, \|f\|_{\widetilde{L}_{1,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{1,\lambda}$ *is independent of* f.

2. If $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$, $1 < p < \infty$ *and* $0 \le \lambda < 2\nu + 2$, *then* $M_\nu f \in \widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_\nu f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)} \le C_{p,\lambda} \, \|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{p,\lambda}$ *depends only on* p, λ *and* n.

We recall the definition of the space $BMO(\mathbb{R}, dm_\nu)$.

Definition 3. Suppose that $b \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$. Let

$$\|b\|_{BMO(\mathbb{R},dm_\nu)} := \sup_{x\in\mathbb{R}, r>0} \frac{1}{m_\nu(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r)}(x)| \; dm_\nu(y),$$

where

$$b_{B(x,r)} := \frac{1}{m_\nu(B(x,r))} \int_{B(x,r)} b(y) \; dm_\nu(y).$$

Define

$$BMO(\mathbb{R}, dm_\nu) := \{b \in L_1^{\rm loc}(\mathbb{R}, dm_\nu) \; : \; \|b\|_{BMO(\mathbb{R},dm_\nu)} < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}, dm_\nu)$ is a Banach space with respect to the norm $\| \cdot \|_{BMO(\mathbb{R},dm_\nu)}$.

We will need the following properties of BMO-functions (see [13]):

$$\|b\|_{BMO(\mathbb{R},dm_\nu)} \approx \sup_{x\in\mathbb{R}, r>0} \left(\frac{1}{m_\nu(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dm_\nu(y) \right)^{\frac{1}{p}}, \quad (7)$$

where $1 \le p < \infty$ and the positive equivalence constants are independent of b, and

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_{BMO(\mathbb{R},dm_\nu)} \ \ln\frac{t}{r} \quad \text{for any} \quad 0 < 2r < t, \tag{8}$$

where the positive constant C does not depend on b, x, r and t.

For any measurable set E with $m_\nu(E) < \infty$ and any suitable function f, the norm $\|f\|_{L(\log L),E}$ is defined by

$$|f\|_{L(\log L),E} = \inf\left\{\lambda > 0 : \frac{1}{m_\nu(E)}\int_E \frac{|f(x)|}{\lambda}\Big(2 + \frac{|f(x)|}{\lambda}\Big)dm_\nu(x) \le 1\right\}.$$

The norm $\|f\|_{\exp L,E}$ is defined by

$$|f\|_{\exp L,E} = \inf\left\{\lambda > 0 : \frac{1}{m_\nu(E)}\int_E \exp\Big(\frac{|f(x)|}{\lambda}\Big)dm_\nu(x) \le 2\right\}.$$

Then, for any suitable functions f and g, the generalized Hölder's inequality holds (see [31]):

$$\frac{1}{m_\nu(E)}\int_E |f(x)||g(x)|dm_\nu(x) \lesssim \|f\|_{\exp L,E}\,\|g\|_{L(\log L),E}. \tag{9}$$

The following John-Nirenberg inequalities on spaces of homogeneous type come from [17, Propositions 6, 7].

Lemma 7. *Let $b \in BMO(\mathbb{R}, dm_\nu)$. Then there exist constants $C_1, C_2 > 0$ such that for every ball $B \subset \mathbb{R}$ and every $\alpha > 0$ we have*

$$m_\nu\big(\{x \in B : |b(x) - b_B| > \alpha\}\big) \le C_1\, m_\nu(B)\exp\Big\{-\frac{C_2}{\|b\|_{BMO(\mathbb{R},dm_\nu)}}\,\alpha\Big\}.$$

By the generalized Hölder's inequality in Orlicz spaces (see [31, page 58]) and John-Nirenberg's inequality, we get (see also [18, (2.14)])

$$\frac{1}{|B|}\int_B \big|b(x) - b_B\big||g(x)|dm_\nu(x) \lesssim \|b\|_{BMO(\mathbb{R},dm_\nu)}\,\|g\|_{L(\log L),B}. \tag{10}$$

We refer, for instance, to [14] and [19] for details on this space and properties.

For a given ball B, we define the following local maximal function:

$$M_{B,\nu}f(x) = \sup_{B\supseteq B'\ni x}\big(m_\nu(B')\big)^{-1}\int_{B'}|f(y)|dm_\nu(y),$$

where the supremum is taken over all balls B' such that $x \in B' \subseteq B$.

For a function b defined on $\mathbb{R}$, we let, for any $x \in \mathbb{R}$,

$$b^-(x) := \begin{cases} 0\ , & \text{if } b(x) \geq 0, \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, for any $x \in \mathbb{R}$, $b^+(x) - b^-(x) = b(x)$.

Lemma 8. *Let $b \in L^1_{\rm loc}(\mathbb{R}, dm_\nu)$. Then the following statements are equivalent:*

1. *$b \in BMO(\mathbb{R}, dm_\nu)$ and $b^- \in L_\infty(\mathbb{R}, dm_\nu)$.*
2. *There exists $s \in [1, \infty)$ such that*

$$\sup_B \frac{\left\| \big(b - M_{B,\nu}(b)\big)\chi_B \right\|_{L_s(\mathbb{R},dm_\nu)}}{\|\chi_B\|_{L_s(\mathbb{R},dm_\nu)}} \leq C. \tag{11}$$

3. *For all $s \in [1, \infty)$ we have* (11).

Proof. Since the proof is similar to the corresponding one in [8], we omit it here. ◄

Lemma 9. *Let $f \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$. Then*

$$M_\nu\big(M_\nu f\big)(x) \approx \sup_{B \ni x} \|f\chi_B\|_{L(1+\log^+ L),\nu}. \tag{12}$$

Proof. Let B be a ball in $\mathbb{R}$. We are going to use weak type estimates (see [34], for instance): there exists a positive constant $c > 1$ such that every $f \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$ and for every $t > \big(1/m_\nu(B)\big) \int_B |f(x)| dm_\nu(x)$ we have

$$\frac{1}{c\,t} \int_{\{x \in B : |f(x)| > t\}} |f(x)| dm_\nu(x) \leq m_\nu(\{x \in B : M_\nu(f\,\chi_B)(x) > t\})$$
$$\leq \frac{c}{t} \int_{\{x \in B : |f(x)| > t/2\}} |f(x)| dm_\nu(x).$$

Then

$$\int_B M_\nu(f\,\chi_B)(x) dm_\nu(x) = \int_0^\infty m_\nu(\{x \in B : M_\nu(f\,\chi_B)(x) > \lambda\}) d\lambda$$
$$= \int_0^{|f|_B} m_\nu(\{x \in B : M_\nu(f\,\chi_B)(x) > \lambda\}) d\lambda$$
$$+ \int_{|f|_B}^\infty m_\nu(\{x \in B : M_\nu(f\,\chi_B)(x) > \lambda\}) d\lambda$$
$$= m_\nu(B)\,|f|_B + \int_{|f|_B}^\infty m_\nu(\{x \in B : M_\nu(f\,\chi_B)(x) > \lambda\}) d\lambda$$

$$
\begin{aligned}
&\geq m_\nu(B)\,|f|_B + \frac{1}{c}\int_{|f|_B}^{\infty}\Big(\int_{\{x\in B:|f(x)|>\lambda\}}|f(x)|dm_\nu(x)\Big)\frac{d\lambda}{\lambda}\\
&= m_\nu(B)\,|f|_B + \frac{1}{c}\int_{\{x\in B:|f(x)|>|f|_B\}}\Big(\int_{|f|_B}^{|f(x)|}\frac{d\lambda}{\lambda}\Big)|f(x)|dm_\nu(x)\\
&= m_\nu(B)\,|f|_B + \frac{1}{c}\int_{\{x\in B:|f(x)|>|f|_B\}}|f(x)|\log\frac{|f(x)|}{|f|_B}\,dm_\nu(x)\\
&\geq \frac{1}{c}\int_B |f(x)|\Big(1+\log^+\frac{|f(x)|}{|f|_B}\Big)dm_\nu(x).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_B M_\nu(f\,\chi_B)(x)dm_\nu(x) = \int_0^\infty m_\nu(\{x\in B: M(f\,\chi_B)(x)>\lambda\})d\lambda\\
&\approx \int_0^\infty m_\nu(\{x\in B: M_\nu(f\,\chi_B)(x)>2\lambda\})d\lambda\\
&= \int_0^{|f|_B} m_\nu(\{x\in B: M_\nu(f\,\chi_B)(x)>2\lambda\})d\lambda\\
&+ \int_{|f|_B}^\infty m_\nu(\{x\in B: M_\nu(f\,\chi_B)(x)>2\lambda\})d\lambda
\end{aligned}
$$

$$
\begin{aligned}
&\leq m_\nu(B)\,|f|_B + c\int_{|f|_B}^{\infty}\Big(\int_{\{x\in B:|f(x)|>\lambda\}}|f(x)|dm_\nu(x)\Big)\frac{d\lambda}{\lambda}\\
&= m_\nu(B)\,|f|_B + c\int_{\{x\in B:|f(x)|>|f|_B\}}|f(x)|\log\frac{|f(x)|}{|f|_B}\,dm_\nu(x)\\
&\leq c\int_B |f(x)|\Big(1+\log^+\frac{|f(x)|}{|f|_B}\Big)dm_\nu(x).
\end{aligned}
$$

Therefore, for all $f\in L_1^{\mathrm{loc}}(\mathbb{R},dm_\nu)$ we get

$$
M_\nu\big(M_\nu f\big)(x)\approx \sup_{B\ni x} m_\nu(B)^{-1}\int_B |f(x)|\Big(1+\log^+\frac{|f(x)|}{|f|_B}\Big)dm_\nu(x). \tag{13}
$$

Since

$$
1\leq \frac{1}{m_\nu(B)}\int_B |f(x)|\Big(1+\log^+\frac{|f(x)|}{|f|_B}\Big)dm_\nu(x),
$$

we have

$$
|f|_B\leq \|f\chi_B\|_{L(1+\log^+ L),\nu}.
$$

Using the inequality $\log^+(ab) \le \log^+ a + \log^+$ with $a, b > 0$, we get

$$\frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{|f|_B}\Big) dm_\nu(x)$$
$$= \frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \Big(\frac{|f(x)|}{\|f\chi_B\|_{L(1+\log^+ L),\nu}} \frac{\|f\chi_B\|_{L(1+\log^+ L),\nu}}{|f|_B}\Big)\Big) dm_\nu(x)$$
$$= \frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{\|f\chi_B\|_{L(1+\log^+ L),\nu}}\Big) dm_\nu(x)$$
$$+ \frac{1}{m_\nu(B)} \int_B |f(x)| \log^+ \frac{\|f\chi_B\|_{L(1+\log^+ L),\nu}}{|f|_B} dm_\nu(x)$$
$$\le \|f\chi_B\|_{L(1+\log^+ L),\nu} + |f|_B \log^+ \frac{\|f\chi_B\|_{L(1+\log^+ L),\nu}}{|f|_B}.$$

Since $\frac{\|f\chi_B\|_{L(1+\log^+ L),\nu}}{|f|_B} \ge 1$ and $\log t \le t$ when $t \ge 1$, we get

$$\frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{|f|_B}\Big) dm_\nu(x) \le 2\|f\chi_B\|_{L(1+\log^+ L),\nu}. \tag{14}$$

On the other hand, since

$$\|f\chi_B\|_{L(1+\log^+ L),\nu} = \frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{\|f\chi_B\|_{L(1+\log^+ L),\nu}}\Big) dm_\nu(x),$$

on using

$$|f|_B \le \|f\chi_B\|_{L(1+\log^+ L),\nu},$$

we get

$$\|f\chi_B\|_{L(1+\log^+ L),\nu} \lesssim \frac{1}{m_\nu(B)} \int_B |f(x)| \Big(1 + \log^+ \frac{|f(x)|}{|f|_B}\Big) dm_\nu(x). \tag{15}$$

Therefore, from (13), (14) and (15) we have (12). ◄

For proving our main results, we need the following estimate.

Lemma 10. [*15, Lemma* 1] *If $b \in BMO(\mathbb{R}, dm_\nu)$, then for any $q \in (0,1)$, there exists a positive constant C such that*

$$M_q^\sharp\big(M_{b,\nu} f\big)(x) \le C\|b\|_{BMO(\mathbb{R},dm_\nu)}\, M_\nu\big(M_\nu f\big)(x) \tag{16}$$

for every $x \in \mathbb{R}$ and for all $f \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$.

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $M_{b,\nu}$ on $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$, when b belongs to the space $BMO(\mathbb{R}, dm_\nu)$.

Theorem 2. *Let* $1 < p < \infty$, $0 \le \lambda < 2\nu + 2$ *and* $0 \le \mu < 2\nu + 2$. *The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}, dm_\nu)$.

(ii) *The operator* $M_{b,\nu}$ *is bounded on* $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}, dm_\nu)$. Combining Theorem 1 and Lemma 10, we get

$$\begin{aligned}\|M_{b,\nu}f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} &\lesssim \|M_q^\sharp\big(M_{b,\nu}f\big)\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}\\ &\lesssim \|b\|_{BMO(\mathbb{R},dm_\nu)}\|M_\nu\big(M_\nu f\big)\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}\\ &\lesssim \|b\|_{BMO(\mathbb{R},dm_\nu)}\|M_\nu f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}\\ &\lesssim \|b\|_{BMO(\mathbb{R},dm_\nu)}\|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that $M_{b,\nu}$ is bounded on $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$. Let $B = B(x,r)$ be a fixed ball. Consider $f = \chi_B$. It is easy to calculate that

$$\begin{aligned}\|\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} &\approx \sup_{y\in\mathbb{R}^n, t>0}\Big([t]_1^{-\lambda}\ [1/t]_1^{\mu}\int_{B(y,t)}\chi_B(z)dz\Big)^{\frac{1}{p}}\\ &= \sup_{y\in\mathbb{R}, t>0}\Big(|B(y,t)\cap B|\ [t]_1^{-\lambda}\ [1/t]_1^{\mu}\Big)^{\frac{1}{p}}\\ &= \sup_{B(y,t)\subseteq B}\Big(|B(y,t)|\ [t]_1^{-\lambda}\ [1/t]_1^{\mu}\Big)^{\frac{1}{p}} = r^{\frac{2\nu+2}{p}}\ [r]_1^{-\frac{\lambda}{p}}\ [1/r]_1^{\frac{\mu}{p}}. \quad (17)\end{aligned}$$

On the other hand, since

$$M_{b,\nu}(\chi_B)(x) \gtrsim \frac{1}{|B|}\int_B |b(z) - b_B|dz \quad \text{for all} \quad x \in B,$$

we have

$$\begin{aligned}\|M_{b,\nu}(\chi_B)\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} &\approx \sup_{B(y,t)}\Big(\ [t]_1^{-\lambda}\ [1/t]_1^{\mu}\int_{B(y,t)}|M_{b,\nu}(\chi_B)(z)|^p dz\Big)^{\frac{1}{p}}\\ &\gtrsim r^{\frac{n}{p}}\ [r]_1^{-\frac{\lambda}{p}}\ [1/r]_1^{\frac{\mu}{p}}\frac{1}{|B|}\int_B|b(z) - b_B|dz. \quad (18)\end{aligned}$$

Since by assumption

$$\|M_{b,\nu}(\chi_B)\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \lesssim \|\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)},$$

by (17) and (18) we get

$$\frac{1}{|B|}\int_B |b(z)-b_B|dz \lesssim 1.$$

◄

From Theorem 2 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

Corollary 3. *Let $1<p<\infty$ and $0\le\lambda<2\nu+2$. The following assertions are equivalent:*

(i) $b\in BMO(\mathbb{R},dm_\nu)$.

(ii) *The operator $M_{b,\nu}$ is bounded on $L_{p,\lambda}(\mathbb{R},dm_\nu)$.*

Corollary 4. *Let $1<p<\infty$ and $0\le\lambda<2\nu+2$. The following assertions are equivalent:*

(i) $b\in BMO(\mathbb{R},dm_\nu)$.

(ii) *The operator $M_{b,\nu}$ is bounded on $\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)$.*

4. Commutator of maximal operator $[b,M_\nu]$ in total Morrey spaces $L_{p,\lambda,\mu}$

In this section, we find necessary and sufficient conditions for the commutator of the maximal operator $[b,M_\nu]$ to be bounded on the spaces $L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)$.

For a function b defined on $\mathbb{R}$, we denote

$$b^-(x):=\begin{cases}0\;, & \text{if } b(x)\ge 0\\ |b(x)|, & \text{if } b(x)<0\end{cases}$$

and $b^+(x):=|b(x)|-b^-(x)$. Obviously, $b^+(x)-b^-(x)=b(x)$.

The following relations between $[b,M_\nu]$ and $M_{b,\nu}$ are valid:

Let b be any non-negative locally integrable function. Then for all $f\in L_1^{\rm loc}(\mathbb{R},dm_\nu)$ and $x\in\mathbb{R}$ the following inequality is valid:

$$\begin{aligned}\big|[b,M_\nu]f(x)\big| &= \big|b(x)M_\nu f(x)-M_\nu(bf)(x)\big|\\ &= \big|M_\nu(b(x)f)(x)-M_\nu(bf)(x)\big|\le M_\nu(|b(x)-b|f)(x)=M_{b,\nu}f(x).\end{aligned}$$

If b is any locally integrable function on $\mathbb{R}$, then

$$|[b,M_\nu]f(x)|\le M_{b,\nu}f(x)+2b^-(x)\,M_\nu f(x),\qquad x\in\mathbb{R}, \tag{19}$$

holds for all $f\in L_1^{\rm loc}(\mathbb{R},dm_\nu)$ (see, for example, [1, 7, 9, 30]).

Obviously, the operators $M_{b,\nu}$ and $[b, M_\nu]$ are essentially different from each other because $M_{b,\nu}$ is positive and sublinear while $[b, M_\nu]$ is neither positive nor sublinear.

Applying Theorem 2, we obtain the following result.

Theorem 3. *Let* $1 < p < \infty$*,* $0 \le \lambda \le 2\nu + 2$ *and* $0 \le \mu \le 2\nu + 2$*. Suppose that* b *is a real valued locally integrable function in* $\mathbb{R}$*. Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}, dm_\nu)$ *such that* $b^- \in L_\infty(\mathbb{R}, dm_\nu)$*.*

(ii) The operator $[b, M_\nu]$ *is bounded on* $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$*.*

Proof. *(i)* $\Rightarrow$ *(ii)*. Suppose that $b \in BMO(\mathbb{R}, dm_\nu)$. Combining Theorems 1 - 2 and the inequality (19), we get

$$\begin{aligned} \|[b, M_\nu]f\|_{L_{p,\lambda,\mu}} &\le \|M_{b,\nu}f + 2b^- \, M_\nu f\|_{L_{p,\lambda,\mu}} \\ &\le \|M_{b,\nu}f\|_{L_{p,\lambda,\mu}} + \|b^-\|_{L_\infty} \, \|M_\nu f\|_{L_{p,\lambda,\mu}} \\ &\lesssim \left(\|b\|_* + \|b^-\|_{L_\infty}\right) \|f\|_{L_{p,\lambda,\mu}}. \end{aligned}$$

(ii) $\Rightarrow$ *(i)*. Assume that $[b, M_\nu]$ is bounded on $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$. Let $B = B(x, r)$ be a fixed ball.

Since

$$M_\nu(b\chi_B)\chi_B = M_{B,\nu}(b) \quad \text{and} \quad M_\nu(\chi_B)\chi_B = \chi_B,$$

we have

$$\begin{aligned} |M_{B,\nu}(b) - b\chi_B| &= |M_\nu(b\chi_B)\chi_B - bM_\nu(\chi_B)\chi_B| \\ &\le |M_\nu(b\chi_B) - bM_\nu(\chi_B)| = |[b, M_\nu]\chi_B|. \end{aligned}$$

Hence

$$\|M_{B,\nu}(b) - b\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \le \|[b, M_\nu]\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}.$$

Thus from (17) we get

$$\begin{aligned} \frac{1}{|B|}\int_B |b - M_{B,\nu}(b)| &\le \left(\frac{1}{|B|}\int_B |b - M_{B,\nu}(b)|^p\right)^{\frac{1}{p}} \\ &\le |B|^{-\frac{1}{p}} \, [r]_1^{\frac{\lambda}{p}} \, [1/r]_1^{-\frac{\mu}{p}} \, \|b\chi_B - M_{B,\nu}(b)\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \\ &\lesssim r^{-\frac{2\nu+2}{p}} \, [r]_1^{\frac{\lambda}{p}} \, [1/r]_1^{-\frac{\mu}{p}} \, \|[b, M_\nu]\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \\ &\lesssim r^{-\frac{2\nu+2}{p}} \, [r]_1^{\frac{\lambda}{p}} \, [1/r]_1^{-\frac{\mu}{p}} \, \|\chi_B\|_{L_{p,\lambda,\mu}} \approx 1. \end{aligned}$$

Denote

$$E := \{x \in B : b(x) \le b_B\}, \quad F := \{x \in B : b(x) > b_B\}.$$

Since

$$\int_E |b(t) - b_B|\, dt = \int_F |b(t) - b_B|\, dt,$$

in view of the inequality $b(x) \le b_B \le M_{B,\nu}(b)$, $x \in E$, we get

$$\begin{aligned} \frac{1}{|B|} \int_B |b - b_B| &= \frac{2}{|B|} \int_E |b - b_B| \\ \le \frac{2}{|B|} \int_E |b - M_{B,\nu}(b)| &\le \frac{2}{|B|} \int_B |b - M_{B,\nu}(b)| \lesssim 1. \end{aligned}$$

Consequently, $b \in BMO(\mathbb{R}, dm_\nu)$.

In order to show that $b^- \in L_\infty(\mathbb{R}, dm_\nu)$, note that $M_{B,\nu}(b) \ge |b|$. Hence

$$0 \le b^- = |b| - b^+ \le M_{B,\nu}(b) - b^+ + b^- = M_{B,\nu}(b) - b.$$

Thus

$$(b^-)_B \le c,$$

and by the Lebesgue differentiation theorem we get

$$b^-(x) = \lim_{m_\nu B \to 0} \int_B b^-(y)\, dm_\nu(y) \le c \quad \text{for a.e. } x \in \mathbb{R}.$$

◄

From Theorem 3 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 5. *Let $1 < p < \infty$ and $0 \le \lambda < 2\nu + 2$. Suppose that b is a real valued locally integrable function in $\mathbb{R}$. Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}, dm_\nu)$ such that $b^- \in L_\infty(\mathbb{R}, dm_\nu)$.

(ii) The operator $[b, M_\nu]$ is bounded on $L_{p,\lambda}(\mathbb{R}, dm_\nu)$.

Corollary 6. *Let $1 < p < \infty$ and $0 \le \lambda < 2\nu + 2$. Suppose that b is a real valued locally integrable function in $\mathbb{R}$. Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}, dm_\nu)$ such that $b^- \in L_\infty(\mathbb{R}, dm_\nu)$.

(ii) The operator $[b, M_\nu]$ is bounded on $\widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$.

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