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On One Class of Banach Function Spaces Defined by Shift Operators

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Abstract. Main aim of this work is to define the class of Banach function spaces which cover rearrangement-invariant and additive-invariant spaces. These spaces are characterized by behavior of additive-shift operators generated by sufficiently small vectors. It is proved that considered spaces are isomorphic to so called additive-invariant spaces.

Key Words and Phrases: Banach function space, Sobolev space, shift operator, substitution operator, extension operator, additive-invariant space, translation-invariant norm.

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1. Introduction

In recent years, it was realized that the non-standard function spaces are needed to solve some problems in different branches of mechanics, mathematics, physics, non-linear elasticity theory, fluid mechanics, mathematical modeling of various physical phenomena, solvability problems for non-linear partial differential equations, etc. Mathematical modeling of electrorheological fluids is among those problems. Electrorheological fluids are those whose viscosity changes (often dramatically) when exposed to an electric field. These fluids are understood experimentally, but a complete theoretical model is still lacking. In fluid dynamics, they are treated as non-Newtonian fluids. In one extensively studied model the energy is given by the integral

$$\int_{\Omega} |Du\,(x)|^{p(x)}\,dx,$$

where Du is the symmetric part of the gradient of the velocity field and the exponent is a function of electric field.

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Morrey spaces, grand-Lebesgue spaces, L_p^*-weak spaces, Marcinkiewicz spaces, Orlicz spaces, variable Lebesgue spaces, etc. all belong to the above non-standard spaces. It is necessary to study these spaces from different points of view. We consider them mostly in the context of partial differential equations. The problems for various elliptic equations (both divergent and non-divergent) have been considered for different Banach spaces of functions (weighted Lebesgue spaces, Orlicz spaces, variable Lebesgue spaces) and this tradition continues today (see, e.g. [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18, 19]).

Rearrangement spaces are one of the largest classes of non-standard spaces. As is known, these spaces are invariant under measure preserving transformation. Along with this, we also consider other classes of spaces with translation invariant norms, i.e. the norms which are invariant with respect to the additive-shift operators. We call such spaces additive-invariant spaces, taking into account that additional conditions may be imposed on the spaces related to composition operators and singular operators.

In general, the Banach function spaces are not separable. Therefore, establishing classical facts in these spaces by the classical methods requires essential modification of the latter and a lot of preparation, concerning correctness of substitution (composition) operator, problems related to the extension operator for corresponding Sobolev spaces, etc. To this aim, based on the additive shift operator $(T_\delta f)(x) = f(x+\delta)$, the corresponding separable subspaces $X_s(\Omega)$ of these spaces are introduced, where the set of infinitely differentiable functions of compact support is dense.

In [4, 14, 15, 16, 17, 18, 19], under some conditions on rearrangement-invariant and additive-invariant Banach function spaces, the boundedness of substitution operators, the existence of extension operators, compactness criteria, Poincare, Fridrichs type inequalities, Schauder-type estimates, Fredholmness of one elliptic operator, and etc. are studied. Main aim of this work is to define wider class of Banach function spaces where the analogs of these results hold true.

2. Needful information

We will use the following standard notations:

N will be the set of natural numbers, Z_+ will denote the set of non-negative integers, $|x| = \sqrt{x_1^2 + ... + x_n^2}$ will mean the norm of $x = (x_1, ..., x_n)$, $m = |E|$ will stand for the Lebesgue measure of the set $E \subset R^n$, supp f will denote the support of the function f, $[X, Y]$ ($[X]$ if $X = Y$) will mean the space of bounded operators acting from Banach function space X to Banach function space Y (the space of bounded operators acting in Banach function space X if $X = Y$), and $\|T\|_{[X,Y]}$ will be the norm of the operator T in $[X, Y]$.

Some monographs have been dedicated to the theory of Banach function spaces. We follow the terminologies and agreements used in [1].

Let's make the following assumptions: $\mathbf{K} = \{(x_1, ..., x_n) : |x_i| < d\} \subset R^n$ will be some cube or $K = R^n$, and (K, m) will be a Lebesgue measure space. We will consider only the class of functions which are finite $m - a.e.$ By χ_E we will denote the characteristic function of m-measurable subset E.

Only relatively compact subdomains Ω of K will be considered, with $\Omega : \bar{\Omega} \subset K$. $\Omega + \delta$ will denote the shift of the domain Ω corresponding to the vector δ, i.e. $\Omega + \delta = \{t + \delta : t \in \Omega\}$, and we will consider only those vectors with $\Omega + \delta \subset \mathbf{K}$.

$X(\mathbf{K})$ will be a Banach function space defined on $\mathbf{K}$ with the norm ρ. For an arbitrary domain $\Omega \subset \mathbf{K} : \overline{\Omega} \subset \mathbf{K}$, $X(\Omega)$ means the space of restrictions of all functions from $X(\mathbf{K})$ to Ω with the corresponding norm, i.e.

$$X(\Omega) = \Big\{ f \in X(\mathbf{K}) : \|f\|_{X(\Omega)} = \|f\chi_\Omega\|_{X(\mathbf{K})} < \infty \Big\}.$$

Depending on circumstances, we will assume that $f \in X(\Omega, m)$ is extended by zero to $\mathbf{K}$ or to the whole of R^n.

The following set will be called a ***possible value*** of the shift vector: $a(\Omega) = \{\delta \in R^n : \Omega - \delta \subset \mathbf{K}\}$, and by $T_\delta : X(\Omega) \to X(\Omega - \delta)$ we denote the *additive-shift operator* defined as follows: $(T_\delta f)(x) = \begin{cases} f(x+\delta), & x + \delta \in \Omega, \\ 0, & x + \delta \notin \Omega. \end{cases}$

We also assume that

$$T_\delta f \in X(K), \forall \Omega : \bar{\Omega} \subset K, \forall \delta \in a(\Omega).$$

It is clear that if the space $X(K)$ is a rearrangement invariant space, then $T_\delta \in [X(\Omega), X(\Omega - \delta)]$, $\forall \delta \in a(\Omega)$, is an isometric operator.

$X_a(\Omega)$ will denote the subspace of all absolutely continuous functions, $X_b(\Omega)$ will be the closure of all bounded functions from $X(\Omega)$, and $C_0^\infty(\Omega)$ will denote the set of all infinitely differentiable functions of compact support on Ω.

3. On separable subspace generated by the shift operator

Let $\Omega \subset \mathbf{K} : \overline{\Omega} \subset \mathbf{K}$ be any domain. By $X_s(\Omega)$ we denote the subspace of all functions from $X(\Omega)$ such that

$\alpha)$ $\|T_\delta(f) - f\|_{X(\mathbf{K})} \to 0$, $\delta \to 0$, where $\delta \in R^n : \Omega - \delta \subset \mathbf{K}$ is a shift vector and $T_\delta f(x) = f(x + \delta)$ is a corresponding shift operator.

We will often assume that the following property holds true: $\beta)$ $\forall E_n \to \emptyset \Rightarrow \|\chi_{E_n}\|_{X(\mathbf{K})} \to 0$.

It provides that $X_b(K) = X_a(K)$. Consequently, $X_a(\Omega) = X_b(\Omega), \forall \Omega \subset K$.

Remark 1. *Let the embedding* $\exists p : 1 \le p < \infty : L_p(\Omega) \subset X(\Omega)$ *hold true. Then the relation* $\|\chi_E\|_X \le \|\chi_E\|_p$ *directly implies that* $X(\Omega)$ *has Property* β)*.*

Let $\omega(\cdot)$ be an infinitely differentiable function which is equal to zero for $t \ge 1$ and takes positive values for $t < 1$, and let $\omega_r(x) = cr^{-n}\omega\left(\frac{|x|^2}{r^2}\right)$, where c is defined by the equality

$$\int_{R^n} \omega_r(x)\,dx = 1,\ \forall r > 0.$$

Let u be any integrable function defined on $\Omega : \overline{\Omega} \subset K$. Consider the average function

$$u_r(x) = (\omega_r * u)(x) = \int_\Omega \omega_r(x-y)\,u(y)\,dy.$$

Here we assume that u is equal to zero on $K \backslash \bar{\Omega}$ and we consider only $r : \operatorname{supp} u_r \subset \mathbf{K}$.

Let us consider the following set of linear operators:

$$I_r : X_s(\Omega) \to X_s(\mathbf{K}) : I_r(u) = u_r;\ \forall r > 0,$$

$$I(u) = u\ \ (\forall u \in X(\Omega))\ \text{being an identity operator.}$$

The following propositions were proved in [14] for rearrangement-invariant spaces. It should be noted that they stay true in general case. For completeness, we state here some of those propositions.

Proposition 1. *Let* $X(\Omega)$ *be a Banach function space and* $f \in X(\Omega)$. *If* $\|T_\delta f - f\|_{X(K)} \to 0,\ \delta \to 0$, *then* $\|f_r - f\|_X \to 0$ *as* $r \to 0$.

Thus, in arbitrary Banach function space, the class $C_0^\infty(\Omega)$ is dense in the subspace $X_s(\Omega)$. Consequently, $X_s(\Omega)$ is always separable.

Proposition 2. *Let* $X(\mathbf{K})$ *be a Banach function space with Property* β)*. Then*

$$X_s(\Omega) = X_a(\Omega) = X_b(\Omega) = \overline{C_0^\infty(\Omega)}.$$

Taking into account that for a rearrangement-invariant space $X(K)$ with Boyd indices α_X, $\beta_X \in (0,1)$ Property β) is fulfilled automatically, by Remark 1 we conclude that it is not necessary to require that the space have Property β).

Let's introduce the spaces of functions $W_X^m(\Omega)$ and $W_{X_s}^m(\Omega)$: $W_X^m(\Omega) = \{f \in X : \partial^p f \in X, \forall p \in Z_+,\ |p| \le m\}$,

$$W_{X_s}^m(\Omega) = \left\{f \in W_X^m : \|T_\delta f - f\|_{W_X^m(\Omega)} \to 0,\ \delta \to 0\right\},$$

with the corresponding norms.

Since the shift operator is continuous on $W_X^m(\Omega)$, $WX_s^m(\Omega)$ is a closed subspace of $W_X^m(\Omega)$.

Consider the subspace $\overset{0}{W}{}_{X_s}^m(\Omega) = \overline{C_0^\infty}(\Omega)$ (closure is taken in the space $W_X^m(\Omega)$). It is clear that $u \in \overset{0}{W}{}_{X_s}^m(\Omega) \Rightarrow u \in \overset{0}{W}{}_{X_s}^m(\Omega_1)$ for every domain $\Omega_1 \supset \Omega$.

Proposition 3. *Let* $X(\mathbf{K})$ *be a Banach function space with Property* $\beta)$ *and* $u \in \overset{0}{W}{}_{X_s}^m(\Omega, m)$. *Then* $u_r \overset{W_X^m(K)}{\to} u$.

Thus, Proposition 3 is true for every Banach function space, not only for rearrangement-invariant space.

Consider the following inequality:

$$\exists C(\delta, \Omega) > 0 : \quad \|T_\delta f\|_{X(\Omega-\delta)} \leq C(\delta, \Omega)\|f\|_{X(\Omega)}, \forall f \in X(\Omega), \forall \delta \in a(\Omega), \qquad (1)$$

i.e. $\forall \delta \in a(\Omega) \Rightarrow T_\delta \in [X(\Omega), X(K)]$.

In particular, if

$$\|T_\delta f\|_{X(K)} = \|f\|_{X(\Omega)}, \forall \Omega \subset K, \forall f \in X(\Omega), \forall \delta \in a(\Omega),$$

then this norm is called a translation-invariant norm. In our previous works, we called such spaces *additive-invariant* spaces (because we consider only possible values of shift vectors).

Example 1. *Let* $X(K)$ *be a rearrangement-invariant space. Then it is clear that*

$$\forall \Omega : \bar{\Omega} \subset K, \forall f \in X(\Omega) \Rightarrow \|f\|_{X(K)} = \|T_\delta f\|_{X(K)}.$$

So (1) *is fulfilled, and we can choose* $C(\delta, \Omega) = 1$, $\forall \Omega$, $\forall \delta \in a(\Omega)$.

Example 2. *Consider the following space:*

$$X(R) = \left\{ f : D(f) = R^1 \& \|f\|_{X(R)} = \sum_{n \in Z} \alpha_n \left\|f \chi_{[n;\, n+1]}\right\|_{L_1(R)}, \alpha_n > 0, \forall n \in Z \right\}.$$

In general, if $\{\alpha_n\}$ is a non-constant bounded sequence, then the considered space is not a rearrangement-invariant space. Moreover, it is not an additive-invariant space. At the same time, its norm may have the property (1). Indeed, if $0 < m \leq \alpha_n \leq M < +\infty$, then it is clear that

$$m\|f\|_{L_1} \leq \|f\|_{X(R)} \leq M\|f\|_{L_1(R)}.$$

Therefore, this space is isomorphic to the space $L_1(R)$, but it is not additive-invariant. At the same time, the relation

$$m\|f\|_{X(\Omega)} \leq \|f(.+\delta)\|_{X(\Omega)} \leq M\|f\|_{X(\Omega)}, \forall\delta \in R, \forall\Omega \subset R, \bar{\Omega} \textit{ is compact},$$

holds true.

Nonetheless, it should be stated that the relation (1) implies that every space with this property is isomorphic to some additive-invariant space.

Statement 1. *Let $X(K)$ be some Banach function space for which the constant in* (1) *is independent of δ and Ω (i.e. the constant in* (1) *is universal). Then for arbitrary bounded domain $\Omega : \overline{\Omega} \subset \mathbf{K}$, $X(\Omega)$ is isometric to some additive-invariant Banach function space.*

Proof. It is sufficient to prove the statement for the case where K is a cube. Indeed, every bounded domain $\Omega : \overline{\Omega} \subset \mathbf{K}$ is contained in some such cube. Therefore, in case $K = R^n$ we can choose some cube containing the given domain.

So, let $K = \{x \in R^n : |x_i| < d\}$ be any cube, $d_0 : 0 < d_0 < d$ be some positive number and $K_{d_0} = \{x \in R^n : |x_i| < d_0\}$. First, let us prove the statement for K_{d_0}. Let's define on $X(\mathbf{K}_{d_0})$ the following norm:

$$\|f\|^+_{K_{d_{d_0}}} = \sup_{\delta:\, K_{d_0}-\delta\subset\mathbf{K}} \|f(.+\delta)\|_{X(\Omega-\delta)} \leq C\|f\|_{X\left(\mathbf{K}_{d_0}\right)}, \forall f \in X(\mathbf{K}). \quad (2)$$

From (2) we have

$$\|f\|_{X\left(K_{d_0}\right)} \leq \|f\|^+_{K_{d_0}} \leq (\text{by } (1))\, C\|f\|_{X\left(K_{d_0}\right)}, \forall f \in X(\mathbf{K}). \quad (3)$$

Therefore, finite sup in (2) exists. Consequently, the norm (2) is defined correctly. It is clear that it is a norm, and it is equivalent to the original norm and it is a translation-invariant norm. Indeed, by definition we have

$$\|f\|^+_{K_{d_0}} = \|T_\delta f\|^+_{K_{d_0}+\delta}, \forall\delta : |\delta| < |d - d_0|.$$

It is not difficult to verify that all axioms of Banach function space are fulfilled in $\left(X(K_{d_0}), \|.\|^+_{K_{d_0}}\right)$.

Therefore, $\left(X(K_{d_0}), \|.\|^+_{K_{d_0}}\right)$ is a Banach function space.

It is clear that for every $\Omega : \overline{\Omega} \subset \mathbf{K}$, $\exists d_0 \Rightarrow \bar{\Omega} \subset K_{d_0}$. Consider $X(\Omega)$ and assume that every function from $X(\Omega)$ is extended by zero to K_{d_0}. Then we have

$$\|f\|_\Omega = \|f\|_{K_{d_0}} \leq \|f\|^+_{K_{d_0}} \leq C\|f\|_{K_{d_0}}.$$

The statement is proved. ◄

Remark 2. *It can be seen from the proof of Statement 1 that if* $X(K)$ *is a Banach space of measurable functions, for which the condition* (1) *is fulfilled, then it is isomorphic to some Banach space of functions with translation-invariant norm.*

Corollary 1. *Suppose that the constant in* (1) *is independent of* δ*, but may depend on* Ω*. Then* $X(\Omega)$ *(in particular,* $X(K_{d_0})$, $\forall d_0 < d$*) is isometric to some Banach function space with translation-invariant norm.*

Proof. Let $\Omega \subset \Omega_1 \subset \mathbf{K}$. Taking into account the embedding $X(\Omega) \subset X(\Omega_1)$, we obtain $\Omega \subset \Omega_1 \subset \mathbf{K} \Rightarrow C(\delta, \Omega) \leq C(\delta, \Omega_1)$.

Therefore, $\forall \Omega \subset \mathbf{K}_{d_0}$ we have $C(\Omega) \leq C(\mathbf{K}_{d_0})$, i.e. with a fixed number d_0, K_{d_0} can be considered instead of K. Consequently, Statement 1 is applicable to K_{d_0}.

The corollary is proved. ◄

Corollary 2. *Let* $X(R^n)$ *have the property* (1) *and* $\Omega : \overline{\Omega} \subset R^n$ *be some bounded domain. Then either* $X(\Omega)$ *is isomorphic to some additive-invariant Banach function space or* $X_s(\Omega)$ *is isomorphic to Banach space with translation-invariant norm.*

As an application, let us consider the following statement, which was proved in [4] for additive-invariant spaces. Therefore, it is also true for the spaces which satisfy the conditions of Statement 1.

Corollary 3. *Let* $X(\Omega)$ *be a Banach function space with the properties* (1) *and* β)*. Then the following assertions are true:*

i) $\lim\limits_{\delta \to 0} \|T_\delta - I\|_{[X_s(\Omega), X(\mathbf{K})]} = 0$*; ii)*$\{I_r\}$ *is a bounded family of bounded operators and* $\lim\limits_{r \to 0} \|I_r - I\|_{[X_s(\Omega), X(\mathbf{K})]} = 0$*; iii) if* $U \subset X_s(\Omega)$ *is some bounded set, then* $\forall \varepsilon > 0\ \exists r_0 > 0\, \forall r < r_0 \Rightarrow \|u - u_r\|_{X(\mathbf{K})} < \varepsilon$*, i.e.* $U_r = \{\omega_r * u : u \in U\}$ *is an* ε*-net for* U*.*

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References

[1] Bennett, C., Sharpley, R. (1988) *Interpolation of Operators*, Academic Press.

[2] Bilalov, B.T., Sadigova S.R. (2021) *Interior Schauder-type estimates for higher-order elliptic operators in grand-Sobolev spaces*, Sahand Communications in Mathematical Analysis, **18(2)**, 129-148.

[3] Bilalov, B.T., Ahmadov, T.M., Zeren, Y., Sadigova, S.R. (2022) *Solution in the small and interior Schauder type Estimate for the m-th order Elliptic operator in Morrey- Sobolev spaces*, Azerb. Jour. Math., **12(2)**, 190-219.

[4] Bilalov, B.T., Mamedov, E.M., Sezer, Y., Nasibova, N. (2025) *Compactness in Banach function spaces. Poincare and Friedrichs inequalities*, Rendiconti del Circolo Matematico di Palermo Series 2, **74(68)**, 1-22.

[5] Bilalov, B.T., Sadigova, S.R. (2022) *On local solvability of higher order elliptic equations in rearrangement invariant spaces*, Sib. Math. J., **63(3)**, 425-437.

[6] Byun, S.S., Lee, M., Ok, J. (2015) $W^2_{p(*)}$ -regularity for elliptic equations in nondivergence form with BMO coefficients. Mathematische Annalen, Springer, 363, 1023-1052.

[7] Byun, S.S., Lee, M., Ok, J. (2018) *Nondivergence parabolic equations in weighted variable exponent spaces*, Transactions of the American Mathematical Society, **370**, 2263-2298.

[8] Byun, S.S., Lee, M. (2015) *Weighted estimates for nondivergence parabolic equations in Orlicz spaces*, Journal of Functional Analysis, **269(8)**, 2530-2563.

[9] Castillo, R.E., Rafeiro, H. (2016) *An introductory course in Lebesgue spaces*, Springer.

[10] Castilio, R.E., Rafeiro, H., Rojas, E.M. (2021) *Unique continuation of the quasilinear elliptic equation on Lebesgue spaces*, Azerb. J. of Math., **11(1)**, 136-153.

[11] Cejas, M.E., Duran, R. (2018) *Weighted a priori estimates for elliptic equations*, Studia Math **243(1)**, 13–24.

[12] Dong, H. Kim, D. (2015) *Elliptic and parabolic equations with measurable coefficients in weighted Sobolev spaces*, Advances in Mathematics, **274**, 681-735.

[13] Dong, H., Kim, D. (2018) *On L_p-estimates for elliptic and parabolic equations with Ap weights*, Trans. Amer. Math. Soc., **370(7)**, 5081-5130.

[14] Mamedov, E.M. (2022) *On substitution and extension operators in Banach-Sobolev function spaces*, Proceedings of the Institute of Math. and Mech., Nat. Ac. of Sciences of Azer., **48(1)**, 88-103.

[15] Mamedov, E.M., Ismailov, N.A. (2023) *On some structural properties in Banach function spaces*, Transactions. Nat. Acd sci. Azerb. Ser. Phys. Tech. Math. Sci. Mathematics, **43(4)**, 114-127.

[16] Mamedov, E.M., Nasibova, N.P., Sezer. Y. (2024) *Some remarks on integral operators in Banach function spaces and representation theorems in Banach-Sobolev spaces*, Azerb. J. of Math., 14(2), 189-204.

[17] Mamedov, E.M., Chetin, S. (2024) *Interior Schauder-type estimates for m-th order elliptic operators in rearrangement-invariant Sobolev spaces*, Turk J Math., **48(4)**, 793-816.

[18] Mamedov, E., Sezer, Y., Safarova, A.R. (2025) *On Solvability Of One Boundary Value Problem For Laplace Equation in Banach-Hardy Classes*, Journal of Contemporary Applied Mathematics, **15(1)**, 25-43.

[19] Sadigova, S.R., Mamedov, E.M., Nasibova, N.P. (2025) *On the index of a problem with an oblique derivative in weighted Sobolev space*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics., **44(4)**, 122-138.

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