UDC 519.622.2 https://doi.org/10.59849/0002-3078.2026.1.3

DETERMINATION OF THE FRACTIONAL ORDER OF OSCILLATORY SYSTEMS USING MASS DISPLACEMENT COORDINATES

Nazila Hajiyeva[1,2], Irada Aliyeva[3], Academician of ANAS Fikret Aliev[1,4]

[1]Institute of Applied Mathematics of Baku State University
[2]Institute of Data Science and Artificial Intelligence, Azerbaijan Technical University
n_hajiyeva@yahoo.com, f_aliev@yahoo.com

[3]Republican Seismic Survey Center of the Azerbaijan National Academy of Sciences
irade.alieva14@gmail.com

[4]Institute of Information Technologies of the Ministry of Science and Education of the Republic of Azerbaijan

Summary. *The present paper addresses the problem of determining the fractional order in an oscillatory process occurring within a Newtonian fluid. Initially, the governing equation is transformed into a Volterra integral equation of the second kind with respect to the displacement of mass coordinates. Subsequently, the solution of this integral equation is represented in the form of a Neumann series. Based on available statistical data, a quadratic functional characterizing the deviation of mass displacement coordinates is constructed, and the unknown fractional order is identified using the least squares method. An efficient computational algorithm for determining the fractional order is then proposed, and illustrative results are presented in tabular form. The outcomes of the computational procedure demonstrate that the identified fractional order satisfies the condition of vanishing first variation of the quadratic functional with an accuracy of* 10^{-8}.

Keywords: *Oscillatory system, fractional derivative, Volterra integral equation of the second kind, the least squares method*

Introduction

Recently, in contrast to classical oscillatory systems [10-15], significant attention has been devoted to fractional-order systems [1-4]. This is primarily due to the fact that oscillatory processes in Newtonian fluids inherently exhibit damping behavior, where a fractional-order derivative replaces the first-order derivative in the second term on the left-hand side of the governing equation [1,6]. Such problems arise, in particular, in the motion of a plunger in an oil well [5,7-9], where the oil acts as a damping medium whose characteristics must be determined. Consequently, the problem of identifying the unknown fractional order naturally emerges and can be formulated as an inverse problem. In practice, the only feasible approach is to utilize statistical data obtained from oil well operations. Based on these data, the fractional order can be identified using the least squares method.

In [6], the corresponding Volterra integral equation of the second kind is first discretized. Taking into account its solution and the available statistical data, a quadratic functional is constructed and minimized by applying a bisection procedure over the interval containing the fractional order. However, in the presented example, the first variation of the functional cannot be reduced below the order of 10^{-4}, indicating that the method proposed in [6] is not sufficiently efficient.

It should also be noted that in [6], when the mass of the plunger in the oscillatory system is sufficiently large, an asymptotic approach is employed by introducing the reciprocal of the mass as a small parameter. In this case, the resulting equation is reduced to a Volterra integral equation of the second kind with respect to the second-order derivative of the phase coordinate. An asymptotic decomposition is then applied, incorporated into the integral equation, and integrated twice to obtain the solution. Based on this solution and statistical data, a quadratic functional is constructed. By applying the least squares method, consistency between theoretical predictions and statistical observations is ensured, leading to the determination of a more accurate fractional order.It is important to emphasize that, unlike the approach in [6], the proposed method ensures that the first variation vanishes with an accuracy of order 10^{-8}, thereby providing significantly improved precision compared to the results reported in [6].

Reduction of the problem to a Volterra integral equation of the second kind

Let the motion of an object be described by the following system of linear differential equations with fractional derivative [1-4]

$$m\ddot{y}(x) + aD^{\alpha}y(x) + by(x) = = f(x), x \geq x_0 > 0, \quad (1)$$

with initial conditions

$$y(x_0) = 0, \dot{y}(x_0) = y_1, \quad (2)$$

$\alpha \in (1,2)$, $y(x)$ is the desired function, m, a, b, y_1, x_0 -given parameters, $f(x)$ is an external force.

Let us compose the following quadratic functional to find α:

$$J(\alpha) = \min_{\alpha}\left(y(l) - \sum_{j=1}^{s}\frac{y_j}{s}\right)^2, \quad (3)$$

where $y_j,\ j=\overline{1,s}$ statistical data for finding $\alpha,\ y(l)$ — the value of the solution to the problem (1)-(2) at the point l.

To solve the problem (1)-(3), we first reduce the problem (1)-(2) to the Volterra integral equation of the second kind with respect to the phase coordinate y(x).

To do this, we first integrate equation (1) from 0 to x [14]:

$$m\int_{x_0}^{x}\ddot{y}(t)dt + a\int_{x_0}^{x}D^{\alpha}y(t)dt + +b\int_{x_0}^{x}y(t)dt = m\dot{y}(x) - m\dot{y}(x_0) + +a\int_{x_0}^{x}D^{\alpha}y(t)dt + b\int_{x_0}^{x}y(t)dt = = \int_{x_0}^{x}f(t)dt. \quad (4)$$

Then we integrate equation (4) again from 0 to x:

$$my(x) - my(x_0) - m\dot{y}(x_0)x + +a\int_{x_0}^{x}dt\int_{x_0}^{t}d\tau\frac{d}{d\tau}\int_{x_0}^{\tau}\frac{(\tau-\xi)^{-\alpha}}{(-\alpha)!}y(\xi)d\xi + +b\int_{x_0}^{x}y(\tau)\,d\tau\int_{\tau}^{x}dt = \int_{x_0}^{x}f(\tau)\,d\tau\int_{\tau}^{x}dt. \quad (5)$$

Let's consider the condition (2) in expression (5) and make several simple transformations:

$$my(x) = -a\int_{x_0}^{x}\frac{(x-\xi)^{1-\alpha}}{(1-\alpha)!}y(\xi)\,d\xi - -b\int_{x_0}^{x}(x-\tau)\,y(\tau)d\tau + +\int_{x_0}^{x}(x-\tau)\,f(\tau)d\tau + my_1x. \quad (6)$$

Let us divide both parts of expression (6) by

$$y(x) = \int_{x_0}^{x} -\frac{1}{m}\left[a\frac{(x-t)^{1-\alpha}}{(1-\alpha)!} + b(x-t)\right]y(t)dt + +\frac{1}{m}\int_{x_0}^{x}(x-t)\,f(t)dt + y_1x.$$

Thus, we have reduced problem (1)-(2) to a Volterra integral equation of the second kind with respect to y(x):

$$y(x) + \int_{x_0}^{x}K(x-t)y(t)\,dt = F(x), \quad (7)$$

where

$$K(x-t)=\frac{1}{m}\left[a\frac{(x-t)^{1-\alpha}}{(1-\alpha)!}+b(x-t)\right]\equiv$$
$$=K_0(x-t), \quad (8)$$

$$F(x)=\frac{1}{m}\int_{x_0}^{x}(x-t)\,f(t)dt+y_1x. \quad (9)$$

To solve the equation (7) using the method of successive approximations, we first perform the first iteration for it. When performing the first iteration in equation (7), we take x=t in (7):

$$y(t)=-\int_{x_0}^{t}K(t-\tau)\,y(\tau)d\tau+F(t). \quad (10)$$

Let us consider expression (10) in equation (7):

$$y(x)+\int_{x_0}^{x}K_1(x,\tau)y(\tau)d\tau=F_1(x), \quad (11)$$

where

$$K_1(x,\tau)=-\int_{\tau}^{x}K(x-t)\,K(t-\tau)dt, \quad (12)$$

$$F_1(x)=F(x)-\int_{x_0}^{x}K(x-t)\,F(t)dt.$$

Now let's perform the second iteration of equation (7). To do this, we write x=τ in (7):

$$y(\tau)=-\int_{x_0}^{\tau}K(\tau-t)\,y(t)dt+F(\tau). \quad (13)$$

Let us consider expression (13) in (11):

$$y(x)+\int_{x_0}^{x}K_2(x,t)y(t)dt=F_2(x),$$

where

$$K_2(x,t)=-\int_{t}^{x}K_1(x,\tau)K(\tau-t)\,d\tau, \quad (14)$$
$$F_2(x)=F(x)-\sum_{j=0}^{1}\int_{x_0}^{x}K_j(x,t)F(t)dt.$$

Now let's evaluate the kernels included in the first and second iterations. Firstly, let's evaluate the kernel $K_1(x,\tau)$. To do this, consider the expression (8) in (12):

$$K_1(x,\tau)=$$
$$-\sum_{k=0}^{2}C_2^k\left(\frac{a}{m}\right)^{2-k}\left(\frac{b}{m}\right)^k\frac{(x-t)^{3-(2-k)\alpha}}{(3-(2-k)\alpha)!}. \quad (15)$$

Now let's evaluate the kernel $K_2(x,t)$. To do this, consider the expressions (8) and (15) in expression (14):

$$K_2(x-t)=-\sum_{k=0}^{3}C_3^k\left(\frac{a}{m}\right)^{3-k}\left(\frac{b}{m}\right)^k\frac{(x-t)^{5-(3-k)\alpha}}{(5-(3-k)\alpha)!}.$$

For the n-th iteration we get:

$$y(x)+\int_{x_0}^{x}K_n(x,t)y(t)dt=F_n(x), \quad (16)$$

where

$$K_n(x-t)=-\sum_{k=0}^{n+1}C_{n+1}^k\left(\frac{a}{m}\right)^{(n+1)-k}\left(\frac{b}{m}\right)^k\times$$
$$\times\frac{(x-t)^{2(n+1)-1-(n+1-k)\alpha}}{(2(n+1)-1-(n+1-k)\alpha)!}, \quad (17)$$

$$F_n(x)=F(x)-\sum_{j=0}^{n-1}\int_0^x K_j(x-t)\,F(t)dt.$$

If we take the limit in (16) for $n\to\infty$, we get:

$$y(x)=F(x)-\sum_{j=0}^{\infty}\int_{x_0}^{x}K_j(x,t)F(t)dt. \quad (18)$$

Thus, problem (1)-(2) is reduced to the Volterra integral equation (7), and its solution has the form of the Neumann series (18).

Finding the fractional order α.

Let us write expression (18) at the point x=l:

$$y(l)=F(l)-\sum_{j=0}^{\infty}\int_{x_0}^{l}K_j(l,t)F(t)dt. \quad (19)$$

Let us consider (19) in functional (3):

$$J(\alpha)=\min_{\alpha}\left(\frac{1}{m}\int_{x_0}^{l}(l-t)f(t)dt+y_1l+\right.$$
$$+\sum_{j=0}^{\infty}\int_0^l\sum_{k=0}^{n+1}C_n^k\left(\frac{a}{b}\right)n-$$
$$\left.k\left(\frac{b}{m}\right)^k\frac{(l-t)^{2n-1-(n-k)\alpha}}{(2n-1-(n-k)\alpha)!}-\sum_{p=1}^{s}\frac{y_p}{s}\right)^2. \quad (20)$$

Using the least squares method, the following condition is checked to determine the parameter α :

$$\frac{\partial J(\alpha)}{\partial \alpha} \approx \frac{J(\alpha+h)-J(\alpha)}{h} \approx 0. \tag{21}$$

So, let us present the following algorithm for solving problems (1)-(3).

Algorithm

1. Enter the values of the parameters $m, a, b, y_1, f, n, k, l, x_0$ included in problem (1)-(2).
2. Substitute the expression for F(t), according to (9).
3. Substitute the expression for $K_n(x-t)$, according to (17).
4. Enter the statistical data $y_p(x), p = \overline{1,s}$.
5. Construct the functional (20).
6. Using the least squares method, we check condition (21) to determine the parameter α.

Example

Let's consider the following example [10]:

$$m = 10^5, a = 3, b = 1, y_1 = 0, f = 8,$$
$$n = 1, k = 1, l = 1, s = 11,$$
$$y_1(x) = 0, y_2(x) = -0.67, y_3(x) = -0.34,$$
$$y_4(x) = 0.81, y_5(x) = 1.22, y_6(x) = 1.44,$$
$$y_7(x) = 1.57, y_8(x) = 1.66, y_9(x) = 1.72,$$
$$y_{10}(x) = 1.77, \ \ y_{11}(x) = 1.81.$$

Then, to solve equation (21), we insert the table:

Table

Defining the fractional order

	$\frac{\partial J(\alpha)}{\partial \alpha}, \alpha = 1.1$	$\frac{\partial J(\alpha)}{\partial \alpha}, \alpha = 1.2$	$\frac{\partial J(\alpha)}{\partial \alpha}, \alpha = 1.4$	$\frac{\partial J(\alpha)}{\partial \alpha}, \alpha = 1.85$
$h = 10^{-1}$	$-0.2443 \cdot 10^{-8}$	$-0.3160 \cdot 10^{-8}$	$-0.5506 \cdot 10^{-7}$	$-0.1472 \cdot 10^{-6}$
$h = 10^{-2}$	$-0.2173 \cdot 10^{-8}$	$-0\ .2804 \cdot 10^{-8}$	$-0.4796 \cdot 10^{-8}$	$-0.5462 \cdot 10^{-7}$
$h = 10^{-3}$	$-0.2148 \cdot 10^{-8}$	$-0\ .2772 \cdot 10^{-8}$	$-0.4734 \cdot 10^{-8}$	$-0.5151 \cdot 10^{-7}$
$h = 10^{-4}$	$-0.2146 \cdot 10^{-8}$	$-0\ .2769 \cdot 10^{-8}$	$-0.4728 \cdot 10^{-8}$	$-0.5121 \cdot 10^{-7}$
$h = 10^{-5}$	$-0.2145 \cdot 10^{-8}$	$-0.2769 \cdot 10^{-8}$	$-0.4727 \cdot 10^{-8}$	$-0.5119 \cdot 10^{-7}$

From the table it is clear that the efficient fractional order is $\alpha = 1.1$.

Conclusion

In this paper, an efficient algorithm for determining the fractional order in oscillatory systems with liquid dampers has been proposed. The approach is based on reducing the governing equation to a Volterra integral equation of the second kind with respect to the displacement of mass coordinates. It is shown that the first variation corresponding to the identified fractional order approaches zero with an accuracy of 10^{-8}, whereas in problem [6] the achieved accuracy is only of order 10^{-4}. This demonstrates the superior efficiency of the proposed algorithm.

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HƏRƏKƏT EDƏN KÜTLƏNİN KOORDİNATLARI VASİTƏSİLƏ RƏQSİ SİSTEMLƏRDƏ TƏRTİBİN TƏYİNİ

Nazilə Hacıyeva, İradə Əliyeva , Fikrət Əliyev

Xülasə. *İşdə Nyuton mayesi daxilində baş verən rəqsi prosesdə kəsr tərtibin müəyyən edilməsi problemi araşdırılır. Əvvəlcə, verilmiş tənlik kütlə koordinatlarının yerdəyişməsinə görə ikinci növ Volterra inteqral tənliyinə gətirilir. Daha sonra bu inteqral tənliyin həlli Neyman sırası şəklində ifadə olunur. Mövcud statistik məlumatlar əsasında kütlə yerdəyişmələrinin kənarlaşmasını xarakterizə edən kvadratik funksional qurulur və naməlum kəsr tərtibi ən kiçik kvadratlar metodu vasitəsilə müəyyən edilir. Kəsr tərtibinin tapılması üçün effektiv hesablama alqoritmi təklif olunur və nümunə üçün nəticələr cədvəl şəklində təqdim edilir. Hesablama nəticələri göstərir ki, müəyyən edilmiş kəsr tərtibi kvadratik funksionalın birinci variasiyasının sıfıra yaxınlaşması şərtini* 10^{-8} *dəqiqliklə ödəyir.*

Açar sözlər: *rəqsi sistem, kəsr törəmə, ikinci növ Volterra inteqral tənliyi, ən kiçik kvadratlar metodu*

ОПРЕДЕЛЕНИЕ ДРОБНОГО ПОРЯДКА КОЛЕБАТЕЛЬНЫХ СИСТЕМ С ИСПОЛЬЗОВАНИЕМ КООРДИНАТ ПЕРЕМЕЩЕНИЯ МАСС

Назиля Гаджиева, Ирада Алиева, Фикрет Алиев

Аннотация: *В настоящей работе рассматривается задача определения дробного порядка в колебательном процессе, происходящем в ньютоновской жидкости. Сначала исходное уравнение преобразуется к интегральному уравнению Вольтерра второго рода относительно перемещения координат масс. Затем решение данного интегрального уравнения представляется в виде ряда Неймана. На основе имеющихся статистических данных строится квадратичный функционал, характеризующий отклонение координат перемещения масс, и неизвестный дробный порядок определяется с использованием метода наименьших квадратов. Далее предлагается эффективный вычислительный алгоритм для определения дробного порядка, а иллюстративные результаты представлены в табличной форме. Результаты вычислительной процедуры показывают, что найденный дробный порядок удовлетворяет условию обращения в нуль первой вариации квадратичного функционала с точностью* 10^{-8}.

Ключевые слова: *колебательная система, дробная производная, интегральное уравнение Вольтерра второго рода, метод наименьших квадратов*